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New Oscillation Criteria For Second Order Quasi-Linear Differential Equations With Sub-Linear Neutral Term*

Natarajan Prabaharan†, Chinnappa Pillai Dharuman‡, John R. Graef§
Ethiraju Thandapani¶

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Abstract

In this paper, the authors obtain some new oscillation criteria for the second order neutral differential equation

$$(a(t)(z'(t))^{\beta})' + q(t)x^{\gamma}(\sigma(t)) = 0,\, t \geq t_0 > 0$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$. Their results extend, unify, and improve some of the results previously reported in the literature. Examples are provided to illustrate the importance of the main results.

1 Introduction

In this paper, we are concerned with the second order neutral differential equation

$$(a(t)(z'(t))^{\beta})' + q(t)x^{\gamma}(\sigma(t)) = 0,\ t \geq t_0 > 0, \tag{1}$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$, subject to the following conditions:

(H_1) $0 < \alpha \leq 1$, β, and $\gamma \geq 1$ are ratios of odd positive integers;

(H_2) $a \in C^1([t_0,\infty),(0,\infty))$, p, $q \in C([t_0,\infty),\mathbb{R})$, $p \geq 0$, $p(t)$ tends to zero as $t \to \infty$, and $q \geq 0$ and is not eventually zero on any half line $[t_*,\infty)$ for $t_* \geq t_0$;

(H_3) $\tau \in C([t_0,\infty),\mathbb{R})$, $\sigma \in C^1([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) \geq 0$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

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†Department of Mathematics, SRM University, Ramapuram Campus, Chennai - 600 089, India
‡Department of Mathematics, SRM University, Ramapuram Campus, Chennai - 600 089, India
§Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
¶Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai-600 005, India

By a solution of equation (1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$, $t_x \geq t_0$, with $a(t)(z'(t))^\beta \in C^1([t_x,\infty),\mathbb{R})$, and which satisfies equation (1) on $[t_x,\infty)$. We consider only those solutions x of equation (1) that are continuable to the right and nontrivial, that is, they satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We also assume that equation (1) possesses such solutions. As is customary, a solution $x(t)$ of equation (1) is called *oscillatory* if it has infinitely many zeros on $[t_x,\infty)$; otherwise, it is said to be *nonoscillatory*. The equation itself is termed oscillatory if all its solutions oscillate.

Following Trench [29], we say that equation (1) is in canonical form if

$$R(t) = \int_{t_0}^{t} a^{-\frac{1}{\beta}}(s)ds \to \infty \text{ as } t \to \infty, \tag{2}$$

and it is in non-canonical form if

$$\int_{t_0}^{\infty} a^{-\frac{1}{\beta}}(s)ds < \infty.$$

In the last few decades there has been a great interest in investigating the oscillatory and asymptotic behavior of solutions of neutral type differential equations because such equations arise in many applications in the natural sciences and technology; see, for example, [14, 23] and the references cited therein.

From a review of the literature, it is observed that there are many results available dealing with the oscillatory and asymptotic behavior of solutions of (1) with $\alpha = 1$ (see [1, 4, 6, 7, 10, 11, 15, 21, 22, 23, 24, 27, 30, 31, 32, 33, 34, 35] and the references therein). Far fewer results are known on the oscillation of equation (1) with $\alpha \neq 1$ and $\beta = 1$ (see [3, 9, 12, 13, 28]), and to the best of our knowledge, there do not appear to be any such results for equation (1) with both β and α different from 1.

