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Well-Posedness And Data Dependence Of Strict Fixed Point For δ-Hardy Roger Type Contraction And Applications*

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Abstract

We first familiarise with δ-Hardy Roger type contraction in the frame work of metric space. Then, well-posedness, data dependence, existence and uniqueness results of strict fixed point for δ-Hardy Roger type contraction are presented. The obtained results generalize the existing results in the literature. Applications to an integral inclusion equation and Fractals concludes the paper.

1 Introduction

Banach contraction principle [1] has been generalised in numerous directions and one such generalisation is due to Nadler [14], who generalised it considering set-valued contraction. There after many results are established for set-valued mappings (see for instance [3]-[8], [16]-[21]). In this paper, considering the fact that Hardy-Rogers type operator is a Ćirić type operator (however the reverse need not be true), we introduce δ-Hardy Roger type contraction and establish strict fixed point for it using iterations of a delta distance which is not even a metric. Also, we present well-posedness and data dependence of strict fixed point problem and utilise it to solve an integral inclusion equation and in presenting a novel iterated function framework via δ-Hardy-Roger type operator to obtain attractor of multifunction system.

Let (X, d) be a metric space,

$$P(X) = \{Y \subset X : Y \neq \emptyset\}, \quad P_b(X) = \{Y \in P(X) : Y \textit{ is bounded}\},$$

$$P_{cl}(X) = \{Y \in P(X) : Y \textit{ is closed}\} \text{ and } P_{cp}(X) = \{Y \in P(X) : Y \textit{ is compact}\}.$$

Define a set-valued operator as $\mathcal{T} : X \to P(X)$ and $\mathcal{T}(Y) = \cup_{x\in Y}\mathcal{T}(x)$ for $Y \in P(X)$. Also, $F_{\mathcal{T}} = \{x \in X : x \in \mathcal{T}(x)\}$ is a set of fixed points and $(SF)_{\mathcal{T}} = \{x \in X : \mathcal{T}x = \{x\}\}$ is a set of strict fixed points of the set-valued operator $\mathcal{T}$. Chifu and Petrusel [10] introduced the δ generalized functional as: $\delta : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\}$,

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

2 Main Results

Firstly, we define δ-Hardy Roger type contraction and establish strict fixed point making use of iterations of a delta distance which is not even a metric.

Definition 1 *If $\mathcal{T} : X \to P_b(X)$ is a set-valued operator of a metric space (X, d) satisfying*

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) \leq a_1 d(x, y) + a_2\delta(x, \mathcal{T}(x)) + a_3\delta(y, \mathcal{T}(y)) + a_4\delta(x, \mathcal{T}(y)) + a_5\delta(y, \mathcal{T}(x)),$$

where $a_i's \in \mathbb{R}^+$, $\sum_{i=1}^{5} a_i < 1$, $x, y \in X$, then $\mathcal{T}$ is called a δ-Hardy Roger type contraction.

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Theorem 1 *If $\mathcal{T}$ is a δ-Hardy Roger type contraction of a complete metric space (X, d), then $(SF)_{\mathcal{T}} = \{u^*\}$.*

Proof. Let $u_0 \in X$. Then there exists $u_1 \in \mathcal{T}(u_0)$ and

$$\delta(u_0, \mathcal{T}(u_0)) \leq qd(u_0, u_1), q > 1 \text{ is arbitrary.}$$

Now,

$$\begin{aligned}
\delta(u_1, \mathcal{T}(u_1)) &\leq \delta(\mathcal{T}(u_0), \mathcal{T}(u_1)) \\
&\leq a_1 d(u_0, u_1) + a_2 \delta(u_0, \mathcal{T}(u_0)) + a_3 \delta(u_1, \mathcal{T}(u_1)) \\
&\quad + a_4 \delta(u_0, \mathcal{T}(u_1)) + a_5 \delta(u_1, \mathcal{T}(u_0)) \\
&\leq a_1 d(u_0, u_1) + a_2 q d(u_0, u_1) + a_3 \delta(u_1, \mathcal{T}(u_1)) \\
&\quad + a_4 \{d(u_0, u_1) + \delta(u_1, \mathcal{T}(u_1))\} + a_5 \delta(u_1, u_1) \\
&\leq (a_1 + a_2 q + a_4) d(u_0, u_1) + (a_3 + a_4) \delta(u_1, \mathcal{T}(u_1)).
\end{aligned}$$

