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Progressively Type-II Right Censored Order Statistics From Doubly Truncated Generalized Exponential Distribution*

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Abstract

In this paper, we establish some new recurrence relations for the single and product moments of progressively Type-II right censored order statistics from the doubly truncated generalized exponential distribution. These relations generalize the results given by Balakrishnan *et al.* [1] for the progressively Type-II right censored order statistics for standard exponential and right truncated exponential distributions. These results, also generalize the corresponding results for usual order statistics due to Saran and Pushkarna [2] for the doubly truncated generalized exponential distribution. In the last section, some deduction and particular cases are also discussed.

1 Introduction

Progressive Type-II censored sampling scheme is a versatile censoring scheme because it allows the experimenter to save time and cost of the life-testing experiment and is quite useful in reliability and life-time studies. This progressive censoring scheme can be briefly described as follows.

Consider an experiment in which n independent and identical items are put on test and their failure times are recorded. These failure times are assumed to be continuous and identically distributed. Suppose a censoring scheme is defined using a set of prefixed integers $\tilde{R} = (R_1, R_2, \ldots, R_m)$ such that at the time of the first failure, R_1 surviving items are removed from the experiment at random from the remaining $n-1$ surviving items, at the time of the second observed failure, R_2 surviving items are removed from the experiment at random from the remaining $n-2-R_1$ surviving items, and so on. The process continues until the $m-th$ failure time at which, all the remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving items are removed from the experiment. We shall denote the m ordered observed failure times by $X_{1:m:n}^{\tilde{R}}, \ldots, X_{m:m:n}^{\tilde{R}}$, and call them the progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$, $m \le n$. If the failure times

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of the n items originally on test are based on the absolutely continuous distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, then the joint probability density function of $X^{\tilde{R}}_{1:m:n}, \ldots, X^{\tilde{R}}_{m:m:n}$ is given by Balakrishnan et al. [1], Balakrishnan and Sandhu [3] and Balakrishnan [4] as

$$f^{X^{\tilde{R}}_{1:m:n},\ldots,X^{\tilde{R}}_{m:m:n}}(x_1,\ldots,x_m) = c(n,m-1)\prod_{j=1}^{m} f(x_j)[1-F(x_j)]^{R_j}, \tag{1}$$

$$Q_1 \le x_1 < x_2 < \cdots < x_m \le P_1,$$

where

$$n = m + \sum_{j=1}^{m} R_j,\ n,m \in \mathrm{N},\ R_j = \mathrm{N}_0,\ 1 \le j \le m,\ \tilde{R} = (R_1,\ldots,R_m),$$

and

$$c(n,m-1) = \prod_{j=1}^{m}\left(n - \sum_{i=1}^{j-1} R_i - j + 1\right) = \begin{cases} \prod_{j=1}^{m} R^*_j, & \text{if } m \ge 1, \\ 1, & \text{if } m < 1, \end{cases} \tag{2}$$

with

$$R^*_j = \sum_{k=j}^{m}(R_k+1).\ \text{ Observe that } R^*_1 = n.$$

The model of ordinary order statistics is contained in the above set up by choosing $\tilde{R} = (0,\ldots,0)$ (*i.e.* $m = n$) as censoring schemes, where no withdrawals are made.

Progressive censoring and associated inferential procedures have been extensively studied in the literature for a number of distributions by Cohen [5, 6, 7, 8, 9], Mann [10, 11], Cohen and Whitten [12], Viveros and Balakrishnan [13], and among others. Aggarwala and Balakrishnan [14] and Balakrishnan and Aggarwala [15] have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distribution and their truncated forms. Also, Saran and Pushkarna [16] have obtained several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from doubly truncated Burr distribution. Mahmoud et al. [17] derived some recurrence relations for single and product moments of progressively Type-II right censored order statistics from linear exponential distribution and also obtained maximum likelihood estimates (MLEs) of the location and scale parameters. Balakrishnan et al. [18] and Balakrishnan and Saleh [19, 20, 21] have established several recurrence relations for single and product moments of progressively Type-II right censored order statistics from logistic, half-logistic, log-logistic and generalized half logistic distributions respectively and the moments so determined are then utilized in inferential method to derive best linear unbiased estimators of the scale and location parameters. For more results, one may refer to Athar et al. [22], Athar and Akhter [23] and references therein.

