

The Equivalence Of Mini-Max Theorem And Existence Of Nash Equilibrium In Asymmetric Three-Players Zero-Sum Game With Two Groups*

Atsuhiko Satoh[†], Yasuhito Tanaka[‡]

Abstract

We consider the relation between Sion's minimax theorem for a continuous function and a Nash equilibrium in an asymmetric three-players zero-sum game with two groups. Two players are in Group A, and they have symmetric payoff functions and strategy space. One player, Player C who is in Group C, does not. We show that the existence of a Nash equilibrium, which is symmetric in Group A, is equivalent to Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.

1 Introduction

We consider the relation between Sion's minimax theorem for a continuous function and existence of a Nash equilibrium in an asymmetric three-players zero-sum game with two groups¹. Two players are in one group (Group A), and they have symmetric payoff functions and the same strategy space, and so their equilibrium strategies, maximin strategies and minimax strategies are the same. One player, Player C who is in the other group (Group C), does not. We will show the following results.

1. The existence of a Nash equilibrium, which is symmetric in Group A, implies Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.
2. Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A implies the existence of a Nash equilibrium which is symmetric in Group A.

Thus, they are equivalent.

An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are three firms, A, B and C in an oligopolistic industry.

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[†]Faculty of Economics, Hokkai-Gakuen University, Toyohira-ku, Sapporo, Hokkaido, 062-8605, Japan

[‡]Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto, 602-8580, Japan

¹In [8] we have analyzed a similar problem in a symmetric zero-sum game in which all players are identical.

Let $\bar{\pi}_A$, $\bar{\pi}_B$ and $\bar{\pi}_C$ be the absolute profits of the firms. Then, their relative profits are

$$\pi_A = \bar{\pi}_A - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C), \quad \pi_B = \bar{\pi}_B - \frac{1}{2}(\bar{\pi}_A + \bar{\pi}_C), \quad \pi_C = \bar{\pi}_C - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C).$$

We see

$$\pi_A + \pi_B + \pi_C = \bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C - (\bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C) = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game². If the oligopoly is fully asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the oligopoly is symmetric for two firms in one group (Group A) in the sense that the demand function is symmetric and two firms have the same cost function, the maximin strategies for those firms with the corresponding minimax strategy of the firm in the other group (Group C) constitute a Nash equilibrium which is symmetric in Group A.

2 The Model and Sion's Minimax Theorem

Consider a three-players zero-sum game with two groups. There are three players, A, B and C. The strategic variables for Players A, B and C are, respectively, s_A , s_B , s_C , and $(s_A, s_B, s_C) \in S_A \times S_B \times S_C$. S_A , S_B and S_C are convex and compact sets in linear topological spaces. The payoff function of each player is $u_i(s_A, s_B, s_C)$, $i = A, B, C$. They are real valued functions on $S_A \times S_B \times S_C$. We assume

u_A , u_B and u_C are continuous on $S_A \times S_B \times S_C$, quasi-concave on S_i for each $s_j \in S_j$, $j \neq i$, and quasi-convex on S_j for $j \neq i$ for each $s_i \in S_i$, $i = A, B, C$.

Three players are partitioned into two groups. Group A and Group C. Group A includes Player A and Player B, and Group C includes only Player C. In Group A two players are symmetric, that is, they have symmetric payoff functions and the same strategy spaces, however, Player C does not. Symmetry of the payoff functions means

$$u_A(s_A, s_B, s_C) = u_B(s_B, s_A, s_C).$$

Thus, the equilibrium strategies, maximin strategies and minimax strategies for Players A and B are the same.

Since the game is a zero-sum game, we have

$$u_A(s_A, s_B, s_C) + u_B(s_A, s_B, s_C) + u_C(s_A, s_B, s_C) = 0,$$

for given (s_A, s_B, s_C) .

Sion's minimax theorem ([9], [2], [1]) for a continuous function is stated as follows.

²About relative profit maximization under imperfect competition please see [3], [4], [5], [6], [10], [11] and [12]

LEMMA 1 (Sion's minimax theorem). Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \rightarrow \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of this theorem in [1].

