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Some Fixed Point Results For A Generalized TAC-Suzuki-Berinde Type F-Contractions In b-Metric Spaces*

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Abstract

In this paper, we introduce some classes of mappings called the TAC-Suzuki-Berinde type F-contraction and TAC-Suzuki-Berinde type rational F-contraction in the frame work of b-metric spaces and prove some fixed point results for these classes of mappings. As an application, we establish the existence of a solution for the following nonlinear integral equation:

$$x(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds,$$

where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Our results improve and extend the corresponding results in the literature.

1 Introduction

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [3] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces

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(see [1, 17, 19, 20, 26, 28] and the references therein). Also, over the years, several authors have developed several iterative schemes for solving fixed point problem for different operators and in different spaces, (see [2, 13, 10, 15, 29, 30, 31, 34, 35] and the references therein). For example, Khan et al. [18] introduced the concept of alternating distance function, which is defined as follows: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

1. $\psi(0) = 0$,
2. ψ is monotonically nondecreasing,
3. ψ is continuous.

They obtained the following result:

THEOREM 1. Let (X, d) be a complete metric space, ψ an alternating distance function and $T : X \to X$ be a self mapping which satisfies the following condition

$$\psi(d(Tx, Ty)) \leq \delta\psi(d(x, y)),$$

for all $x, y \in X$, where $\delta \in (0, 1)$. Then T has a unique fixed point.

REMARK 1. Clearly, if we take $\psi(x) = x$, for all $x \in X$, we obtain the Banach contraction mapping.

Berinde [6, 7] introduced and studied a class of contractive mappings, which is defined as follows:

DEFINITION 1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [32] introduced a class of mappings satisfying condition (C), known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

DEFINITION 1.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

THEOREM 1.2. Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

In 2012, Wardowski [37] introduced the notion of F-contractions, which is defined as follows:

DEFINITION 1.3. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F_1) F is strictly increasing;

(F_2) for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

He established the following result:

THEOREM 1.3. Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.

REMARK 1.2 ([37]). If we suppose that $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction mapping.

In [22], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

THEOREM 1.4. Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

In 2016, Chandok et al. [9] introduced a new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces.

DEFINITION 1.4 ([9]). Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β)-admissible mapping, if

1. $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
2. $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

DEFINITION 1.5 ([9]). Let (X,d) be a metric space and let $\alpha,\beta : X \to [0,\infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq f(\psi(d(x,y)),\phi(d(x,y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty}\phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0,\infty)^2 \to \mathbb{R}$ is continuous, $f(a,t) \leq a$ and $f(a,t) = a \Rightarrow a = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

THEOREM 1.5 ([9]). Let (X,d) be a complete metric space and let $T : X \to X$ be a cyclic (α,β)-admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:

1. T is continuous,
2. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

One of the most interesting generalizations of metric spaces is the concept of b-metric spaces (to be defined in Section 2) introduced by Czerwik in [11]. He proved the Banach contraction principle in this setting with the fact that d need not to be continuous. Thereafter, several results have been extended from metric spaces to b-metric spaces. In addition, a lot of results have been published on the fixed point theory of various classes of single-valued and multi-valued operators in the frame work of b-metric spaces (see [4, 8, 11, 25, 39] and the references therein). Yamaod and Sintunawarat [39] introduced the notion of (α,β)-(ψ,φ)-contraction mapping in the frame work of b-metric spaces as follows:

DEFINITION 1.6. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $\alpha,\beta : X \to [0,\infty)$ be two given mappings. We say that $T : X \to X$ is an (α,β)-(ψ,φ)-contraction mapping if the following conditions holds: for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ implies that

$$\psi(s^3 d(Tx,Ty)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y)),$$

where

$$M_s(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}$$

and $\psi,\varphi : [0,\infty) \to [0,\infty)$ are alternating distance functions.

THEOREM 1.6. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ an (α,β)-(ψ,φ)-contraction mapping. Suppose that one of the following conditions holds:

1. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$,
2. there exists $y_0 \in X$ such that $\alpha(y_0) \geq 1$,

and the following hold:

1. T is continuous,
2. T is cyclic (α, β)-admissible.

Then T has a fixed point.

Recently, Babu et al. [4] generalized the result of Chandok et al. [9] by introducing a generalized TAC-contractive mapping in the frame work of b-metric spaces.