This implies

$$(1 - a_3 - a_4)\delta(u_1, \mathcal{T}(u_1)) \leq (a_1 + a_2 q + a_4) d(u_0, u_1),$$

or

$$\delta(u_1, \mathcal{T}(u_1)) \leq \left[\frac{a_1 + a_2 q + a_4}{1 - (a_3 + a_4)}\right] d(u_0, u_1).$$

Also, $u_1 \in \mathcal{T}(u_0)$, $\exists$ $u_2 \in \mathcal{T}(u_1)$ and

$$\delta(u_1, \mathcal{T}(u_1)) \leq qd(u_1, u_2).$$

Therefore,

$$\begin{aligned}
\delta(u_2, \mathcal{T}(u_2)) &\leq \delta(\mathcal{T}(u_1), \mathcal{T}(u_2)) \\
&\leq a_1 d(u_1, u_2) + a_2 \delta(u_1, \mathcal{T}(u_1)) + a_3 \delta(u_2, \mathcal{T}(u_2)) + a_4 \delta(u_1, \mathcal{T}(u_2)) \\
&\quad + a_5 \delta(u_2, \mathcal{T}(u_1)) \\
&\leq a_1 d(u_1, u_2) + a_2 q d(u_1, u_2) + a_3 \delta(u_2, \mathcal{T}(u_2)) + a_4 \{d(u_1, u_2) \\
&\quad + \delta(u_2, \mathcal{T}(u_2))\} + a_5 \delta(u_2, u_2) \\
&= (a_1 + a_2 q + a_4) d(u_1, u_2) + (a_3 + a_4) \delta(u_2, \mathcal{T}(u_2)).
\end{aligned}$$

This implies

$$(1 - (a_3 + a_4))\, \delta(u_2, \mathcal{T}(u_2)) \leq (a_1 + a_4 + a_2 q) d(u_1, u_2),$$

or

$$\begin{aligned}
\delta(u_2, \mathcal{T}(u_2)) &\leq \left[\frac{a_1 + a_4 + a_2 q}{1 - (a_3 + a_4)}\right] d(u_1, u_2) \\
&\leq \left[\frac{a_1 + a_4 + a_2 q}{1 - (a_3 + a_4)}\right] \delta(u_1, \mathcal{T}(u_1)) \\
&\leq \left[\frac{a_1 + a_4 + a_2 q}{1 - (a_3 + a_4)}\right]^2 d(u_0, u_1).
\end{aligned}$$

Following the similar pattern, we construct a sequence $(u_n)_{n \in \mathbb{N}}$ satisfying the following properties:

(i) $u_n \in \mathcal{T}(u_{n-1})$;

(ii) $d(u_n, u_{n+1}) \leq \delta(u_n, \mathcal{T}(u_n)) \leq \left[\frac{a_1 + a_4 + a_2 q}{1 - a_3 - a_4}\right]^n d(u_0, u_1)$.