Let s_B be given, then $u_A(s_A, s_B, s_C)$ is a function of s_A and s_C . We can apply Lemma 1 to such a situation, and get the following equation.

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B, s_C). \tag{1}$$

We do not require

$$\max_{s_C \in S_C} \min_{s_A \in S_A} u_C(s_A, s_B, s_C) = \min_{s_A \in S_A} \max_{s_C \in S_C} u_C(s_A, s_B, s_C),$$

nor

$$\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s_C) = \min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s_C) \text{ given } s_C.$$

We assume that

$$\arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B, s_C)$$

and

$$\arg \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B, s_C)$$

are unique, that is, single-valued. By the maximum theorem they are continuous in s_B . Throughout this paper we also assume that the maximin strategy and the minimax strategy of players in any situation are unique, and the best responses of players in any situation are unique. Similarly, we obtain

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A, s_B, s_C),$$

given s_A ³.

Let $s_B = s$. Consider the following function.

$$s \rightarrow \arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s, s_C).$$

Since u_A is continuous, S_A and S_C are compact and $S_A = S_B$, this function is also continuous. Thus, there exists a fixed point. Denote it by \tilde{s} . \tilde{s} satisfies

$$\arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \tilde{s}. \tag{2}$$

³We do not require

$$\begin{aligned} \max_{s_C \in S_C} \min_{s_B \in S_B} u_C(s_A, s_B, s_C) &= \min_{s_B \in S_B} \max_{s_C \in S_C} u_C(s_A, s_B, s_C), \\ \max_{s_B \in S_B} \min_{s_A \in S_A} u_B(s_A, s_B, s_C) &= \min_{s_A \in S_A} \max_{s_B \in S_B} u_B(s_A, s_B, s_C). \end{aligned}$$

From (1), \tilde{s} satisfies

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \quad (3)$$

From symmetry for Players A and B, \tilde{s} also satisfies

$$\arg \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(\tilde{s}, s_B, s_C) = \tilde{s}, \quad (4)$$

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(\tilde{s}, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(\tilde{s}, s_B, s_C). \quad (5)$$

We summarize the arguments in the following lemma.

LEMMA 2 (Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A). There exists \tilde{s} such that (2)–(5) are satisfied.

3 The Main Results

Consider a Nash equilibrium of a three-players zero-sum game. Let s_A^* , s_B^* , s_C^* be the values of s_A , s_B , s_C which, respectively, maximize u_A given s_B and s_C , maximize u_B given s_A and s_C , maximize u_C given s_A and s_B in $S_A \times S_B \times S_C$. Then,

$$u_A(s_A^*, s_B^*, s_C^*) \geq u_A(s_A, s_B^*, s_C^*) \text{ for all } s_A \in S_A,$$

$$u_B(s_A^*, s_B^*, s_C^*) \geq u_B(s_A^*, s_B, s_C^*) \text{ for all } s_B \in S_B,$$

$$u_C(s_A^*, s_B^*, s_C^*) \geq u_C(s_A^*, s_B^*, s_C) \text{ for all } s_C \in S_C.$$

They mean

$$\arg \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) = s_A^*,$$

$$\arg \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) = s_B^*,$$

$$\arg \max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*.$$

We assume that the Nash equilibrium is symmetric in Group A, that is, it is symmetric for Player A and Player B. Then, $s_B^* = s_A^*$, and $u_A(s_A^*, s_B^*, s_C^*) = u_B(s_A^*, s_B^*, s_C^*)$. We also have

$$u_A(s_A^*, s_B^*, s_C) = u_B(s_A^*, s_B^*, s_C) \text{ for any } s_C.$$

Since the game is zero-sum,

$$u_A(s_A^*, s_B^*, s_C) + u_B(s_A^*, s_B^*, s_C) = 2u_A(s_A^*, s_B^*, s_C) = 2u_B(s_A^*, s_B^*, s_C) = -u_C(s_A^*, s_B^*, s_C).$$

Thus,

$$\arg \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) = \arg \max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*,$$

$$\arg \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) = \arg \max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*.$$

They imply

$$\begin{aligned} \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) &= u_A(s_A^*, s_B^*, s_C^*) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*), \\ \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) &= u_B(s_A^*, s_B^*, s_C^*) = \max_{s_B \in S_B} u_B(s_A, s_B^*, s_C^*). \end{aligned}$$

First we show the following theorem.