Hosking [24, 25] suggested, discussed and applied many generalizations of standard distributions. The family of generalized exponential distribution is an example of such a generalization. This distribution is from an IFR (Increasing Failure Rate) family and is widely used as a life-span model (cf. Cohen and Whitten [12]). The truncated form of life-span models are often of great interest in reliability studies, for more detail one may see Cohen [9].

Let $X^{\tilde{R}}_{1:m:n} < X^{\tilde{R}}_{2:m:n} < \cdots < X^{\tilde{R}}_{m:m:n}$ be the m ordered observed failure times in a sample of size n under progressive Type-II right censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$ from the doubly truncated generalized exponential distribution with *pdf*,

$$f(x) = \frac{(1-\alpha x)^{\frac{1}{\alpha}-1}}{P-Q}, \quad Q_1 \leq x \leq P_1 \text{ and } 0 \leq \alpha < 1, \tag{3}$$

with the corresponding *df*,

$$F(x) = \frac{(1-\alpha Q_1)^{\frac{1}{\alpha}} - (1-\alpha x)^{\frac{1}{\alpha}}}{P-Q}, \quad Q_1 \leq x \leq P_1, \tag{4}$$

where $(1-P)$ and Q respectively, are the proportion of truncation on the right and the left of the distribution.

The exponential distribution is considered as special cases of (3), when the shape parameter $\alpha \to 0$.

Here

$$Q_1 = \frac{1-(1-Q)^{\alpha}}{\alpha} \quad \text{and} \quad P_1 = \frac{1-(1-P)^{\alpha}}{\alpha}.$$

Let

$$P_2 = \frac{(1-P)}{(P-Q)} \quad \text{and} \quad Q_2 = \frac{(1-Q)}{(P-Q)}.$$

It is noted that from (3) and (4),

$$(1-\alpha x) f(x) = P_2 + [1 - F(x)]. \tag{5}$$

The relation in (5) is the "characterizing differential equation" for the distribution in (3).

Now, we shall denote

$$\mu^{(R_1,\ldots,R_m)^{(k_1)}}_{r:m:n} = E[\{X^{\tilde{R}}_{r:m:n}\}^{k_1}] = \mu^{(R_1,\ldots,R_{r-1},R^*_r-1)^{(k_1)}}_{r:r:n}, \tag{6}$$

where $1 \leq r \leq m \leq n,\ k_1 \geq 0,\ R^*_r - 1 \geq m - r \geq 0$, and

$$\mu^{(R_1,\ldots,R_m)^{(k_1,k_2)}}_{r,s:m:n} = E[\{X^{\tilde{R}}_{r:m:n}\}^{k_1}\{X^{\tilde{R}}_{r:m:n}\}^{k_2}] = \mu^{(R_1,\ldots,R_{s-1},R^*_s-1)^{(k_1,k_2)}}_{r,s:s:n}, \tag{7}$$

where $1 \leq r < s \leq m \leq n,\ k_1, k_2 \geq 0$, as given by Balakrishnan et al. [1].

Saran and Pushkarna [2] utilized the general results obtained by Khan et al. [26, 27] and established some recurrence relations for single and product moments of order statistics from the doubly truncated generalized exponential distribution.

The progressively type II right censored order statistics of the distribution given in (3) to the best of our knowledge have not been studied in the literature. The main aim of this paper is to fill this gap. Therefore, in this article, we have obtained the recurrence relations for the single and product moments of progressively Type-II right censored order statistics from a doubly truncated generalized exponential distribution. These relations, when used in a systematic manner, enable the recursive computation of moments for all sample sizes and all possible progressive censoring schemes. Further, the contents of the paper also generalize the results given by Balakrishnan et al. [1] for the progressively Type-II right censored order statistics for standard exponential and right truncated exponential distributions. These results, also generalize the corresponding results for usual order statistics due to Saran and Pushkarna [2] for the doubly truncated generalized exponential distribution.

The rest of the paper is organized as follows. In Section 2, we derive expressions for the single moments of progressively Type-II right censored order statistics from the doubly truncated generalized exponential distribution. Section 3 is devoted to expressions for product moments. In Section 4, some deduction and particular cases are discussed. Section 5 ends with some concluding remarks.