It is a well established method that to find oscillation criteria for second order neutral differential equations with a positive linear neutral term, the following important relation between x and z

$$x(t) \geq (1 - p(t))z(t)$$

has been used in the case where $x(t)$ is positive and $z(t)$ is positive and increasing. However, for equations with a positive sub-linear neutral term, finding such a relation between x and z is more difficult. In [28], the authors used the relation

$$x(t) \geq (M^{1-\alpha} - p(t))z(t)$$

where $z(t) \geq M > 0$ and $0 < \alpha \leq 1$. In [3], the authors used the relation

$$x(t) \geq \left(1 - \left(\alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M}\right) p(t)\right) z(t)$$

where again $x(t)$ is positive, $z(t)$ is positive and increasing, $z(t) \geq M > 0$, and $0 < \alpha \leq 1$. In [12, 13] the authors used a relationship of the form

$$x(t) \geq \left(1 - p(t) - \frac{1}{M}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} p(t)\right) z(t)$$

where $0 < \alpha < 1$. But in this paper, we make use of a different type of relation between x and z to obtain our results. Thus, our results in this paper for equation (1) are different from those in [3, 12, 13, 28] even if $\beta = 1$. Hence, we are able to improve and extend some known results in the literature [2, 3, 6, 7, 9, 11, 12, 13, 15, 21, 22, 23, 24, 27, 28, 30, 31, 32, 33, 34, 35, 36]. The results here also complement those in [5, 18, 19, 20].

2 Some Preliminary Lemmas

In this section, we present some lemmas that will be used to prove our main results. Due to the assumptions and the form of our equation, we only need to give proofs for the case of eventually positive solutions since the proofs for eventually negative solutions would be similar.

We begin with the following lemma, which can be found in [16, Theorem 40].

LEMMA 1. If a and b are nonnegative and $0 < \alpha \leq 1$, then

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b.$$

To simplify our notations, for any positive continuous function $\rho : [t_0, \infty) \to \mathbb{R}^+$, we set:

$$\begin{aligned} P(t) &= \left(1 - \alpha p(t) - \frac{1}{\rho(t)}(1-\alpha)p(t)\right), \\ Q(t) &= q(t) P^{\gamma}(\sigma(t)), \\ \overline{R}(t) &= R^{\gamma}(t) + \frac{\gamma}{\beta}\int_{t_1}^{t} R^{\gamma}(s) R^{\beta}(\sigma(s)) \rho^{\gamma-\beta}(s) Q(s) ds, \end{aligned}$$

and

$$R_1(t) = \exp\left(-\beta \int_{\sigma(t)}^{t} \frac{ds}{(\overline{R}(s) a^{\frac{\gamma}{\beta}}(s))^{\frac{1}{\gamma}}}\right)$$

for $t \geq t_1$, where $t_1 \geq t_0$ is large enough.

The following two lemmas will be used to prove our main results. The first one can be found in [6, Lemma 3].

LEMMA 2. Let condition (2) hold and assume that $x(t)$ is a positive solution of equation (1) on $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that for $t \geq t_1$,

$$z(t) > 0, \quad z'(t) > 0, \quad \text{and} \quad (a(t)(z'(t))^{\beta})' \leq 0. \tag{3}$$

LEMMA 3. Let $x(t)$ be a positive solution of equation (1) on $[t_0, \infty)$ and assume that (3) holds. Then there exists $t_1 \geq t_0$ such that

$$z(t) \geq R(t) a^{\frac{1}{\beta}}(t) z'(t) \tag{4}$$

and

$$\frac{z(t)}{R(t)} \text{ is decreasing for } t \geq t_1. \tag{5}$$

PROOF. Assume that $x(t)$ is a positive solution of equation (1) and let $x(t) > 0$ and (3) holds for all $t \geq t_1 \geq t_0$. Since $a(t)(z'(t))^\beta$ is decreasing,

$$z(t) \geq \int_{t_1}^{t} \frac{a^{\frac{1}{\beta}}(s)z'(s)}{a^{\frac{1}{\beta}}(s)} ds \geq a^{\frac{1}{\beta}}(t)z'(t)R(t).$$

Moreover, using the last inequality, we have

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{a^{\frac{1}{\beta}}(t)z'(t)R(t) - z(t)}{a^{\frac{1}{\beta}}(t)R^2(t)} \leq 0.$$

So $\frac{z(t)}{R(t)}$ is decreasing, and this proves the lemma.

