

Cylindrically Symmetric Fractional Helmholtz Equation*

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Abstract

Cylindrically symmetric fractional Helmholtz equation is analytically solved in an isotropic medium. Caputo's definition of the fractional derivative is followed at the solution approach. The general solution utilizes fractional Bessel functions attached to particular azimuthal and longitudinal exponents, it is represented in orthogonal and completeness basis likewise the ordinary form. The derived solution could be implemented at fractional modes of Bessel light as well as time-independent fractional diffusion.

1 Introduction

Helmholtz equation is a partial differential equation that describes time independent physical evolution in space. It is used in several applications in physics and applied mathematics. The solution of Helmholtz equation, which is commonly derived by separation of variables, resolves essential phenomena in nature. For instance, propagation of axial symmetric light beam through an isotropic medium, like the space, satisfies the equation. Time-independent diffusion of neutrons inside cylindrical reactor is another direct application of Helmholtz equation.

Fractional Helmholtz equation (FHE) is based on implementing the fractional differentiation of the ordinary equation [1]. It does not only generalize the ordinary Helmholtz equation, but also considers fractional field of the regular form. It is compatible with the ordinary equation at non fractional order [2], beside that, it is available in Curvilinear and Cartesian coordinates. Several approaches have been employed to find a convenient solution of FHE; for instance, fractional solutions have been proposed using differintegral relations [3]. Hilfer definition of Riemann-Liouville fractional derivative is also used to present fractional solution of Helmholtz equation [4]. In addition, FHE can also be solved by numerical approaches[5, 6], however, most of the approaches are performed in Cartesian coordinates.

This study drives sufficient analytical solution of the cylindrically symmetric FHE with Caputo fractional derivative. Our solution utilizes cylindrical coordinates due

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to its symmetrical feature, however the solution applicable in spherical coordinates as well. The derived results enriches the literature by adding efficient analytical solution away from rectangular space. Moreover, the followed procedure could be extended to deal with similar equations such as Poisson, Heat and Schrodinger equations.

2 The derivation

FHE of a fractional order $\alpha \in \mathcal{R}$ reads [4]

$$(\nabla_\alpha^2 + k^2)U(r, \phi, z) = 0. \tag{1}$$

The Laplace operator in cylindrical coordinates for isotropic medium, which is azimuthally independent of fractional differentiation, is given by [7]

$$\nabla_\alpha^2 = \frac{1}{r^\alpha}D_\alpha(r^\alpha D_\alpha) + \frac{\Gamma^2(\alpha + 1)}{r^{2\alpha}}\frac{\partial^2}{\partial \phi^2} + D_\alpha^2, \tag{2}$$

where D_α represents fractional derivative operator. It is defined for a p-differentiable $f(x)$ at right side of arbitrary constant a . by [8]

$$D_\alpha f(x) = \frac{1}{\Gamma(p - \alpha)} \int_a^x \frac{f^{(p)}(\tau) d\tau}{(x - \tau)^{\alpha+1-p}}, \tag{3}$$

where $p = [\alpha + 1]$. Separation of variables, $U^\alpha(r, \phi, z) = R(r)\Phi(\phi)Z(z)$ along with Laplace operator illustrate FHE to

$$\frac{1}{Rr^\alpha}D_\alpha(r^\alpha D_\alpha R) + \frac{\Gamma^2(\alpha + 1)}{r^{2\alpha}}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}D_\alpha^2 Z(z) + k^2 = 0. \tag{4}$$

Eq. 4 can be decomposed into the followings:

$$\frac{d^2 Z}{dz^2} + B^{2\alpha} Z = 0, \tag{5a}$$

$$\frac{d^2 \Phi}{d\phi^2} + \eta^2 \Phi = 0, \tag{5b}$$

$$r^\alpha D_\alpha(r^\alpha D_\alpha R) + (k_B^2 r^{2\alpha} - \eta^2 \Gamma^2(\alpha + 1))R = 0, \tag{5c}$$

where B and η are constants, and $k_B = \sqrt{k^2 - B^{2\alpha}}$. Eqs 5a and 5b are identical to the non-fractional solution terms, they have the sinusoidal form solutions $Z = e^{\pm iB^\alpha z}$ and $\Phi = e^{i\eta\phi}$ respectively, where both B and $\eta \in \mathcal{R}$ unless for evanescent wave.