Next, consider

$$\begin{aligned} d(u_n, u_{n+p}) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+p-1}, u_{n+p}) \\ &\leq \left[\left(\frac{a_1 + a_4 + a_2 q}{1 - a_3 - a_4} \right)^n + \left(\frac{a_1 + a_4 + a_2 q}{1 - a_3 - a_4} \right)^{n+1} \right. \\ &\quad \left. + ... + \left(\frac{a_1 + a_4 + a_2 q}{1 - a_3 - a_4} \right)^{n+p-1} \right] d(u_0, u_1). \end{aligned}$$

Assume that $\alpha = \frac{a_1+a_4+a_2q}{1-a_3-a_4}$. Then

$$d(u_n, u_{n+p}) \leq \alpha^n [1 + \alpha + ... + \alpha^{p-1}] d(u_0, u_1) = \alpha^n \frac{\alpha^p - 1}{\alpha - 1} d(u_0, u_1).$$

Choose $q < \frac{1-a_1-a_3-2a_4}{a_2}$ then $\alpha < 1$. Letting $n \to \infty$, we get

$$d(u_n, u_{n+p}) \to 0,$$

i.e., $(u_n)_{n \in N}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u^* \in X$ such that $u_n \to u^*$ as $n \to \infty$. Now, we shall demonstrate that $u^* \in (SF)_{\mathcal{T}}$. Consider,

$$\begin{aligned} \delta(u^*, \mathcal{T}(u^*)) &\leq d(u^*, u_n) + \delta(u_n, \mathcal{T}(u_n)) + \delta(\mathcal{T}(u_n), \mathcal{T}(u^*)) \\ &\leq d(u^*, u_n) + \delta(u_n, \mathcal{T}(u_n)) + a_1 d(u_n, u^*) + a_2 \delta(u_n, \mathcal{T}(u_n)) \\ &\quad + a_3 \delta(u^*, \mathcal{T}(u^*)) + a_4 \delta(u_n, \mathcal{T}(u^*)) + a_5 \delta(u^*, \mathcal{T}(u_n)) \\ &\leq d(u^*, u_n) + \delta(u_n, \mathcal{T}(u_n)) + a_1 d(u_n, u^*) + a_2 \delta(u_n, \mathcal{T}(u_n)) \\ &\quad + a_3 \delta(u^*, \mathcal{T}(u^*)) + a_4 \left(d(u_n, u^*) + \delta(u^*, \mathcal{T}(u^*)) \right) \\ &\quad + a_5 \left(d(u^*, u_n) + \delta(u_n, \mathcal{T}(u_n)) \right) \\ &= (1 + a_1 + a_4 + a_5) \, d(u^*, u_n) + (1 + a_2 + a_5) \, \delta(u_n, \mathcal{T}(u_n)) \\ &\quad + (a_3 + a_4) \delta(u^*, \mathcal{T}(u^*)). \end{aligned}$$

This implies

$$(1 - a_3 - a_4) \delta(u^*, \mathcal{T} u^*) \leq (1 + a_1 + a_4 + a_5) d(u^*, u_n) + (1 + a_2 + a_5) \delta(u_n, \mathcal{T}(u_n)),$$

or

$$\delta(u^*, \mathcal{T} u^*) \leq \left[\frac{1 + a_1 + a_4 + a_5}{1 - a_3 - a_4} \right] d(u^*, u_n) + \left[\frac{1 + a_2 + a_5}{1 - a_3 - a_4} \right] \delta(u_n, \mathcal{T}(u_n)).$$

Since $\delta(u_n, \mathcal{T}(u_n)) \leq \alpha^n d(u_0, u_1)$, we see that $\delta(u^*, \mathcal{T} u^*) = 0$. It implies that $T(u^*) = \{u^*\}$, i.e., $u^* \in (SF)_{\mathcal{T}}$. For uniqueness, assume that there exists two distinct points $u^*, v^* \in (SF)_{\mathcal{T}}$. So