DEFINITION 1.7. Let (X, d) be a b-metric space, $\alpha, \beta : X \to [0, \infty)$ be two given mappings and T be a self map on X. The mapping T is said to be generalized TAC-contrative map in b-metric spaces, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(s^3 d(Tx, Ty)) \leq f(\psi(M_s(x, y)), \phi(M_s(x, y))),$$

where

$$M_s(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\},$$

ψ is an alternating distance function, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0, \infty)^2 \to \mathbb{R}$ is continuous with $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

THEOREM 1.7. Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a generalized TAC-contraction mapping. Suppose the following conditions hold:

1. T is a cyclic (α, β)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point.

Motivated by the research works described above and the recent interest in this direction of research, in this paper we first generalize the concept of cyclic (α, β)-admissible mapping, by introducing the concept of cyclic (α_s, β_s)-admissible mapping and cyclic (α, β)-admissible type S mapping in the frame work of b-metric spaces. In addition, we introduce TAC-Suzuki-Berinde type F-contraction, TAC-Suzuki type F-contraction and TAC-Suzuki-Berinde type rational F-contraction mapping, establish some fixed point results regarding these classes of mappings and finally applied our result to the solution of a nonlinear integral equation in the frame work of b-metric spaces.

2 Premilinaries

In this section, we introduce some concepts and present some results that will be needed in the sequel.

DEFINTION 2 ([11]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d). It is clear that, the class of b-metric spaces is larger than that of metric spaces. If $s = 1$, a b-metric become a metric.

EXAMPLE 2 ([4]). Let $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. It is easy to see that (x, d) is a b-metric space with coefficient $s = 2$, but (X, d) is not a metric space.

DEFINITION 2.1 ([8]). Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is said to be

1. b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
2. b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

DEFINITION 2.2 ([8]). Let (X, d) be a b-metric space. Then X is said to be complete if every b-Cauchy sequence in X is b-convergent.

LEMMA 2 ([25]). Suppose that (X, d) is a b-metric space and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and

1. $\epsilon \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$,
2. $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) \leq s^2\epsilon$,
3. $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_{k+1}}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_{k+1}}, x_{n_k}) \leq s^2\epsilon$,
4. $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq s^3\epsilon$.

LEMMA 2.1 ([27]). Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:

1. if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$;
2. if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Furthermore, the authors in [27] replaced the condition F_2 in the definition of F-contraction with the following condition.

(F_*) $\inf F = -\infty$

or, also by

(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

3 Main Result

In this section, we introduce the notions of cyclic (α_s, β_s)-admissible mapping, cyclic (α, β)-admissible type S mapping, TAC-Suzuki-Berinde type F-contraction, TAC-Suzuki type F-contraction, TAC-Suzuki-Berinde type rational F-contraction mapping and establish some fixed point results regarding these classes of mappings.

DEFINITION 3. Let X be a nonempty set and $s \geq 1$. Let $T : X \to X$ and $\alpha, \beta : X \to [0, \infty)$ be given mappings. The mapping T is said to be cyclic (α_s, β_s)-admissible mapping, if

1. $\alpha(x) \geq s^3$ for some $x \in X$ implies that $\beta(Tx) \geq s^3$,
2. $\beta(x) \geq s^3$ for some $x \in X$ implies that $\alpha(Tx) \geq s^3$.

REMARK 3. Clearly, if $s = 1$, then Definition 3 reduces to Definition 1.4.

LEMMA 3. Let X be a nonempty set and $T : X \to X$ be a cyclic (α_s, β_s)-admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$. Define the sequence $x_{n+1} = Tx_n$, then $\alpha(x_m) \geq s^3$ implies that $\beta(x_n) \geq s^3$ and $\beta(x_m) \geq s^3$ implies that $\alpha(x_n) \geq s^3$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.

PROOF. Using the fact that T is a cyclic (α_s, β_s)-admissible mapping and our hypothesis, we have that there exists $x_0 \in X$ such that

$$\alpha(x_0) \geq s^3 \Rightarrow \beta(Tx_0) = \beta(x_1) \geq s^3$$

and

$$\beta(x_0) \geq s^3 \Rightarrow \alpha(Tx_0) = \alpha(x_1) \geq s^3.$$

Continuing this way, we obtain that

$$\alpha(x_n) \geq s^3 \Rightarrow \beta(Tx_n) = \beta(x_{n+1}) \geq s^3$$

and

$$\beta(x_n) \geq s^3 \Rightarrow \alpha(Tx_n) = \alpha(x_{n+1}) \geq s^3$$

for all $n \in \mathbb{N} \cup \{0\}$. Using similar approach, we obtain that

$$\alpha(x_m) \geq s^3 \Rightarrow \beta(Tx_m) = \beta(x_{m+1}) \geq s^3$$

and

$$\beta(x_m) \geq s^3 \Rightarrow \alpha(Tx_m) = \alpha(x_{m+1}) \geq s^3,$$

for all $m \in \mathbb{N} \cup \{0\}$. In addition, since

$$\alpha(x_m) \geq s^3 \Rightarrow \beta(x_{m+1}) \geq s^3 \Rightarrow \alpha(x_{m+2}) \geq s^3 \cdots$$

with $m < n$, we deduce that

$$\alpha(x_m) \geq s^3 \Rightarrow \beta(x_n) \geq s^3.$$

Using similar approach, we have that

$$\beta(x_m) \geq s^3 \Rightarrow \alpha(x_n) \geq s^3.$$

We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions

$(F_1^{'})$ F is strictly increasing,

$(F_2^{'})$ $\inf F = -\infty$,

or, also by,

$(F_3^{'})$ there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

$(F_4^{'})$ F is continuous on $(0, \infty)$.