2.1 Fractional Bessel Function

Eq. 5c is a fractional differential equation with a singular point at $r^\alpha = 0$, without loss of generality, this equation can be solved by fractional series expansion of the form: $R(r) = r^{\alpha\nu} \sum_{n=0}^\infty C_n r^{\alpha n}$. Substitution in Eq. 5c and use:

$$D_\alpha(x^\beta) = (\Gamma(\beta + 1)/\Gamma(\beta - \alpha + 1))(x)^{\beta-\alpha}, \tag{6}$$

where $\alpha, \beta \notin N$. The first coefficient in the fractional series can be written as:

$$C_0 \left[\left(\frac{\Gamma(\alpha\nu + 1)}{\Gamma(\alpha\nu - \alpha + 1)} \right)^2 - \eta^2 \Gamma^2(\alpha + 1) \right] = 0. \quad (7)$$

The surface, which is oriented by α, ν and η , is defined for $C_0 \neq 0$, is illustrated in Fig. 1. The recursion formula of the fractional series is determined by considering the

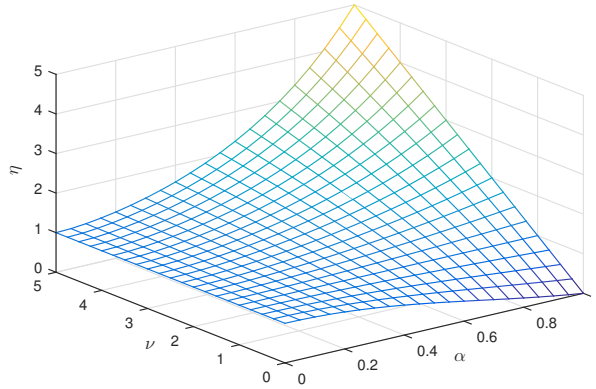


Figure 1: Meshed surface composite of ν, η and α .

n th terms as follows:

$$C_n \left[\left(\frac{\Gamma(\alpha\nu + \alpha n + 1)}{\Gamma(\alpha\nu + \alpha n - \alpha + 1)} \right)^2 - \eta^2 \Gamma^2(\alpha + 1) \right] + C_{n-2} k_B^2 = 0. \quad (8)$$

Thus,

$$C_n = -k_B^2 \left[\left(\frac{\Gamma(\alpha\nu + \alpha n + 1)}{\Gamma(\alpha\nu + \alpha n - \alpha + 1)} \right)^2 - \left(\frac{\Gamma(\alpha\nu + 1)}{\Gamma(\alpha\nu - \alpha + 1)} \right)^2 \right]^{-1} C_{n-2}. \quad (9)$$

Using the asymptotic formula of Gamma function [9]

$$\Gamma(ax + b) \sim \sqrt{2\pi} e^{-ax} (ax)^{ax+b-\frac{1}{2}}, \quad (10)$$

to simplify the ratio in Eq. 9 as follows

$$\frac{\Gamma(\alpha\nu + 1)}{\Gamma(\alpha\nu - \alpha + 1)} \equiv (\alpha\nu)^\alpha, \quad \left(\frac{\Gamma(\alpha\nu + \alpha n + 1)}{\Gamma(\alpha\nu + \alpha n - \alpha + 1)} \right) \equiv (\alpha\nu + \alpha n)^\alpha. \quad (11)$$

Thus the n th term coefficient C_n reduces to:

$$C_n = \frac{-k_B^2}{(n^2 \alpha^2 + 2\alpha\nu n)^\alpha} C_{n-2}. \quad (12)$$

Since odd terms vanish, only even indices are considered

$$C_{2n} = \frac{-k_B^2}{[2^2 n(\alpha n + \nu)]^\alpha} C_{2n-2}, \quad (13)$$

taking $C_0 = 1$,

$$C_{2n} = \frac{(-1)^n k_B^{2n}}{[2^{2n}(\alpha n)!(\alpha n + \nu)!]^\alpha}. \tag{14}$$

Hence, the particle solution of Eq. 5c is

$$R(r) = r^{\alpha\nu} \sum_{n=0}^{\infty} (-1)^n \left(\frac{k_B}{2^\alpha}\right)^{2n} \frac{r^{2\alpha n}}{[(\alpha n)!(\alpha n + \nu)!]^\alpha}. \tag{15}$$

The generated series is a Bessel function of fractional mode: $J_{\alpha\nu}(k_B r)$, it is compatible with the ordinary form when $\alpha = 1$. The fractional Bessel function of first and second modes are respectively plotted at different fractional orders in Figs. 2 (a and b). There are no fractional orders of zero Bessel function according to Eq. 15.