$$\begin{aligned} d(u^*, v^*) &= \delta(\mathcal{T}(u^*), \mathcal{T}(v^*)) \\ &\leq a_1 d(u^*, v^*) + a_2 \delta(u^*, \mathcal{T}(u^*)) + a_3 \delta(v^*, \mathcal{T}(v^*)) + a_4 \delta(u^*, \mathcal{T}(v^*)) + a_5 \\ &\quad \delta(v^*, \mathcal{T}(u^*)) \\ &\leq a_1 d(u^*, v^*) + a_2 \delta(u^*, \mathcal{T}(u^*)) + a_3 \delta(v^*, \mathcal{T}(v^*)) + a_4 (d(u^*, v^*) + \delta(v^*, \\ &\quad \mathcal{T}(v^*))) + a_5 \left(d(v^*, u^*) + \delta(u^*, \mathcal{T}(u^*)) \right) \\ &\leq (a_1 + a_4 + a_5) d(u^*, v^*) + a_2 \delta(u^*, \mathcal{T} u^*) + a_3 \delta(v^*, \mathcal{T} v^*) + a_4 \delta(v^*, \mathcal{T} v^*) + \\ &\quad a_5 \delta(u^*, \mathcal{T} u^*). \end{aligned}$$

This implies

$$(1 - a_1 - a_4 - a_5) d(u^*, v^*) \leq a_2 \delta(u^*, \mathcal{T} u^*) + a_3 \delta(v^*, \mathcal{T} v^*) + a_4 \delta(v^*, \mathcal{T} v^*) + a_5 \delta(u^*, \mathcal{T} u^*),$$

or

$$(1 - a_1 - a_4 - a_5)d(u^*, v^*) \leq 0,$$

or $1 - a_1 - a_4 - a_5 \leq 0$ or $a_1 + a_4 + a_5 \geq 1$, a contradiction to the fact that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Hence $u^* = v^*$. ■

Example 1 *Let $X = [0, 3]$ be a complete metric space and the mapping $\mathcal{T} : X \to P_b(X)$ be defined as*

$$\mathcal{T}(x) = \begin{cases} [0, 1], & 0 \leq x < 2, \\ \{2\}, & 2 \leq x \leq 3. \end{cases}$$

Taking $a_1 = \frac{3}{10}, a_2 = 0, a_3 = \frac{1}{20}, a_4 = \frac{1}{4}, a_5 = \frac{7}{20}$, $a_1 + a_2 + a_3 + a_4 + a_5 = \frac{19}{20} < 1$. Now we have following cases:

Case I: *When $x, y \in [0, 2)$,*

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) = 1 \leq a_1.2 + a_2.2 + a_3.2 + a_4.2 + a_5 \leq 2.\frac{19}{20}.$$

Case II: *When $x \in [0, 2)$ and $y \in [2, 3]$,*

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) = 2 \leq a_1.3 + a_2.2 + a_3.1 + a_4.2 + a_5.3 \leq \frac{45}{20}.$$

Case III: *When $x \in [2, 3]$ and $y \in [0, 2)$,*

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) = 2 \leq a_1.2 + a_2.1 + a_3.2 + a_4.3 + a_5.2 \leq \frac{43}{20}.$$

Case IV: *When $x, y \in [2, 3]$,*

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) = 0 \leq a_1 + a_2 + a_3 + a_4 + a_5 \leq \frac{19}{20}.$$

Subsequently, all the hypotheses of Theorem 1 are verified and $x = 2$ is the only strict fixed point of a discontinuous set-valued operator $\mathcal{T}$.

Next, we try to establish sufficient conditions for the well-posedness of a strict fixed point problem for the set-valued operator.

Theorem 2 *If $\mathcal{T}$ is a δ-Hardy Roger type contraction of a complete metric space (X, d), then the strict fixed point is well-posed for $\mathcal{T}$ with respect to δ_d.*