DEFINITION 3.1. Let (X, d) be a b-metric space with $s \geq 1$, $\alpha, \beta : X \to [0, \infty)$ be two functions and T be a self map on X. The mapping T is said to be TAC-Suzuki-Berinde type F-contraction if $F \in \mathcal{F}$, $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$

$$\frac{1}{2s} d(x, Tx) \leq d(x, y) \text{ and } d(Tx, Ty) > 0, \tag{2}$$

then

$$\tau + F(\alpha(x)\beta(y)d(Tx, Ty)) \leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

THEOREM 3. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a TAC-Suzuki-Berinde type F-contraction mapping. Suppose the following conditions hold:

1. T is a cyclic (α_s, β_s)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. T is continuous.

Then T has a fixed point.

PROOF. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic (α_s, β_s)-admissible mapping and $\alpha(x_0) \geq s^3$, we have $\beta(x_1) = \beta(Tx_0) \geq s^3$ and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq s^3$, continuing the process, we have

$$\alpha(x_{2k}) \geq s^3 \quad \text{and} \quad \beta(x_{2k+1}) \geq s^3, \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{3}$$

Using similar argument, we have that

$$\beta(x_{2k}) \geq s^3 \quad \text{and} \quad \alpha(x_{2k+1}) \geq s^3, \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{4}$$

It follows from (3) and (4) that $\alpha(x_n) \geq s^3$ and $\beta(x_n) \geq s^3$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n)\beta(x_{n+1}) \geq s^3$ and $\frac{1}{2s}d(x_n, Tx_n) = \frac{1}{2s}d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, we obtain from (2)

$$\begin{aligned}
& \tau + F(d(x_{n+1}, x_{n+2})) \\
= \; & \tau + F(d(Tx_n, Tx_{n+1})) \\
\leq \; & \tau + F(\alpha(x_n)\beta(x_{n+1})d(Tx_n, Tx_{n+1})) \\
\leq \; & F(d(x_n, x_{n+1})) + L\min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\} \\
= \; & F(d(x_n, x_{n+1})),
\end{aligned}$$

which imples that

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

Using similar approach, it is easy to see that

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau.$$

Thus by inductively, we obtain

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau, \quad \forall \; n \in \mathbb{N} \cup \{0\}. \tag{5}$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$ in (5), we have

$$\lim_{n \to \infty} F(d(x_n, x_{n+1})) = -\infty.$$

It follows from (F_3') and Lemma 2 that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

In what follows, we now show that $\{x_n\}$ is a b-Cauchy sequence. Suppose that $\{x_n\}$ is not a b-Cauchy sequence, then by Lemma 2, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and (1)–(4) of Lemma 2 hold. Since $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$, using Lemma 3, we obtain that $\alpha(x_{m_k})\beta(x_{n_k}) \geq s^3$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{1}{2s}d(x_{m_k}, Tx_{m_k}) < \frac{\epsilon}{2s} < \epsilon \leq d(x_{m_k}, x_{n_k}).$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned} & \tau + F(d(x_{m_{k+1}}, x_{n_{k+1}})) \\ \leq \; & \tau + F(\alpha(x_{m_k})\beta(x_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\ \leq \; & F(d(x_{m_k}, x_{n_k})) \\ & +L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}. \end{aligned} \tag{7}$$

Using Lemma 2, $(F_4^{'})$ and (6), we have that

$$\begin{aligned} \tau + F(s\epsilon) = \tau + F(s^3\frac{\epsilon}{s^2}) &\leq \tau + F(\limsup_{k\to\infty}[\alpha(x_{m_k})\beta(x_{n_k})d(Tx_{m_k}, Tx_{n_k})]) \\ &= \limsup_{k\to\infty} F(\alpha(x_{m_k})\beta(x_{n_k})d(x_{m_k}, x_{n_k})) \\ &\leq F(\limsup_{k\to\infty} d(x_{m_k}, x_{n_k})) \\ &\leq F(s\epsilon). \end{aligned}$$

That is

$$\tau + F(s\epsilon) \leq F(s\epsilon)$$

which is a contradiction. We therefore have that $\{x_n\}$ is b-Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point.

THEOREM 3.1. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a TAC-Suzuki-Berinde type F-contraction type mapping. Suppose the following conditions hold:

1. T is a cyclic (α_s, β_s)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s^3$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s^3$.