2 Single Moments

In this section, we shall exploit the relation (5) to derive recurrence relations for the single moments of progressively Type-II right censored order statistics from the doubly truncated generalized exponential distribution. In view of (1) the k_1^{th} single moment of the r^{th} progressively Type-II right censored order statistic is given as

$$\begin{aligned}\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(k_1)}} &= E[\{X_{r:m:n}^{\tilde{R}}\}^{k_1}] \\ &= c(n,m-1)\underset{Q_1\le x_1<\cdots<x_m\le P_1}{\iint\cdots\int} x_r^{k_1} f(x_1)[1-F(x_1)]^{R_1} f(x_2)[1-F(x_2)]^{R_2} \\ &\quad \times\cdots\times f(x_m)[1-F(x_m)]^{R_m}\,dx_1\cdots dx_m, \qquad (8)\end{aligned}$$

where $c(n, m-1)$ is defined in (2). The single moments of progressively Type-II right censored order statistics in (8) satisfy the following recurrence relations.

THEOREM 2.1. For the doubly truncated generalized exponential distribution as given in (3) and for $2 \le m \le n-1$ and $k_1 \ge 0$,

$$\begin{aligned}\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1+1)}} &= \frac{1}{[R_m^*+\alpha(k_1+1)]}\Bigg[(i+1)\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1)}} \\ &\quad +P_2\left\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(k_1+1)}}\right. \\ &\quad \left.-\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(k_1+1)}}\right\} \\ &\quad +R_m^*\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(k_1+1)}}\Bigg]. \qquad (9)\end{aligned}$$

PROOF. From (8), we have

$$\begin{aligned}
& \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k_1)}} - \alpha\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}} \\
= \; & c(n,m-1)\iint\cdots\int_{Q_1\le x_1<\cdots<x_{m-1}\le P_1} Z(x_{m-1})f(x_1)[1-F(x_1)]^{R_1} \\
& \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}dx_1\cdots dx_{m-1}, \qquad (10)
\end{aligned}$$

where

$$Z(x_{m-1}) = \int_{x_{m-1}}^{P_1} x_m^{k_1}[1-\alpha x_m]f(x_m)[1-F(x_m)]^{R_m}dx_m. \qquad (11)$$

Now using relation (5) in (11), we get

$$\begin{aligned}
Z(x_{m-1}) &= \int_{x_{m-1}}^{P_1} x_m^{k_1}\{P_2+[1-F(x_m)]\}[1-F(x_m)]^{R_m}dx_m \\
&= P_2\int_{x_{m-1}}^{P_1} x_m^{k_1}[1-F(x_m)]^{R_m}dx_m + \int_{x_{m-1}}^{P_1} x_m^{k_1}[1-F(x_m)]^{R_m+1}dx_m \qquad (12) \\
&= \frac{1}{(k_1+1)}\Bigg[P_2\Bigg\{-x_{m-1}^{k_1+1}[1-F(x_{m-1})]^{R_m} \\
&\quad +R_m\int_{x_{m-1}}^{P_1} x_m^{k_1+1}[1-F(x_m)]^{R_m-1}f(x_m)dx_m\Bigg\} \\
&\quad -x_{m-1}^{k_1+1}[1-F(x_{m-1})]^{R_m+1} \\
&\quad +(R_m+1)\int_{x_{m-1}}^{P_1} x_m^{k_1+1}[1-F(x_m)]^{R_m}f(x_m)dx_m\Bigg]. \qquad (13)
\end{aligned}$$

Substituting the resultant expression of $Z(x_{m-1})$ from (13) in (10), we get

$$\begin{aligned}
& \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k_1)}} - \alpha\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}} \\
= \; & \frac{1}{(k_1+1)}\Bigg[P_2\Bigg\{-c(n,m-1)\iint\cdots\int_{Q_1\le x_1<\cdots<x_{m-1}\le P_1} x_{m-1}^{k_1+1}f(x_1)[1-F(x_1)]^{R_1} \\
& \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m}dx_1\cdots dx_{m-1} \\
& +R_m c(n,m-1)\iiint\cdots\int_{Q_1\le x_1<\cdots<x_m\le P_1} x_m^{k_1+1}f(x_1)[1-F(x_1)]^{R_1} \\
& \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}f(x_m)[1-F(x_m)]^{R_m-1}dx_1\cdots dx_m\Bigg\} \\
& -c(n,m-1)\int\int\cdots\int_{Q_1\le x_1<\cdots<x_{m-1}\le P_1} x_{m-1}^{k_1+1}f(x_1)[1-F(x_1)]^{R_1} \\
& \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m+1}dx_1\cdots dx_{m-1}
\end{aligned}$$