3 Oscillation Results

In this section, we present some new oscillation results for equation (1).

THEOREM 1. Let $\gamma \geq \beta$, condition (2) holds, and assume there is a positive, continuous, decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$ such that $P(t) > 0$ for all large t. If the first order delay differential equation

$$w'(t) + Q(t)\overline{R}(t)w^{\frac{\gamma}{\beta}}(\sigma(t)) = 0 \tag{6}$$

is oscillatory, then every solution of equation (1) is oscillatory.

PROOF. Let $x(t)$ be a positive solution of equation (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, $P(t) > 0$, and (3) holds for $t \geq t_1$. Since $z(t)$ is increasing, applying Lemma 1 with $a = z(t)$ and $b = 1$, we obtain

$$x(t) \geq z(t) - p(t)z^\alpha(t) \geq (1 - \alpha p(t))z(t) - (1 - \alpha)p(t), \ t \geq t_1. \tag{7}$$

Now $z(t)$ is increasing and $\rho(t)$ is positive, decreasing, and tending to zero, so we have $z(t) \geq \rho(t)$ for $t \geq t_2$ for some $t_2 \geq t_1$. Using this in (7), we obtain

$$x(t) \geq P(t)z(t)$$

for $t \geq t_3$ for some $t_3 \geq t_2$. This, together with equation (1), implies that

$$(a(t)(z'(t))^\beta)' \leq -Q(t)z^\gamma(\sigma(t)) \tag{8}$$

for $t \geq t_3$. Now a simple computation, together with (4) and the fact that $\gamma \geq 1$, shows that

$$\left(z^\gamma(t) - R^\gamma(t)a^{\frac{\gamma}{\beta}}(t)(z'(t))^\gamma\right)' \geq -R^\gamma(t)\left(a^{\frac{\gamma}{\beta}}(t)(z'(t))^\gamma\right)'. \tag{9}$$

Also, it is easy to see that

$$\left(a^{\frac{\gamma}{\beta}}(t)(z'(t))^{\gamma}\right)' = \frac{\gamma}{\beta}\left(a(t)(z'(t))^{\beta}\right)^{\frac{\gamma}{\beta}-1}(a(t)(z'(t))^{\beta})'. \tag{10}$$

Combining (8), (9), and (10), we obtain

$$\left(z^{\gamma}(t) - R^{\gamma}(t)a^{\frac{\gamma}{\beta}}(t)(z'(t))^{\gamma}\right)' \geq \frac{\gamma}{\beta}R^{\gamma}(t)Q(t)\left(a(t)(z'(t))^{\beta}\right)^{\frac{\gamma}{\beta}-1}z^{\gamma}(\sigma(t)) \geq 0$$

for $t \geq t_3$. Integrating the last inequality from t_3 to t, we have

$$z^{\gamma}(t) \geq R^{\gamma}(t)a^{\frac{\gamma}{\beta}}(t)(z'(t))^{\gamma} + \frac{\gamma}{\beta}\int_{t_1}^{t} R^{\gamma}(s)Q(s)\left(a(s)((z'(t))^{\beta}\right)^{\frac{\gamma}{\beta}-1}z^{\gamma}(\sigma(s))ds. \tag{11}$$

Since $\gamma \geq \beta$, we have $z^{\gamma-\beta}(\sigma(t)) \geq \rho^{\gamma-\beta}(t)$ and $z^{\beta}(\sigma(t)) \geq R^{\beta}(\sigma(t))\ a(t)(z'(t))^{\beta}$ for $t \geq t_3$. Using these inequalities in (11), we obtain

$$z^{\gamma}(t) \geq a^{\frac{\gamma}{\beta}}(t)(z'(t))^{\gamma}\left[R^{\gamma}(t) + \frac{\gamma}{\beta}\int_{t_1}^{t} R^{\gamma}(s)R^{\beta}(\sigma(s))\rho^{\gamma-\beta}(s)Q(s)ds\right], \quad t \geq t_3,$$

where we have used that $a^{\frac{1}{\beta}}(t)z'(t)$ is nonincreasing. Hence,

$$z^{\gamma}(\sigma(t)) \geq \left(a^{\frac{1}{\beta}}(\sigma(t))z'(\sigma(t))\right)^{\gamma}\overline{R}(\sigma(t)). \tag{12}$$