Then T has a fixed point.

PROOF. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. In Theorem 3, we have establish that $\{x_n\}$ is b-Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. More so, using the condition that $\beta(x_n) \geq s^3$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\beta(x) \geq s^3$. We now establish that T has a fixed point.

Claim: We claim that

$$\frac{1}{2s} d(x_n, x_{n+1}) < d(x_n, x)$$

or

$$\frac{1}{2s} d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x).$$

Proof of claim. Suppose on the contrary that there exists $m \in \mathbb{N} \cup \{0\}$, such that

$$\frac{1}{2s} d(x_m, x_{m+1}) \geq d(x_m, x) \quad \text{and} \quad \frac{1}{2s} d(x_{m+1}, x_{m+2}) \geq d(x_{m+1}, x). \tag{8}$$

Now observe that

$$2sd(x_m, x) \leq d(x_m, x_{m+1}) \leq sd(x_m, x) + sd(x, x_{m+1}), \tag{9}$$

which implies that $d(x_m, x) \leq d(x, x_{m+1})$. It follows from (8) and (9), that

$$d(x_m, x) \leq d(x, x_{m+1}) \leq \frac{1}{2s} d(x_{m+1}, x_{m+2}). \tag{10}$$

Since $\frac{1}{2s} d(x_m, x_{m+1}) < d(x_m, x_{m+1})$, we have that

$$\begin{aligned}
& \tau + F(d(x_{m+1}, x_{m+2})) \\
\leq \; & \tau + F(\alpha(x_m)\beta(x_{m+1}) d(Tx_m, Tx_{m+1})) \\
\leq \; & F(d(x_m, x_{m+1})) \\
& + L \min\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2}), d(x_m, x_{m+2}), d(x_{m+1}, x_{m+1})\} \\
= \; & F(d(x_m, x_{m+1})).
\end{aligned} \tag{11}$$

It follows that

$$\tau + F(d(x_{m+1}, x_{m+2})) \leq F(d(x_m, x_{m+1})).$$

Using the fact that F is strictly increasing, we have that

$$d(x_{m+1}, x_{m+2}) < d(x_m, x_{m+1}).$$

Using this fact, (10) and (8), we have

$$\begin{aligned}
d(x_{m+1}, x_{m+2}) &< d(x_m, x_{m+1}) \\
&\leq sd(x_m, x) + sd(x, x_{m+1}) \\
&\leq \frac{1}{2} d(x_{m+1}, x_{m+2}) + \frac{1}{2} d(x_{m+1}, x_{m+2}) \\
&= d(x_{m+1}, x_{m+2}),
\end{aligned} \tag{12}$$

which is a contradiction. Thus we must have that

$$\frac{1}{2s}d(x_n, x_{n+1}) < d(x_n, x)$$

or

$$\frac{1}{2s}d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x).$$

Hence, we have

$$\begin{aligned}
\tau + F(d(x_{n+1}, Tx)) &= \tau + F(d(Tx_n, Tx)) \\
&\leq \tau + F(\alpha(x_n)\beta(x)d(Tx_n, Tx)) \\
&\leq F(d(x_n, x)) + L\min\{d(x_n, x_{n+1}), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\}.
\end{aligned}$$

Using the fact that $F \in \mathcal{F}$ and Lemma 2, we have that

$$\lim_{n\to\infty} F(d(Tx_n, Tx)) = -\infty$$

and so

$$\lim_{n\to\infty} d(Tx_n, Tx) = 0.$$

Now, observe that

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0.$$

Clearly, we have that

$$d(x, Tx) = 0 \Rightarrow x = Tx.$$

Hence, T has a fixed point.

THEOREM 3.2. Suppose that the hypothesis of Theorem 3.1 holds and in addition suppose $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.

PROOF. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Since, $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$, we have $\alpha(x)\beta(y) \geq s^3$ and $\frac{1}{2s}d(x, Tx) = 0 \leq d(x, y)$, we obtain that

$$\begin{aligned}
F(d(x, y)) = F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) &\leq \tau + F(\alpha(x)\beta(y)d(Tx, Ty)) \\
&\leq F(d(x, y)) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \\
&\leq F(d(x, y)),
\end{aligned}$$

which implies that

$$F(d(x, y)) < F(d(x, y)).$$

Clearly, we get a contradiction. Thus T has a unique fixed point.

DEFINITION 3.2. Let (X, d) be a b-metric space with $s \geq 1$, $\alpha, \beta : X \to [0, \infty)$ be two functions and T be a self map on X. The mapping T is said to be TAC-Suzuki type F-contraction, if $F \in \mathcal{F}$, $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$

$$\frac{1}{2s} d(x, Tx) \leq d(x, y) \text{ and } d(Tx, Ty) > 0,$$

then

$$\tau + F(\alpha(x)\beta(y)d(Tx, Ty)) \leq F(d(x, y)).$$

THEOREM 3.3. Let (X, d) be a b-complete b-metric space with $s \geq 1$ and $T : X \to X$ be a TAC-Suzuki type F-contraction type mapping. Suppose the following conditions hold:

1. T is a cyclic (α_s, β_s)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s^3$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s^3$.