Proof. Using Theorem 1, $(SF)_{\mathcal{T}} = \{u^*\}$. Suppose $u_n \in X, n \in N$ satisfying

$$\delta_d(u_n, \mathcal{T}(u_n)) \to 0 \quad \text{as } n \to \infty.$$

We next prove that $u_n \to u^*$ as $n \to \infty$. Now,

$$\begin{aligned} d(u_n, u^*) &\leq \delta_d(u_n, \mathcal{T}(u_n)) + \delta_d(\mathcal{T}(u_n), \mathcal{T}(u^*)) \\ &\leq \delta_d(u_n, \mathcal{T}(u_n)) + a_1 d(u_n, u^*) + a_2 \delta_d(u_n, \mathcal{T}(u_n)) + a_3 \delta_d(u^*, \mathcal{T}(u^*)) + a_4 \\ &\quad \delta_d(u_n, \mathcal{T}(u^*)) + a_5 \delta_d(u^*, \mathcal{T}(u_n)) \\ &\leq \delta_d(u_n, \mathcal{T}(u_n)) + a_1 d(u_n, u^*) + a_2 \delta_d(u_n, \mathcal{T}(u_n)) + a_3 \delta_d(u^*, \mathcal{T}(u^*)) + a_4 \\ &\quad (d(u_n, u^*) + \delta_d(u^*, \mathcal{T}(u^*)) + a_5(d(u^*, u_n) + \delta_d(u_n, \mathcal{T}(u_n)). \end{aligned}$$

This implies

$$(1 - a_1 - a_4 - a_5)d(u_n, u^*) \leq (1 + a_2 + a_5)\delta_d(u_n, \mathcal{T}(u_n)),$$

or

$$d(u_n,u^*) \le \left(\frac{1+a_2+a_5}{1-a_1-a_4-a_5}\right)\delta_d(u_n,\mathcal{T}(u_n)) \to 0 \ \ as \ \ n\to\infty,$$

i.e., $u_n \to u^*$ as $n \to \infty$. ■

Now, we establish a data dependence result.

Theorem 3 *If $\mathcal{T}_1$ is a δ-Hardy Roger type contraction of a complete metric space (X,d) and $\mathcal{T}_2 : X \to P_b(X)$ is a set-valued operator such that*

(i) $(SF)_{\mathcal{T}_2} \neq \emptyset$;

(ii) $\exists\ \eta > 0$ *satisfying* $\delta(\mathcal{T}_1(x),\mathcal{T}_2(x)) \le \eta$, $x \in X$,

then

$$\delta(u_1^*,(SF)_{\mathcal{T}_2}) \le \left(\frac{1+a_3+a_4}{1-a_1-a_4-a_5}\right)\eta.$$

Proof. Let $u_2^* \in (SF)_{(\mathcal{T}_2)}$. So, $\delta(u_2^*,\mathcal{T}_2(u_2^*)) = 0$. Now,

$$\begin{aligned}
d(u_1^*,u_2^*) &= \delta(\mathcal{T}_1(u_1^*),\mathcal{T}_2(u_2^*))\\
&\le \delta(\mathcal{T}_1(u_1^*),\mathcal{T}_1(u_2^*)) + \delta(\mathcal{T}_1(u_2^*),\mathcal{T}_2(u_2^*))\\
&\le a_1 d(u_1^*,u_2^*) + a_2\delta(u_1^*,\mathcal{T}_1(u_1^*)) + a_3\delta(u_2^*,\mathcal{T}_1(u_2^*)) + a_4\delta(u_1^*,\mathcal{T}_1(u_2^*))\\
&+ a_5\delta(u_2^*,\mathcal{T}_1(u_1^*)) + \delta(\mathcal{T}_1(u_2^*),\mathcal{T}_2(u_2^*))\\
&\le a_1 d(u_1^*,u_2^*) + a_3\delta(u_2^*,\mathcal{T}_1(u_2^*)) + a_4(d(u_1^*,u_2^*) + \delta(u_2^*,\mathcal{T}_1(u_2^*)))\\
&+ a_5(d(u_2^*,u_1^*) + \delta(u_1^*,\mathcal{T}_1(u_1^*))) + \delta(\mathcal{T}_1(u_2^*),\mathcal{T}_2(u_2^*)).
\end{aligned}$$