Using (12) in (8), and in view of (3), one can see that $w(t) = a(t)(z'(t))^{\beta}$ is a positive solution of the first order delay differential inequality

$$w'(t) + Q(t)\overline{R}(\sigma(t))w^{\frac{\gamma}{\beta}}(\sigma(t)) \leq 0. \tag{13}$$

But by Theorem 1 of Philos [25], the associated delay differential equation

$$w'(t) + Q(t)\overline{R}(\sigma(t))w^{\frac{\gamma}{\beta}}(\sigma(t)) = 0$$

must also have a positive solution, which is a contradiction. This completes the proof of the theorem.

Using the results in [10] and [26], one can easily obtain the following corollaries to Theorem 1.

COROLLARY 1. Let the conditions of Theorem 1 hold with $\gamma = \beta$. If

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\overline{R}(\sigma(s))ds > \frac{1}{e} \tag{14}$$

then every solution of equation (1) is oscillatory.

COROLLARY 2. Let the conditions of Theorem 1 hold with $\gamma > \beta$. If $\sigma(t) = t - \delta$, $\delta > 0$, and

$$\liminf_{t\to\infty}\left(\frac{\gamma}{\beta}\right)^{-\delta}\log(Q(t)\overline{R}(t-\delta)) > 0, \tag{15}$$

then every solution of equation (1) is oscillatory.

Our next result is for the case $\gamma < \beta$.

THEOREM 2. Assume that $\gamma < \beta$, condition (2) holds, and there exists a positive, continuous, decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero such that $P(t) > 0$ for all large t. If

$$\int_{t_0}^{\infty} Q(t)\left(R^\gamma(\sigma(t)) + \frac{\gamma}{\beta M^{\beta-\gamma}} \int_{t_0}^{\sigma(s)} R^\gamma(s)R^\gamma(\sigma(s))Q(s)ds\right)dt = \infty \tag{16}$$

for any constant $M > 0$, then every solution of equation (1) is oscillatory.

PROOF. Assume that equation (1) has a positive solution, say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, $P(t) > 0$, and (3) holds for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, from (11) we have

$$z^\gamma(t) \geq \left(a(t)(z'(t))^\beta\right)^{\frac{\gamma}{\beta}} \left(R^\gamma(t) + \frac{\gamma}{\beta}\int_{t_1}^{t} R^\gamma(s)Q(s)(a(s)((z'(t))^\beta)^{-1} z^\gamma(\sigma(s))ds\right) \tag{17}$$

for $t \geq t_3$. Since $\frac{z(t)}{R(t)}$ is decreasing, there exists a constant $M > 0$ such that $\frac{z(t)}{R(t)} \leq M$ for all $t \geq t_3$, and from (3) and (4), we have

$$z^\beta(\sigma(t)) \geq R^\beta(\sigma(t))a(\sigma(t))(z'(\sigma(t)))^\beta \geq R^\beta(\sigma(t))a(t)(z'(t))^\beta$$

for $t \geq t_3$. Since $\gamma < \beta$, using these inequalities in (17) yields

$$z^\gamma(t) \geq (a(t)((z'(t))^\beta)^{\frac{\gamma}{\beta}} \left(R^\gamma(t) + \frac{\gamma}{\beta M^{\beta-\gamma}} \int_{t_1}^{\sigma(t)} R^\gamma(s)R^\gamma(\sigma(s))Q(s)ds\right),$$

for $t \geq t_3$. Using the last inequality in (8) and setting $w(t) = a(t)(z'(t))^\beta > 0$, we have that w is a positive solution of the delay differential inequality

$$w'(t) + Q(t)\left(R^\gamma(\sigma(t)) + \frac{\gamma}{\beta M^{\beta-\gamma}} \int_{t_1}^{\sigma(t)} R^\gamma(s)R^\gamma(\sigma(s))Q(s)ds\right) w^{\frac{\gamma}{\beta}}(\sigma(t)) \leq 0.$$