$$+(R_m+1)c(n,m-1)\underset{Q_1\le x_1<\cdots<x_m\le P_1}{\iint\cdots\int} x_m^{k_1+1} f(x_1)[1-F(x_1)]^{R_1}$$

$$\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} f(x_m)[1-F(x_m)]^{R_m}dx_1\cdots dx_m\Bigg]$$

$$= \frac{1}{(k_1+1)}\Bigg[P_2\Bigg\{-\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(k_1+1)}}$$

$$+\frac{c(n,m-1)}{c(n-1,m-1)}R_m\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(k_1+1)}}\Bigg\}$$

$$-(n-R_1-\cdots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k_1+1)}}$$

$$+(R_m+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}}\Bigg],$$

or

$$\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1)}}-\alpha\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1+1)}}$$

$$= \frac{1}{(k_1+1)}\Bigg[P_2\Bigg\{-\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(k_1+1)}}$$

$$+\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(k_1+1)}}\Bigg\}$$

$$-R_m^*\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(k_1+1)}}+R_m^*\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1+1)}}\Bigg].$$

After rearranging the terms, we get

$$\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1+1)}} = \frac{1}{[R_m^*+\alpha(k_1+1)]}\Bigg[(k_1+1)\mu_{m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1)}}$$

$$+P_2\Bigg\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(k_1+1)}}$$

$$-\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(k_1+1)}}\Bigg\}$$

$$+R_m^*\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(k_1+1)}}\Bigg],$$

and, hence the theorem.

THEOREM 2.2. For $2\le r\le m-1$, $m\le n-1$ and $k_1\ge 0$,

$$\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}} = \mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(k_1+1)}}$$

$$= \frac{1}{[R_r^*+\alpha(k_1+1)]}\Bigg[(k_1+1)\mu_{r:r:n}^{(R_1,\ldots,R_r^*-1)^{(k_1)}}$$

$$+P_2\left\{\frac{c(n,r-1)}{c(n-1,r-2)}\mu_{r-1:r-1:n-1}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-2)^{(k_1+1)}}\right.$$
$$\left.-\frac{c(n,r-1)}{c(n-1,r-1)}(R_r^*-1)\mu_{r:r:n-1}^{(R_1,\ldots,R_{r-1},R_r^*-2)^{(k_1+1)}}\right\}$$
$$\left.+R_r^*\mu_{r-1:r-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-1)^{(k_1+1)}}\right], \tag{14}$$

where $R_r^*-2 \geq m-r-1 \geq 0$.

PROOF. The theorem can be proved on the lines of Theorem 2.1 and using (6).

COROLLARY 2.1. Under the conditions as stated in Theorem 2.2

$$\begin{aligned}
&\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}}\\
= \;&\mu_{1:1:n}^{(R_1^*-1)^{(k_1+1)}}\\
= \;&\frac{1}{[n+\alpha(k_1+1)]}\left[(k_1+1)\mu_{1:1:n}^{(R_1^*-1)^{(k_1)}} - nP_2\mu_{1:1:n-1}^{(R_1^*-2)^{(k_1+1)}} + nQ_2Q_1^{k_1+1}\right] \quad (15)
\end{aligned}$$

and subsequently for $n=1$,

$$\mu_{1:1:1}^{(0)^{(k_1+1)}} = \frac{1}{[1+\alpha(k_1+1)]}\left[(k_1+1)\mu_{1:1:1}^{(0)^{(k_1)}} - P_2P_1^{k_1+1} + Q_2Q_1^{k_1+1}\right]. \tag{16}$$

PROOF. We shall denote by convention

$$c(n,0) = R_1^* = n,$$

$$c(n,-1) = 1$$

and

$$\mu_{0:0:n}^{(R_0^*-1)^{(k_1+1)}} = \mu_{0:0:n}^{(R_0^*-2)^{(k_1+1)}} = Q_1^{k_1+1}.$$

Therefore, at $r=1$ in (14), we get

$$\begin{aligned}
\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}} &= \mu_{1:1:n}^{(R_1^*-1)^{(k_1+1)}}\\
&= \frac{1}{[n+\alpha(k_1+1)]}\Bigg[(k_1+1)\mu_{1:1:n}^{(R_1^*-1)^{(k_1)}}\\
&\quad +P_2\left\{nQ_1{}^{(k_1+1)} - \frac{n}{(n-1)}(n-1)\mu_{1:1:n-1}^{(R_1^*-2)^{(k_1+1)}}\right\} + nQ_1^{(k_1+1)}\Bigg].
\end{aligned}$$

After rearranging the terms, yields (15). The equation (16) is obvious.