This implies

$$\begin{aligned}
(1-a_1-a_4-a_5)d(u_1^*,u_2^*) &\le a_3\delta(u_2^*,\mathcal{T}_1(u_2^*)) + a_4\delta(u_2^*,\mathcal{T}_1(u_2^*)) + \delta(\mathcal{T}_1(u_2^*),\mathcal{T}_2(u_2^*))\\
&\le a_3\eta + a_4\eta + \eta\\
&\le (1+a_3+a_4)\eta,
\end{aligned}$$

or

$$d(u_1^*,u_2^*) \le \left(\frac{1+a_3+a_4}{1-a_1-a_4-a_5}\right)\eta.$$

By taking supremum, it follows that

$$\delta(u_1^*,(SF)_{\mathcal{T}_2}) \le \left(\frac{1+a_3+a_4}{1-a_1-a_4-a_5}\right)\eta.$$

■

3 Applications

3.1 Application to Volterra Integral Inclusion.

Now, we utilise Theorem 1 to solve the following Volterra integral inclusion

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)F(s,x(s))ds, \tag{1}$$

for $t \in J$, $\sigma : J \to J, k : J \times J \to R$, $q : J \to E$ are continuous, $F : J \times E \to C(E)$, E is a real Banach space with norm $\|.\|_E$, $C(E)$ is the class of all non-empty closed subsets of E and $J = [0, 1]$ in R is a closed and bounded interval. Let $C(J, E)$ is the space of all continuous E-valued functions on J and $\|x\| = sup_{t \in J} |x(t)|_E$. We use the following definitions.

Definition 2 *A set-valued function $\beta : J \times E \to 2^E$ is Carathèodory if*

(i) *$t \to \beta(t, x)$ is measurable, $x \in E$ and*

(ii) *$x \to \beta(t, x)$ is upper semi-continuous a. e. for $t \in J$.*

Definition 3 *A Carathèodory multifunction $F(t, X)$ is L^1-Carathèodory if for every real number $r > 0$ $\exists$ a function $h_r \in L^1(J, R)$ satisfying $\|F(t, x)\| \leq h_r t$ for almost every $t \in J$ and $x \in E$ and $\|x\|_E \leq r$. Denote $\|F(t, x(t))\| = \sup\{\|u\|_E : u \in F(t, x(t))\}$, $\mathcal{T}_F^1 = \{v \in B(J, E) : v(t) \in F(t, x(t))$ a.e. $t \in J\}$, where $B(J, E)$ is the space of all E-valued Bochner-integrable functions on J.*

Lemma 1 ([13]) *If diam$(E) < \infty$ and $F : J \times E \to 2^E$ is L^1- Carathèodory, then $\mathcal{T}_F^1 \neq \phi$, $x \in E$.*

Lemma 2 ([13]) *Let E be a Banach space, $L : L^1(J, E) \to C(J, E)$ a continuous linear mapping and F a Carathèodory set-valued mapping such that $\mathcal{T}_F^1 \neq \phi$. Then $Lo\mathcal{T}_F^1 : C(J, E) \to 2^{C(J,E)}$ is a closed graph operator on $C(J, E) \times C(J, E)$.*

Theorem 4 *Suppose that the following set of hypotheses hold.*

(i) *The function $k(t, s)$ is non-negative on $J \times J$ and $M = \sup_{t,s \in J}[k(t, s)]$;*

(ii) *the set-valued function $F(t, x)$ is Carathèodory;*

(iii) *the set-valued function $F(t, x)$ is nondecreasing in x a.e. for $t \in J$;*

(iv) *$|F(s, x(s)) - F(s, y(s))| \leq \frac{1}{M}(\Theta(x, y))$, $s \in J, x \in E$, where*

$$\Theta(x, y) = a_1 d(x, y) + a_2 \delta(x, \mathcal{T}(x)) + a_3 \delta(y, \mathcal{T}(y)) + a_4 \delta(x, \mathcal{T}(y)) + a_5 \delta(y, \mathcal{T}(x)),$$

$a_i' s \in \mathbb{R}^+$, $\sum_{i=1}^{5} a_i < 1$;

(v) *$\mathcal{T}_F^1 \neq \phi$, $x \in C(J, E)$.*

Then the integral inclusion (1) has a solution in J.