But by Theorem 1 in [25], the associated delay differential equation

$$w'(t) + Q(t)\left(R^\gamma(\sigma(t)) + \frac{\gamma}{\beta M^{\beta-\gamma}} \int_{t_1}^{\sigma(t)} R^\gamma(s)R^\gamma(\sigma(s))Q(s)ds\right) w^{\frac{\gamma}{\beta}}(\sigma(t)) = 0 \tag{18}$$

must also have a positive solution. On the other hand, by Theorem 2 in [17], condition (16) implies that equation (18) is oscillatory. This contradiction completes the proof.

In our next theorem, we employ a Riccati substitution technique to obtain new oscillation criteria for equation (1). We will need the following lemma in the proof.

LEMMA 4. ([36, Lemma 2.3] Let $g(u) = Bu - Au^{\frac{\omega+1}{\omega}}$ where A and B are constants and ω is a ratio of odd positive integers. Then g attains it maximum value at $u^* = \left(\frac{B\omega}{A(\omega+1)}\right)^{\omega}$ and $g(u^*) = \frac{\omega^{\omega}}{(\omega+1)^{\omega+1}} \frac{B^{\omega+1}}{A^{\omega}}$.

THEOREM 3. Let $\gamma \geq \beta$, condition (2) holds, and assume there exists a positive, continuous, decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero such that $P(t) > 0$ for all large t. If there exists a positive, nondecreasing, differentiable function $\mu : [t_0, \infty) \to \mathbb{R}^+$ such that for all sufficiently large $T \geq t_0$,

$$\limsup_{t\to\infty} \int_T^t \left(\mu(s)Q(s)R_1(s)\rho^{\gamma-\beta}(s) - \frac{a(s)(\mu'(s))^{1+\beta}}{(\beta+1)^{\beta+1}\mu^{\beta}(s)} \right) ds = \infty, \tag{19}$$

then every solution of equation (1) is oscillatory.

PROOF. Let $x(t)$ be a positive solution of equation (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, $P(t) > 0$, and (3) holds for $t \geq t_1$. Define a Riccati type transformation by

$$w(t) = \mu(t)a(t) \left(\frac{z'(t)}{z(t)} \right)^{\beta}, \ t \geq t_1. \tag{20}$$

Then, $w(t) > 0$ for all $t \geq t_1$, and

$$w'(t) = \frac{\mu'(t)}{\mu(t)} w(t) + \mu(t) \frac{(a(t)((z'(t))^{\beta})'}{z^{\beta}(t)} - \beta\mu(t)a(t) \left(\frac{z'(t)}{z(t)} \right)^{\beta+1}. \tag{21}$$

As in the proof of Theorem 1, we have

$$z^{\gamma}(t) \geq \overline{R}(t) \left(a^{\frac{1}{\beta}}(t) z'(t) \right)^{\gamma}$$

or

$$\frac{z'(t)}{z(t)} \leq \frac{1}{\overline{R}^{\frac{1}{\gamma}}(t) a^{\frac{1}{\beta}}(t)}.$$

Integrating the last inequality from $\sigma(t)$ to t, yields

$$\frac{z(\sigma(t))}{z(t)} \geq \exp\left(- \int_{\sigma(t)}^t \frac{ds}{\overline{R}^{\frac{1}{\gamma}}(s) a^{\frac{1}{\beta}}(s)} \right). \tag{22}$$

Combining (8) and (22), we have

$$\begin{aligned} \frac{(a(t)((z'(t))^{\beta})'}{z^{\beta}(t)} &\leq -Q(t) \left(\frac{z(\sigma(t))}{z(t)} \right)^{\beta} z^{\gamma-\beta}(\sigma(t)) \\ &\leq -Q(t) \exp\left(-\beta \int_{\sigma(t)}^t \frac{ds}{\overline{R}^{\frac{1}{\gamma}}(s) a^{\frac{1}{\beta}}(s)} \right) \rho^{\gamma-\beta}(t), \end{aligned} \tag{23}$$

where we have used the facts that $\gamma \geq \beta$ and $z(t) \geq \rho(t)$ for all $t \geq t_1$.