Then T has a fixed point.

PROOF. The proof follows directly from Theorem 3 and Theorem 3.1 by taking $L = 0$.

THEOREM 3.4. Suppose that the hypothesis of Theorem 3.3 holds and in addition suppose $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.

PROOF. The proof follows directly from Theorem 3.2 by taking $L = 0$.

DEFINITION 3.3. Let X be a nonempty set and $s \geq 1$. Let $T : X \to X$ and $\alpha, \beta : X \to [0, \infty)$ be mappings. The mapping T is said to be cyclic (α, β)-admissible type S mapping if

1. $\alpha(x) \geq s$ for some $x \in X$ implies that $\beta(Tx) \geq s$,
2. $\beta(x) \geq s$ for some $x \in X$ implies that $\alpha(Tx) \geq s$.

REMARK 3.1. Clearly, if $s = 1$, then Definition 3.3 reduces to Definition 1.4. We also note that Definition 3 is a generalization of Definition 3.3.

THEOREM 3.5. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping with $d(Tx, Ty) > 0$ which satisfies the following statement: if

$$\frac{1}{2s} d(x, Tx) \leq d(x, y) \quad \text{and} \quad \alpha(x)\beta(y) \geq s,$$

then

$$F(s^3 d(Tx,Ty)) \leq F(d(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Suppose the following conditions hold:

1. T is a cyclic (α,β)-admissible type S mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s$ and $\beta(x_0) \geq s$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s$.

Then T has a fixed point.

PROOF. The proof follow similar approach as in Theorem 3 and Theorem 3.1, thus we omit it.

THEOREM 3.6. Let (X,d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping with $d(Tx,Ty) > 0$ which satisfies the following statement: if

$$\frac{1}{2s} d(x,Tx) \leq d(x,y) \quad \text{and} \quad \alpha(x)\beta(y) \geq s,$$

then

$$F(s^3 d(Tx,Ty)) \leq F(d(x,y)).$$

Suppose the following conditions hold:

1. T is a cyclic (α,β)-admissible type S mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s$ and $\beta(x_0) \geq s$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s$.

Then T has a fixed point.

PROOF. The proof follow similar approach as in Theorem 3 and Theorem 3.1 by taking $L = 0$, thus we omit it.

COROLLARY 3.1. Let (X,d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping with $d(Tx,Ty) > 0$ which satisfies the following statement: if

$$\frac{1}{2s} d(x,Tx) \leq d(x,y) \quad \text{and} \quad \alpha(x)\beta(y) \geq 1,$$

then

$$F(s^3 d(Tx,Ty)) \leq F(d(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Suppose the following conditions hold:

1. T is a cyclic (α, β)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point.

COROLLARY 3.2. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping with $d(Tx, Ty) > 0$ which satisfies the following statement: if

$$\frac{1}{2s} d(x, Tx) \leq d(x, y),$$

then

$$F(s^3 d(Tx, Ty)) \leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has a unique fixed point.

DEFINITION 3.4. Let (X, d) be a b-metric space with $s \geq 1$, $\alpha, \beta : X \to [0, \infty)$ be two functions and T be a self map on X. The mapping T is said to be TAC-Suzuki-Berinde type rational F-contraction if $F \in \mathcal{F}, \tau > 0$ and $L \geq 0$ such that if

$$\frac{1}{2s} d(x, Tx) \leq d(x, y) \quad \text{and} \quad d(Tx, Ty) > 0 \quad \text{for all } x, y \in X,$$

then

$$F(\alpha(x)\beta(y)d(Tx, Ty)) \leq F(M(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}, \frac{d(x, Tx)d(y, Ty)}{s + d(x, y)}, \frac{d(y, Tx)[1 + d(x, Tx)]}{s + d(x, y)}\right\}.$$

THEOREM 3.7. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a TAC-Suzuki-Berinde type rational F-contraction type mapping. Suppose the following conditions hold:

1. T is a cyclic (α_s, β_s)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. T is continuous.

Then T has a fixed point.