Proof. A continuous function $x : J \to E$ is a solution of the integral inclusion (1), if

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t, s) v(s) ds,$$

where $v \in B(J, E)$ such that $v(t) \in F(t, x(t))$. Define the set-valued mapping $\mathcal{T} : [0, 1] \to 2^X$ as

$$\mathcal{T}(x) = q(t) + \int_0^{\sigma(t)} k(t, s) v(s) ds,$$

where $v \in \mathcal{T}_F^1(x)$ for every $t \in [0, 1]$. Clearly $\mathcal{T}$ is well-defined, since, from (v), $\mathcal{T}_F^1 \neq \phi$. For all $t \in [0, 1]$ by (ii) and (iv), we get

$$|\mathcal{T}(x) - \mathcal{T}(y)| = \left| \int_0^{\sigma(t)} \Big(k(t, s) v_1(s) - k(t, s) v_2(s) \Big) ds \right|_E, v_1, v_2 \in T_F^1(x).$$

Taking supremum on both sides

$$\delta(\mathcal{T}(x), \mathcal{T}(y)) \leq M |F(s, x(s)) - F(s, y(s))| \leq \Theta(x, y),$$

i.e., the operator $\mathcal{T}$ verify the hypotheses of the Theorem 1 on $[0, 1]$ and consequently, the given integral inclusion has a unique solution. ∎

3.2 Application to Fractals.

Fixed point theory performs a significant role in fractals that are the self-similar sets. Iterated function systems define fractals as attractors in discrete dynamical frameworks and can be applied to wavelet analysis, quantum physics, computer graphics and different applied sciences. This concept was first introduced by Hutchinson [11] and popularized by Barnsley [2] as a natural generalization of the celebrated Banach contraction principle. Now, we present novel iterated function framework utilizing the δ-Hardy Rogers type operators which covers a large range of operators. The operator

$$\mathcal{T} : P_{cp}(X) \to P_{cp}(X), \mathcal{T}(Y) = \cup_{i=1}^{m} \mathcal{T}_i(Y), Y \in P(X)$$

is the multifractal operator generated by $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_m)$, such that $\mathcal{T}_i : X \to P_{cp}(X)$. A fixed point $V^* \in P_{cp}(X)$ of $\mathcal{T}$ is an attractor of the iterated multifunction system $\mathcal{T}$. Next, we establish existence of an attractor.

Theorem 5 *Let $\mathcal{T}_i : X \to P_{cp}(X)$, $i \in \{1, ..., m\}$ be a finite family of set-valued operator of a complete metric space (X, d) such that*

$$\begin{aligned} \delta(\mathcal{T}_i(x), \mathcal{T}_i(y)) &\leq A_1 d(x,y) + A_2 \delta(x, \mathcal{T}_i(x)) + A_3 \delta(y, \mathcal{T}_i(y)) + A_4 \delta(x, \mathcal{T}_i(y)) \\ &\quad + A_5 \delta(y, \mathcal{T}_i(x)), \end{aligned}$$

$A_j's \in \mathbb{R}^+$, $\sum_{j=1}^{5} A_j < 1$, $x, y \in X$. Then the operator $\mathcal{T} : P_{cp}(X) \to P_{cp}(X)$ defined by $\mathcal{T}(B) = \cup_{i=1}^{m} \mathcal{T}i(B)$ for all $B \in P_{cp}(X)$ satisfies:

$$\begin{aligned} \delta(\mathcal{T}(B), \mathcal{T}(C)) &\leq A_1 d(B,C) + A_2 \delta(B, \mathcal{T}(B)) + A_3 \delta(C, \mathcal{T}(C)) + A_4 \delta(B, \mathcal{T}(C)) \\ &\quad + A_5 \delta(C, \mathcal{T}(B)), \end{aligned}$$

where $B, C \in P_{cp}(X)$ and has attractor A in $(P_{cp}(X), \delta(d))$ such that $A = \mathcal{T}(A) = \cup_{i=1}^{m} \mathcal{T}_i(A)$, $A = \lim_{n\to\infty} \mathcal{T}^{o^n}(B)$ and $B \in P_{cp}(X)$.