From (21) and (23), it follows that

$$w'(t) \leq \frac{\mu'(t)}{\mu(t)} w(t) - \mu(t)Q(t)R_1(t)\rho^{\gamma-\beta}(t) - \frac{\beta}{(\mu(t)a(t))^{\frac{1}{\beta}}} w^{1+\frac{1}{\beta}}(t). \tag{24}$$

Letting $B = \frac{\mu'(t)}{\mu(t)}$ and $A = \frac{\beta}{(\mu(t)a(t))^{\frac{1}{\beta}}}$ in Lemma 4, it follows from (24) that

$$w'(t) \leq -\mu(t)Q(t)R_1(t)\rho^{\gamma-\beta}(t) + \frac{a(t)(\mu'(t))^{1+\beta}}{(\beta+1)^{\beta+1}\mu^{\beta}(t)}. \tag{25}$$

Let $T \geq t_1$ be sufficiently large and integrate (25) from T to t to obtain

$$\int_T^t \left[\mu(s)Q(s)R_1(s)\rho^{\gamma-\beta}(s) - \frac{a(s)(\mu'(t))^{1+\beta}}{(\beta+1)^{\beta+1}\mu^{\beta}(s)}\right] ds \leq w(T),$$

which contradicts (19). This completes the proof of the theorem.

Our final result is for the case $\gamma < \beta$.

THEOREM 4. Let $\gamma < \beta$, condition (2) holds, and assume that there exists a positive, continuous, decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero such that $P(t) > 0$ for all large t. If there exists a positive, nondecreasing, differentiable function $\mu : [t_0, \infty) \to \mathbb{R}^+$ such that, for all sufficiently large $T \geq t_0$,

$$\limsup_{t\to\infty} \int_T^t \left(\mu(s)Q(s)R_1(s)R^{\gamma-\beta}(\sigma(s)) - \frac{M^{\beta-\gamma}a(s)(\mu'(t))^{1+\beta}}{(\beta+1)^{\beta+1}\mu^{\beta}(s)}\right) ds = \infty \tag{26}$$

for every constant $M > 0$, then every solution of equation (1) is oscillatory.

PROOF. The proof is similar to that of Theorem 3 except that inequality (23) is replaced by

$$\frac{(a(t)((z'(t))^{\beta})'}{z^{\beta}(t)} \leq -M^{\gamma-\beta}Q(t)R_1(t)R^{\gamma-\beta}(\sigma(t)),$$

where we have used $\frac{z(t)}{R(t)} \leq M$ for all $t \geq t_1$ and $\gamma < \beta$. We omit the details.

4 Examples

In this section, we present two examples to illustrate our main results.

EXAMPLE 1. Consider the second order neutral differential equation

$$((z'(t))^{\beta})' + \frac{q_0}{t^{1+\beta}} x^{\beta}(\lambda t) = 0, \ t \geq 1 \tag{27}$$

where $z(t) = x(t) + \frac{p_0}{t^{1-\alpha}} x^{\alpha}(\tau(t))$, α and β are ratio of odd positive integers with $0 < \alpha \leq 1$ and $\beta \geq 1$, $p_0 \in (0,1)$, $\tau(t) \leq t$, $q_0 > 0$, and $\lambda \in (0,1)$. By taking