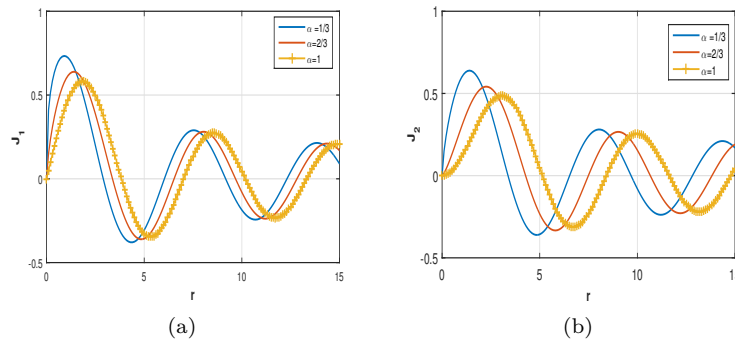


Figure 2: Fractional Bessel Function of first (a) and second (b) modes at particular fraction orders as well as non-fraction order ($\alpha = 1$).

2.2 General Solution

The general solution of the FHE is given by

$$U^\alpha = J_{\alpha\nu}(k_B r) e^{i\eta\phi} e^{\pm iB^\alpha z}. \tag{16}$$

Azimuthal coefficient can be extracted from Eq. 7 as $\eta = (\alpha\nu)^\alpha / \Gamma(\alpha + 1)$, thus Eq. 16 can be written as:

$$U^\alpha = J_{\alpha\nu}(k_B r) \exp\left[i \frac{(\alpha\nu)^\alpha}{\Gamma(\alpha + 1)} \pm B^\alpha z\right]. \tag{17}$$

Eq. 17 is not only compatible with ordinary solution at $\alpha = 1$, but also it expands the Helmholtz solution to fractional modes and bases.

2.3 Orthogonality and Completeness

This subsection characterizes the orthogonality and completeness of the fractional Helmholtz general solution. Without loss of generality, the transverse components of the general solution can be expanded by orthogonal basis such that

$$J_{\alpha\nu}(k_B r) \exp\left[i \frac{(\alpha\nu)^\alpha}{\Gamma(\alpha+1)}\right] = \sum_{m=-\infty}^{\infty} C_m^\alpha J_m(k_B r) e^{im\phi}. \quad (18)$$

The expansion coefficients can be determined by the orthogonality of $(J_m(k_B r) e^{im\phi})$ [9] as follows

$$C_m^\alpha = \frac{1}{2\pi} \int_0^\infty J_{\alpha\nu}(kr) J_m(kr) \frac{dr}{r} \int_0^{2\pi} \exp[i(\nu/\Gamma(\alpha) - m)\phi] d\phi; \quad (19)$$

evaluating the integrals in Eq. 19 one gets

$$C_m^\alpha = \frac{2}{\pi^2} \frac{(i)^{\nu/\Gamma(\alpha)} \sin(\pi\nu/\Gamma(\alpha)) \sin[\pi(\alpha\nu - m)/2]}{(\nu/\Gamma(\alpha) - m) [(\alpha\nu)^2 - m^2]}; \quad (20)$$

when $\alpha = 1$, $\nu \rightarrow m$, then $C_m^\alpha = 1$, hence the transverse FHE reduces to the ordinary form. Moreover, the completeness of $(J_m(k_B r) e^{im\phi})$ states that [10]

$$\begin{aligned} & \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty |C_m^\alpha|^2 \int_0^\infty dk_B k_B [J_m(kr) e^{im\phi}] [J_m(kr') e^{im\phi'}]^* \\ &= \frac{1}{r} \delta(r - r') \delta(\phi - \phi'). \end{aligned} \quad (21)$$

Assuming axial periodic boundary conditions with finite normalization length, unless evanescent waves, the general solution of the FHE can be written as

$$U_\pm^\alpha(r, \phi, z) = \sum_{m=-\infty}^{\infty} C_m^{\alpha\nu} J_m(k_B r) e^{im\phi} e^{\pm iB^\alpha z} \quad (22)$$

where $\alpha \notin N$, $m = 0, \pm 1, \pm 2, \dots$ and $k_B = \sqrt{k^2 - B^{2\alpha}}$, Eq. 22 represents orthogonal and complete set solution.