PROOF. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic (α_s, β_s)-admissible mapping and $\alpha(x_0) \geq s^3$, we have $\beta(x_1) = \beta(Tx_0) \geq s^3$ and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq s^3$, continuing the process, we have

$$\alpha(x_{2k}) \geq s^3 \quad \text{and} \quad \beta(x_{2k+1}) \geq s^3 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{13}$$

Using similar argument, we have that

$$\beta(x_{2k}) \geq s^3 \quad \text{and} \quad \alpha(x_{2k+1}) \geq s^3 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{14}$$

It follows from (13) and (14) that $\alpha(x_n) \geq s^3$ and $\beta(x_n) \geq s^3$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n)\beta(x_{n+1}) \geq s^3$ and $\frac{1}{2s}d(x_n, Tx_n) = \frac{1}{2s}d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, we have

$$\begin{aligned}
& \tau + F(d(x_{n+1}, x_{n+2})) \\
= \;& \tau + F(d(Tx_n, Tx_{n+1})) \leq \tau + F(\alpha(x_n)\beta(x_{n+1})d(Tx_n, Tx_{n+1})) \\
\leq \;& F(M(x_n, x_{n+1})) \\
& + L\min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\} \\
= \;& F(M(x_n, x_{n+1})),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
M(x_n, x_{n+1}) &= \max\Big\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\
&\qquad \frac{d(x_n, Tx_{n+1}) + d(x_{n+1,}, Tx_n)}{2s}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{s + d(x_n, x_{n+1})}, \\
&\qquad \frac{d(x_{n+1}, Tx_n)[1 + d(x_n, Tx_n)]}{s + d(x_n, x_{n+1})}\Big\} \\
&= \max\Big\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1,}, x_{n+1})}{2s}, \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{s + d(x_n, x_{n+1})}, \\
&\qquad \frac{d(x_{n+1}, x_{n+1})[1 + d(x_n, x_{n+1})]}{s + d(x_n, x_{n+1})}\Big\} \\
&= \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2s}, \\
&\qquad \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{s + d(x_n, x_{n+1})}, 0\Big\}.
\end{aligned}$$

Since $\frac{d(x_n, x_{n+1})}{s + d(x_n, x_{n+1})} < 1$, we obtain

$$\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{s + d(x_n, x_{n+1})} < d(x_{n+1}, x_{n+2}).$$

We therefore have that

$$M(x_n, x_{n+1}) = \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}.$$

If we suppose that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}),$$

we then have that (15) becomes

$$\tau + F(d(x_{n+1}, x_{n+2})) \leq F(d(x_{n+1}, x_{n+2})),$$

which contradict the fact that F is strictly increasing and $\tau > 0$. Therefore, we must have that

$$M(x_n, x_{n+1}) = \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} = d(x_n, x_{n+1}),$$

which implies that

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau,$$

using similar approach, it is easy to see that

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau$$

and inductively, we obtain

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \quad n \in \mathbb{N} \cup \{0\}. \tag{16}$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$ of (16), we have

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty.$$

It follows from $(F_3^{'})$ and Lemma 2 that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

In what follows, we now show that $\{x_n\}$ is a b-Cauchy sequence. Suppose that $\{x_n\}$ is not a b-Cauchy sequence, then by Lemma 2, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and $(1) - (4)$ of Lemma 2 hold. Since $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$, using Lemma 3, we obtain that $\alpha(x_{m_k})\beta(x_{n_k}) \geq s^3$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{1}{2s} d(x_{m_k}, Tx_{m_k}) < \frac{\epsilon}{2s} < \epsilon \leq d(x_{m_k}, x_{n_k}).$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned}
& \tau + F(d(x_{m_{k+1}}, x_{n_{k+1}})) \\
\leq\ & \tau + F(\alpha(x_{m_k})\beta(x_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\
\leq\ & F(M(x_{m_k}, x_{n_k})) \\
& + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\},
\end{aligned}$$

where

$$\begin{aligned}
M(x_{m_k}, x_{n_k}) = \max\Big\{ & d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \\
& \frac{d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2s} \frac{d(x_{m_k}, x_{m_{k+1}})d(x_{n_k}, x_{n_{k+1}})}{s + d(x_{m_k}, x_{n_k})}, \\
& \frac{d(x_{m_k}, x_{n_{k+1}})[1 + d(x_{n_k}, x_{n_{k+1}})]}{s + d(x_{m_k}, x_{n_k})}\Big\}.
\end{aligned}$$

Using Lemma 2 and $(F_4^{'})$, we have that

$$\begin{aligned}
\tau + F(s\epsilon) = \tau + F(s^3\frac{\epsilon}{s^2}) & \leq \tau + F(\limsup_{k\to\infty}[\alpha(x_{m_k})\beta(x_{n_k})d(Tx_{m_k}, Tx_{n_k})]) \\
& = \limsup_{k\to\infty} F(\alpha(x_{m_k})\beta(x_{n_k})d(x_{m_k}, x_{n_k})) \\
& \leq F(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k})) \\
& \leq F(s\epsilon),
\end{aligned}$$

we obtain

$$\tau + F(s\epsilon) \leq F(s\epsilon)$$

which is a contradiction. We therefore have that $\{x_n\}$ is b-Cauchy. Since (X, d) is b-complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point.