3 Product Moments

The product moments of r^{th} and s^{th} progressively Type-II right censored order statistics in view of (1) is given as

$$\begin{aligned}\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2)}} &= E[\{X_{r:m:n}^{\tilde{R}}\}^{k_1}\{X_{s:m:n}^{\tilde{R}}\}^{k_2}] \\ &= c(n,m-1)\iint\limits_{Q_1\le x_1<\cdots<x_m\le P_1}\cdots\int x_r^{k_1}x_s^{k_2}f(x_1)[1-F(x_1)]^{R_1}f(x_2)[1-F(x_2)]^{R_2} \\ &\quad\times\cdots\times f(x_m)[1-F(x_m)]^{R_m}dx_1\cdots dx_m. \qquad (17)\end{aligned}$$

The following theorems contain the recurrence relations for the product moments of progressively Type-II right censored order statistics from the doubly truncated generalized exponential distribution.

THEOREM 3.1. For $1\le r<m\le n-1$ and $k_1,k_2\ge 0$,

$$\begin{aligned}\mu_{r,m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1,k_2+1)}} &= \frac{1}{[R_m^*+\alpha(k_2+1)]}\Bigg[(k_2+1)\mu_{r,m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1,k_2)}} \\ &\quad+P_2\left\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{r,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(k_1,k_2+1)}}\right. \\ &\quad\left.-\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{r,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(k_1,k_2+1)}}\right\} \\ &\quad+R_m^*\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(k_1,k_2+1)}}\Bigg]. \qquad (18)\end{aligned}$$

PROOF. From (17), we have

$$\begin{aligned}&\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2)}}-\alpha\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2+1)}} \\ &= c(n,m-1)\iint\limits_{Q_1\le x_1<\cdots<x_{m-1}\le P_1}\cdots\int x_r^{k_1}Z(x_{m-1})f(x_1)[1-F(x_1)]^{R_1} \\ &\quad\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}dx_1\cdots dx_{m-1}, \qquad (19)\end{aligned}$$

where $Z(x_{m-1})$ is same as in (11).

Now substituting the value of $Z(x_{m-1})$ from (13) in (19) after replacing k_1 with k_2, we get

$$\begin{aligned}&\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2)}}-\alpha\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2+1)}} \\ &= \frac{1}{(k_2+1)}\Bigg[P_2\Bigg\{-c(n,m-1)\iint\limits_{Q_1\le x_1<\cdots<x_{m-1}\le P_1}\cdots\int x_r^{k_1}x_{m-1}^{k_2+1}f(x_1)[1-F(x_1)]^{R_1} \\ &\quad\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m}dx_1\cdots dx_{m-1}\end{aligned}$$

$$+R_m c(n,m-1)\iint_{Q_1\le x_1<\cdots<x_m\le P_1}\cdots\int x_r^{k_1}x_m^{k_2+1}f(x_1)[1-F(x_1)]^{R_1}$$
$$\left.\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}f(x_m)[1-F(x_m)]^{R_m-1}dx_1\cdots dx_m\right\}$$
$$-c(n,m-1)\int\int_{Q_1\le x_1<\cdots<x_{m-1}\le P_1}\cdots\int x_r^{k_1}x_{m-1}^{k_2+1}f(x_1)[1-F(x_1)]^{R_1}$$
$$\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m+1}dx_1\cdots dx_{m-1}$$
$$+(R_m+1)c(n,m-1)\iint_{Q_1\le x_1<\cdots<x_m\le P_1}\cdots\int x_r^{k_1}x_m^{k_2+1}f(x_1)[1-F(x_1)]^{R_1}$$
$$\left.\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}f(x_m)[1-F(x_m)]^{R_m}dx_1\cdots dx_m\right]$$
$$= \frac{1}{(k_2+1)}\left[P_2\left\{-\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{r,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(k_1,k_2+1)}}\right.\right.$$
$$\left.+\frac{c(n,m-1)}{c(n-1,m-1)}R_m\mu_{r,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(k_1,k_2+1)}}\right\}$$
$$-(n-R_1-\cdots-R_{m-1}-m+1)\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k_1,k_2+1)}}$$
$$\left.+(R_m+1)\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2+1)}}\right]$$

or

$$\mu_{r,m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1,k_2)}}-\alpha\mu_{r,m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1,k_2+1)}}$$
$$= \frac{1}{(k_2+1)}\left[P_2\left\{-\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{r,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(k_1,k_2+1)}}\right.\right.$$
$$\left.+\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{r,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(k_1,k_2+1)}}\right\}$$
$$\left.-R_m^*\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(k_1,k_2+1)}}+R_m^*\mu_{r,m:m:n}^{(R_1,\ldots,R_m^*-1)^{(k_1,k_2+1)}}\right]$$

and after rearranging the terms, we get the result in (18).