$\rho(t)=\frac{p_0}{t^{1-\alpha}}$, we see that $P(t)=\alpha(1-\frac{p_0}{t^{1-\alpha}})>0$ for $t\geq 1$, $Q(t)=\frac{q_0\alpha^\beta}{t^{1+\beta}}(1-\frac{p_0}{(\lambda t)^{1-\alpha}})^\beta$, and $R(t)=t-1$. It is easy to see that by Corollary 1, equation (27) is oscillatory if

$$q_0\alpha^\beta\lambda^\beta\left(1-\frac{p_0}{\lambda^{1-\alpha}}\right)^\beta\left(1+\frac{q_0\alpha^\beta\lambda^\beta(1-\frac{p_0}{\lambda^{1-\alpha}})^\beta}{\beta}\right)\ln\frac{1}{\lambda}>\frac{1}{e}.$$

If we assume $\alpha=1$ in equation (27), then we have

$$((z'(t))^\beta)'+\frac{q_0}{t^{1+\beta}}x^\beta(\lambda t)=0,\ t\geq 1 \tag{28}$$

where $z(t)=x(t)+p_0x(\tau(t))$, which is same as the equation considered in Example 1 in [11]. Now, by our results the equation (28) is oscillatory if

$$q_0\lambda^\beta\left(1-p_0\right)^\beta\left(1+\frac{q_0\lambda^\beta(1-p_0)^\beta}{\beta}\right)\ln\frac{1}{\lambda}>\frac{1}{e}$$

which is different from the condition obtained in [11] for the case $\beta>1$.

As a special case of equation (27) or (28), we have

$$((x'(t))^3)'+\frac{q_0}{t^4}x^3(t/2)=0,\ t\geq 1, \tag{29}$$

and the equation (29) is oscillatory provided

$$q_0>3.6843.$$

But by the known related criterion for (4.3) based on comparison with a first order delay differential equation (see, e.g., [6, Theorem 5]) gives $q_0>16.9847$ which is a significantly weaker result.

EXAMPLE 2. Consider the second order neutral differential equation

$$((z'(t))^3)'+\frac{q_0}{t^4}x^3(\lambda t)=0,\ t\geq 1, \tag{30}$$

where $z(t)=x(t)+\frac{p_0}{t}x^\alpha(\tau(t))$, $0<\alpha\leq 1$ is a ratio of odd positive integers, $p_0\in(0,1)$, $q_0>0$, $\lambda\in(0,1)$, and $\tau(t)\leq t$. By taking $\rho(t)=\frac{p_0}{t}$, we see that $P(t)=\alpha(1-\frac{p_0}{t})>0$ for $t\geq 1$, and $Q(t)=\frac{q_0}{t^4}\alpha^3(1-\frac{p_0}{\lambda t})^3$. A simple calculation gives $R_1(t)=\lambda^{3\lambda_1}$ where $\lambda_1=\left(\frac{3}{3+q_0\alpha^3\lambda^3(1-\frac{p_0}{\lambda})^3}\right)^{1/3}$. Now by taking $\mu(t)=t^3$, we see that condition (19) is satisfied if

$$q_0\alpha^3\left(1-\frac{p_0}{\lambda}\right)^3\lambda^{3\lambda_1}>\frac{81}{256}. \tag{31}$$

Hence, by Theorem 3, every solution of (30) is oscillatory provided condition (31) is satisfied.

We conclude this paper with the following remark.

REMARK 1.

1. It is important to note that none of the results in the literature can be applied to equations (27) and (30) to yield this conclusion.

2. Note that the results obtained in this paper generalize and are different from those of in [6, 11] where $\alpha = 1$ and $\beta \neq 1$. Also, the results established here extend those in [3, 12, 13, 28] where $\beta = 1$ and $0 < \alpha \leq 1$. They also extend and complement results in [2, 3, 6, 7, 9, 11, 15, 21, 22, 23, 24, 27, 30, 31, 32, 33, 34, 35, 36] that were obtained for $\alpha = \beta = 1$. Thus, the results presented in this paper extend, improve, and complement to many known results in the literature.

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