THEOREM 3.8. Let (X, d) be a b-complete b-metric space with $s \geq 1$ and $T : X \to X$ be a TAC-Suzuki-Berinde type rational F-contraction type mapping. Suppose the following conditions hold:

1. T is a cyclic (α_s, β_s)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s^3$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s^3$.

Then T has a fixed point.

PROOF. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. In Theorem 3, we have establish that $\{x_n\}$ is b-Cauchy. Since (X, d) is b-complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. More so, using the condition that $\beta(x_n) \geq s^3$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\beta(x) \geq s^3$. We now establish that T has a fixed point.

Claim: We claim that
$$d(x_n, x) < \frac{1}{2s} d(x_n, x_{n+1})$$
or
$$d(x_{n+1}, x) < \frac{1}{2s} d(x_{n+1}, x_{n+2}).$$
Proof of Claim. Then using the fact that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, we have
$$\begin{aligned} d(x_n, x_{n+1}) &\leq sd(x_n, x) + sd(x, x_{n+1}) \\ &< \frac{1}{2} d(x_n, x_{n+1}) + \frac{1}{2} d(x_{n+1}, x_{n+2}) \\ &= d(x_n, x_{n+1}). \end{aligned}$$
The above inequality is a contradiction, thus, we must have that
$$d(x_n, x) \geq \frac{1}{2s} d(x_n, x_{n+1}) \quad \text{or} \quad d(x_{n+1}, x) \geq \frac{1}{2s} d(x_{n+1}, x_{n+2}).$$
Hence, we have
$$\begin{aligned} &\tau + F(d(x_{n+1}, Tx)) \\ &= \tau + F(d(Tx_n, Tx)) \\ &\leq \tau + F(\alpha(x_n)\beta(x) d(Tx_n, Tx)) \\ &\leq F(M(x_n, x)) + L \min\{d(x_n, Tx), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\}, \end{aligned}$$
where
$$\begin{aligned} M(x_n, x) \quad = \quad \max \Bigg\{ & d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2s}, \\ & \frac{d(x_n, Tx_n) d(x, Tx)}{s + d(x_n, x)}, \frac{d(x, Tx_n)[1 + d(x_n, Tx_n)]}{s + d(x_n, x)} \Bigg\}. \end{aligned}$$
Using the fact that $F \in \mathcal{F}$ and Lemma 2, we have that
$$\lim_{n\to\infty} F(d(Tx_n, Tx)) = -\infty$$
so that
$$\lim_{n\to\infty} d(Tx_n, Tx) = 0.$$
Now, observe that
$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0.$$

Clearly, we have that

$$d(x,Tx)=0 \Rightarrow x=Tx.$$

Hence, T has a fixed point.

THEOREM 3.9. Suppose that the hypothesis of Theorem 3.8 holds and in addition suppose $\alpha(x)\geq s^3$ and $\beta(y)\geq s^3$ for all $x,y\in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.

PROOF. The proof follows similar approach as in Theorem 3.2.

4 Example and Application

In this section, we give an example and application on the existence of a solution for the following nonlinear integral equation:

$$x(t)=g(t)+\int_a^b M(t,s)K(t,x(s))ds, \tag{17}$$

where $M:[a,b]\times[a,b]\to\mathbb{R}^+$, $K:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous functions. Let $X=C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. We defined $d:X\times X\to[0,\infty)$ by $d(x,y)=\sup_{t\in[a,b]}|x(t)-y(t)|^2$. Clearly, (X,d) is a complete b-metric space with $s=2$.

EXAMPLE 4. Let $X=[0,\infty)$ and $d:X\times X\to[0,\infty)$ be defined as $d(x,y)=|x-y|^2$ for all $x,y\in X$. It is clear that (X,d) is a b-metric space with $s=2$. We defined $T:X\to X$ by

$$Tx=\begin{cases}\frac{x}{16} & \text{if } x\in[0,1],\\ 4x & \text{if } x\in(1,\infty),\end{cases}$$

and $\alpha,\beta:X\to[0,\infty)$ by

$$\alpha(x)=\beta(x)=\begin{cases}8 & \text{if } x\in[0,1],\\ 0 & \text{if } x\in(1,\infty),\end{cases}$$

and $F(t)=\frac{-1}{t}+t$. Then T satisfy conditions in Theorem 3.2 and T is TAC-Suzuki-Berinde type F-contraction mapping.