THEOREM 3.2. For $1\le r<s\le n-1$ and $k_1,k_2\ge 0$,

$$\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2+1)}} = \mu_{r,s:s:n}^{(R_1,\ldots,R_{s-1},R_s^*-1)^{(k_1,k_2+1)}}$$
$$= \frac{1}{[R_s^*+\alpha(k_2+1)]}\left[(k_2+1)\mu_{r,s:s:n}^{(R_1,\ldots,R_s^*-1)^{(k_1,k_2)}}\right.$$
$$+P_2\left\{\frac{c(n,s-1)}{c(n-1,s-2)}\mu_{r,s-1:s-1:n-1}^{(R_1,\ldots,R_{s-2},R_{s-1}^*-2)^{(k_1,k_2+1)}}\right.$$

$$- \frac{c(n,s-1)}{c(n-1,s-1)}(R_s^*-1)\mu_{r,s:s:n-1}^{(R_1,\ldots,R_{s-1},R_s^*-2)^{(k_1+1)}}\Bigg\}$$
$$+R_s^*\mu_{r,s-1:s-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}^*-1)^{(k_1,k_2+1)}}\Bigg]. \quad (20)$$

PROOF. This follows from Theorem 3.1 and (7).

4 Special Cases

REMARK 4.1. By letting $P \to 1,\ Q \to 0$ and shape parameter $\alpha \to 0$, we can get the corresponding recurrence relations for the exponential distribution and at $Q \to 0$ and $\alpha \to 0$, we can get the corresponding recurrence relations for the right truncated exponential distribution as obtained by Balakrishnan et al. [1], by noting down that

$$P_1 = \frac{1-(1-P)^\alpha}{\alpha}$$

or

$$(1-P) = (1-\alpha P_1)^{\frac{1}{\alpha}} \to e^{-P_1} \quad \text{as } \alpha \to 0 \text{ or } \frac{1}{\alpha} \to \infty.$$

REMARK 4.2. By letting $P \to 1,\ Q \to 0$ in the Theorems, we can get the corresponding recurrence relations for the generalized exponential distribution as,

$$\begin{aligned} & \mu_{r:m:n}^{(R_1,\ldots,R_m)^{(k_1+1)}} \\ = \; & \mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(k_1+1)}} \\ = \; & \frac{1}{[R_r^*+\alpha(k_1+1)]}\left[(k_1+1)\mu_{r:r:n}^{(R_1,\ldots,R_r^*-1)^{(k_1)}} + R_r^*\mu_{r-1:r-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-1)^{(k_1+1)}}\right] \end{aligned}$$

and

$$\begin{aligned} & \mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(k_1,k_2+1)}} \\ = \; & \mu_{r,s:s:n}^{(R_1,\ldots,R_{s-1},R_s^*-1)^{(k_1,k_2+1)}} \\ = \; & \frac{1}{[R_s^*+\alpha(k_2+1)]}\left[(k_2+1)\mu_{r,s:s:n}^{(R_1,\ldots,R_s^*-1)^{(k_1,k_2)}} + R_s^*\mu_{r,s-1:s-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}^*-1)^{(k_1,k_2+1)}}\right]. \end{aligned}$$

REMARK 4.3. For a special case $R_1 = R_2 = \cdots = R_m = 0$, so that $m = n$, the recurrence relations established in section 2 and 3 reduce to the recurrence relations for the single and product moments of order statistics from the doubly truncated generalized exponential distribution by Saran and Pushkarna [2]

REMARK 4.4. Theorem 3.2 reduces to Theorem 2.2 as $k_1 \to 0$.

5 Conclusions

The above study demonstrates that the recurrence relations developed here can be used to compute the single as well as product moments and hence means, variances and covariances of progressively Type-II right censored order statistics for all sample sizes and all censoring schemes in a simple recursive way from doubly truncated generalized exponential distribution for different values of P and Q.

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