Proof. Let $F \in P_{cp}(X)$, then F is a non-empty and compact in X. Clearly $\mathcal{T}(F)$ is non-empty. Now we establish that $\mathcal{T}(F)$ is compact in X. If $\{y_n\} \subset \mathcal{T}(F)$, then there is a sequence $\{u_n\} \subset F$ satisfying $y_n = \mathcal{T} u_n (n = 1, 2, ...)$. Compactness of F implies that there is a subsequence $\{x_{n_k}\} \subset \{u_n\}$ such that $\{x_{n_k}\} \to x \in F$. Since T is continuous, $\{y_{n_k}\} = \mathcal{T} x_{n_k} \to \mathcal{T} x \in \mathcal{T}(F)$. Hence, $\mathcal{T}(F)$ is compact in X. Definition of δ-Hardy Roger type contraction shows that $\mathcal{T}$ satisfies

$$\begin{aligned} \delta(\mathcal{T}(B), \mathcal{T}(C)) &\leq A_{1_1} d(B,C) + A_{2_1} \delta(B, \mathcal{T}(B)) + A_{3_1} \delta(C, \mathcal{T}(C)) + A_{4_1} \delta(B, \mathcal{T}(C)) \\ &\quad + A_{5_1} \delta(C, \mathcal{T}(B)), \quad \forall \quad B, C \in P_{cp}(X). \end{aligned}$$

Now, we shall use the principle of mathematical induction. The statement is clearly true for $m = 1$. Now, for $m = 2$,

$$\begin{aligned} \delta\left(\mathcal{T}(B), \mathcal{T}(C)\right) &= \delta\left(\mathcal{T}_1(B) \cup \mathcal{T}_2(B), \mathcal{T}_1(C) \cup \mathcal{T}_2(C)\right) \\ &\leq \max\{A_{1_2}\} \delta(B,C) + \max\{A_{2_2}\} \delta(B, \mathcal{T}_1(B)) \\ &\quad + \max\{A_{3_2}\} \delta(C, \mathcal{T}_1(C)) + \max\{A_{4_2}\} \delta(B, \mathcal{T}_1(C)) \\ &\quad + \max\{A_{5_2}\} \delta(C, \mathcal{T}_1(B)). \end{aligned}$$

Hence by induction inequality is true for all $i \in 1, ..., m$, i.e.,

$$\begin{aligned} \delta(\mathcal{T}(B), \mathcal{T}(C)) &\leq A_1 d(B,C) + A_2 \delta(B, \mathcal{T}_n(B)) + A_3 \delta(C, \mathcal{T}_n(C)) + A_4 \delta(B, \mathcal{T}_n(C)) \\ &\quad + A_5 \delta(C, \mathcal{T}_n(B)), \end{aligned}$$

where $A_1 = \max\{A_{1j}\}$, $A_2 = \max\{A_{2j}\}, ..., A_5 = \max\{A_{5j}\}$, i.e., the operator T satisfy all the hypotheses of Theorem 1 and consequently T has a attractor $A = T(A) = \cup_{i=1}^{m} T_i(A)$ and $A = \lim T^{o^n}(B)$, $B \in P_{cp}(X)$. ∎

4 Remarks

Remark 1 *Choosing suitably the values of constants in Theorems 1, 2 and 3, similar results can be established for Kannan [12], Chatterjee [9], Reich [15] and Banach [1] type set-valued contractions.*

Remark 2 *In Theorem 5, we established the attractors of δ-Hardy Roger type set-valued iterated function systems, which generalizes the celebrated Hutchinson iterated function systems.*

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