3 Application

3.1 Fractional Bessel Light

The scalar wave amplitude of Bessel light, mathematically speaking, is the solution of the cylindrically symmetric Helmholtz equation. The modes of the transverse terms, which are first kind Bessel functions attached to exponentials of complex azimuthal terms, are basically integers. So its spatial wave is characterized by helical structure as well as phase singularity at the center [11]. Moreover, the magnitude of its wave number, k , composites of transverse parameter, k_B , as well as longitude parameter, B .

Fractional Bessel light could be theoretically derived in several ways, it could be generated by using fractional calculus of raising operators or fractional Fourier transforms [12]. In addition, it is generated practically for non-complete cycle of the azimuthal phase [13]. Our procedure is based on direct derivation as an Eigen state of the cylindrically symmetric FHE , in an analogy to the familiar ordinary form of Bessel light. Thus, the amplitude of monochromatic fractional Bessel light that propagates in free space can be expressed as

$$U_{\pm}^{\alpha}(r, \phi, z) = \sum_{m=-\infty}^{\infty} \int_0^k dk_B k_B U_{+}^{\alpha} + \sum_{m=-\infty}^{\infty} \int_0^k dk_B k_B U_{-}^{\alpha} \quad (23)$$

where \pm indicates traveling along the $+z$ or $-z$ axis, respectively. Similar to ordinary modes, fractional Bessel modes are non diffracting light. The modulus of its transverse amplitude, is independent of the propagation distance, that is

$$|U^{\alpha}(r, \phi, z = 0)|^2 = |U^{\alpha}(r, \phi, z = z_0)|^2 \quad (24)$$

As an example, fractional orders of second Bessel light modes, $\nu = 2$, are illustrated in Fig. 3. On the other hand, fractional order Bessel light represents a destruction in the wave amplitude as can be seen in Fig. 3. The rings of ordinary Bessel light is deformed as fractional order goes away, again their formulas are compatible with ordinary forms at non-fractional modes. The corresponding phase angles, which are suspended between imaginary and real values of Eq. 23, are also defined in fractional space. Its distribution represents intermediate states between the ordinary states of integer modes as shown in Fig. 4, where it is clear the effect of the fractional orders. Readers should note that both Figs. 3 and 4 are plotted in transverse scale in units of k^{-1} .

3.2 Cylindrically Symmetric Diffusion

The equation of time-independent fractional diffusion at uniform azimuthal symmetry inside a cylinder of finite length can be also described by FHE. Eq. 22 can be modified to represent fractional diffusion in this case, the fractional flux of such system is given by:

$$U_{\pm}^{\alpha\nu}(r, \phi, z) = \sum_{m=-\infty}^{\infty} C_m^{\alpha\nu} J_m(k_B r) e^{\pm i B^{\alpha} z} \quad (25)$$

where $C_m^{\alpha\nu}$ is defined according to the orthonormality property as

$$C_m^{\alpha\nu} = \frac{2 \sin[\pi(\alpha\nu - m)/2]}{\pi [(\alpha\nu)^2 - m^2]} \quad (26)$$

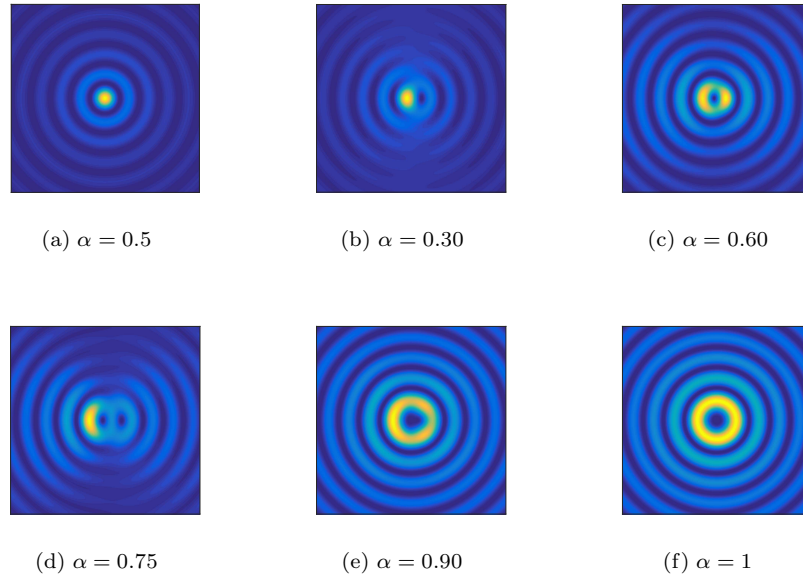


Figure 3: Images of transverse amplitude of second mode Bessel light at different fraction orders.