PROOF. Since for any $x\in[0,1]$, we have that $\alpha(x)=s^3=8$, $Tx=\frac{x}{16}$, and $\beta(Tx)=\beta(\frac{x}{16})=8$. Since $\alpha(x)=\beta(x)$, it is easy to see that T is cyclic (α_s,β_s)-admissible mapping. For any $x_0\in[0,1]$, we have that $\alpha(x_0)=8$ and $\beta(x_0)=8$. Let $\{x_n\}$ be sequence in X with $\beta(x_n)\geq 8$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to\infty$, using the definition of β, we must have that $\{x_n\}\subset[0,1]$ and thus $x\in[0,1]$, which implies that $\beta(x)=8$. Since T is cyclic (α_s,β_s)-admissible mapping, if $x\in[0,1]$, we need to show that T is TAC-Suzuki-Berinde type F-contraction mapping for any $x,y\in[0,1]$ with $\frac{1}{2s}d(x,Tx)\leq d(x,y)$. Let $x,y\in[0,1]$ and without loss of generality we suppose that $x\leq y$. We then have that $\frac{1}{2s}d(x,Tx)=\frac{1}{4}|x-\frac{x}{16}|=\frac{15x}{64}$. Thus for

$\frac{1}{4}d(x,Tx) \leq d(x,y)$, we must have that $\frac{79x}{64} \leq y$. Observe that, for $\tau = 1$ and $L > 1$, it is easy to see that

$$\begin{aligned}
& \tau + F(\alpha(x)\beta(y)d(Tx,Ty)) \\
= \;& 1 + F(64|\frac{y}{16} - \frac{x}{16}|^2) \\
= \;& 1 + F(\frac{1}{4}|y-x|^2) = 1 + \frac{|y-x|^2}{4} - \frac{4}{|y-x|^2} \\
\leq \;& |y-x|^2 - \frac{1}{|y-x|^2} \\
& + L\min\left\{\left(\frac{15x}{16}\right)^2, \left(\frac{15y}{16}\right)^2, \left(\frac{16x-y}{16}\right)^2, \left(\frac{16y-x}{16}\right)^2\right\} \\
= \;& F(d(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.
\end{aligned}$$

Thus T is TAC-Suzuki-Berinde type F-contraction mapping and T also satisfy all the hypothesis of Theorem 3 with $x = 0$ as the unique fixed point of T.

THEOREM 4. Let $X = C([a,b],\mathbb{R})$ and $T : X \to X$ be the operator given by

$$Tx(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds$$

for all $t, s \in [a,b]$, where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. Furthermore, suppose the following conditions hold:

1. there exists a continuous mapping $\mu : X \times X \to [0,\infty)$ such that

$$|K(s,x(s)) - K(s,y(s))| \leq \mu(x,y)|x(s) - y(s)|$$

   for all $s \in [a,b]$ and $x, y \in X$;

2. there exists $\tau > 0$ and $\alpha, \beta : X \to [0,\infty]$ such that $\alpha(x) \geq s^3 \Rightarrow \beta(Tx) \geq s^3$ and $\beta(x) \geq s^3 \Rightarrow \alpha(Tx) \geq s^3$ for all $x \in X$, we have

$$\int_a^b M(t,s)\mu(x,y) \leq \sqrt{\frac{e^{-\tau}}{\alpha(x)\beta(y)}};$$

3. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$;

4. for any sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$ and $\beta(x_n) \geq s^3$ for each $n \in \mathbb{N} \cup \{0\}$, we have $\beta(x) \geq s^3$.

Then the integral equation (17) has a solution.

PROOF. We define $\alpha(x) = \beta(x) = 9$ for all $x \in X$. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)|^2 : s \in [a,b]\} \geq \sup\{|Tx(s) - x(s)|^2 : s \in [a,b]\},$$

which implies that

$$d(y,x) \geq d(Tx,x) \geq \frac{1}{4}d(Tx,x).$$

Thus, we have that

$$\begin{aligned}
|Ty(s) - Tx(s)|^2 &\leq \left(\int_a^b |M(t,s)[K(t,y(s)) - K(t,x(s))]|ds \right)^2 \\
&\leq \left(\int_a^b M(t,s)\mu(x,y)|y(s) - x(s)|ds \right)^2 \\
&\leq \left(\sup_{s\in[a,b]} |y(s) - x(s)| \int_a^b M(t,s)\mu(x,y)ds \right)^2 \\
&\leq d(y,x)\frac{e^{-\tau}}{81}.
\end{aligned}$$

Thus, we have that

$$81d(Tx,Ty) \leq d(x,y)e^{-\tau},$$

which implies that

$$\alpha(x)\beta(y)d(Tx,Ty) \leq d(x,y)e^{-\tau}.$$

Suppose that $F(t) = \ln t$, we have that

$$\tau + \ln(\alpha(x)\beta(y)d(Tx,Ty)) \leq \ln(d(x,y)).$$

Clearly, all the conditions in Theorem 3.3 are satisfied, and so T has a fixed point. Thus the integral equation (17) has a solution.

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