Table 1: Numerical values of critical radii at different fractional modes of higher fluxes.

Flux Order	α	$r_c(k_B)$
$\nu = 1$	0.25	2.7809
	0.5	3.1416
	0.75	3.4910
	1	3.8317
$\nu = 2$	0.25	3.1416
	0.5	3.8317
	0.75	4.4934
	1	5.1356

Eq. 25 is reduced to ordinary flux equation which is compatible with the familiar solution of spatial diffusion [14]. The flux is fractionalized at higher orders ($\nu \geq 1$) and the critical fractional flux is determined at the zeroes of Eq. 25. On the other hand, the critical radii are determined at the zeroes of the fractional Bessel function, which vary according to its mode. Certain numerical values of the critical radius, r_c , for different values of α are illustrated in Table 1, the data used in the calculation is just for the first two high orders of the fluxes. Also the critical length is determined when the exponential term of Eq. 25 vanishes, that is

$$e^{iB^\alpha z} = 0 \rightarrow B^\alpha = m\pi/L_c, m = 0, \pm 1, \pm 2, \dots \quad (27)$$

thus, the critical length of the cylinder could be specified. The data in Table 1, in

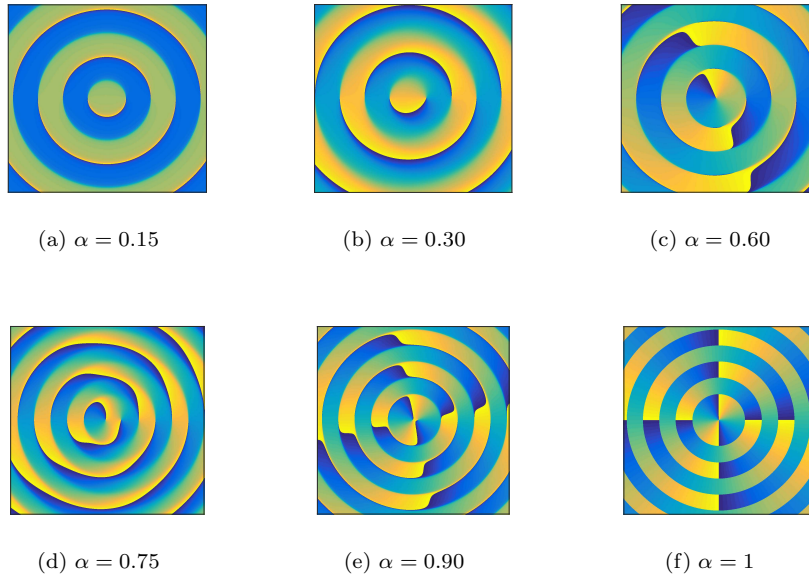


Figure 4: Phase angle of second mode Bessel light at different fractional orders.

parallel with Eq. 27, could be used to determine the corresponding fractional buckling according to the condition in Eq. 22:

$$k^2 = k_B^2 + m^2\pi^2/L_c^2 \tag{28}$$

Table 2: Fractional buckling of certain critical lengths and radii.

$L_c(cm)$	$r_c(cm)$	α	$k(cm)^{-1}$
15	11.3144	0.25	0.3591
		0.50	0.3816
		0.75	0.4047
20	9.4788	0.25	0.3328
		0.50	0.3668
		0.75	0.4003
30	8.6066	0.25	0.3397
		0.50	0.3798
		0.75	0.4189
50	8.2440	0.25	0.3431
		0.50	0.3862
		0.75	0.4281
100	8.1042	0.25	0.3446
		0.50	0.3889
		0.75	0.4319

Table 2 illustrates chosen typical values of critical lengths and critical radii [15, 16, 9]

which generates numerical values of the buckling of a cylinder at different fractional first mode. It also shows the variation of buckling as the fractional order changes, note that the buckling is slightly reduced at small fractional modes.

4 Conclusion

This study applies a fractional differentiation of the cylindrically symmetric Helmholtz equation for isotropic medium. It provides an exact solution based on fractional Bessel functions attached to particular exponents. The solution does not only compatible with the ordinary form at non fractional orders, but also it considers the intermediate cases as well. Moreover; the derived solution is represented as orthogonal and complete set. In this work, we illustrate two direct applications of the cylindrically symmetric FHE, spatial wave amplitudes and their phase angles of fractional Bessel light modes are characterized. Also time independent fractional diffusion in a cylindrically symmetric isotropic medium is analyzed. In addition, critical heights and radii where corresponding fluxes vanish, are determined.

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