THEOREM 1. The existence of a Nash equilibrium, which is symmetric in Group A, implies Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.

PROOF. Let (s_A^*, s_B^*, s_C^*) be a Nash equilibrium of a three-players zero-sum game. This means

$$\begin{aligned} \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) &\leq \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) \\ &= \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) \leq \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C), \end{aligned} \tag{6a}$$

for Player A.

$$\begin{aligned} \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) &\leq \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) \\ &= \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) \leq \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C), \end{aligned} \tag{6b}$$

for Player B.

On the other hand, since

$$\min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \leq u_A(s_A, s_B^*, s_C^*),$$

we have

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \leq \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*).$$

This inequality holds for any s_C . Thus,

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \leq \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C).$$

With (6a), we obtain

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C). \tag{7a}$$

Similarly, for Player B we can show

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C). \tag{7b}$$

Then (6a), (6b), (7a) and (7b) imply

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*),$$

$$\begin{aligned}\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) &= \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C), \\ \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) &= \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*), \\ \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) &= \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C).\end{aligned}$$

From

$$\min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \leq u_A(s_A, s_B^*, s_C^*),$$

and

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*),$$

we have

$$\arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \arg \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) = s_A^*.$$

From

$$\max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) \geq u_A(s_A^*, s_B^*, s_C),$$

and

$$\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C),$$

we get

$$\arg \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \arg \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) = s_C^*.$$

Similarly, we can show

$$\arg \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) = \arg \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) = s_B^* = s_A^*,$$

and

$$\arg \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) = \arg \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) = s_C^*.$$

Therefore,

$$\arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \arg \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C),$$

and

$$\arg \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \arg \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C).$$

Q.E.D.

Next we show the following theorem.

THEOREM 2. Sion's minimax theorem with symmetry in Group A implies the existence of a Nash equilibrium which is symmetric in Group A.

PROOF. Let \tilde{s} be a value of s_A such that

$$\tilde{s} = \arg \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C).$$

Then, we have

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \quad (8)$$

Since

$$u_A(\tilde{s}, \tilde{s}, s_C) \leq \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C),$$

and

$$\min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C),$$

we get

$$\arg \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \arg \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \quad (9)$$

Since the game is zero-sum,

$$u_A(\tilde{s}, \tilde{s}, s_C) + u_B(\tilde{s}, \tilde{s}, s_C) = 2u_A(\tilde{s}, \tilde{s}, s_C) = -u_C(\tilde{s}, \tilde{s}, s_C).$$

Therefore,

$$\arg \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \arg \max_{s_C \in S_C} u_C(\tilde{s}, \tilde{s}, s_C).$$

Let

$$\hat{s}_C = \arg \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \arg \max_{s_C \in S_C} u_C(\tilde{s}, \tilde{s}, s_C). \quad (10)$$

Then, from (8) and (9)

$$\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C) = \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \max_{s_A \in S_A} u_A(s_A, \tilde{s}, \hat{s}_C) = u_A(\tilde{s}, \tilde{s}, \hat{s}_C). \quad (11)$$

Similarly, we can show

$$\max_{s_B \in S_B} u_B(\tilde{s}, s_B, \hat{s}_C) = u_B(\tilde{s}, \tilde{s}, \hat{s}_C). \quad (12)$$

(10), (11) and (12) mean that $(s_A, s_B, s_C) = (\tilde{s}, \tilde{s}, \hat{s}_C)$ is a Nash equilibrium which is symmetric in Group A. Q.E.D.

4 Concluding Remarks

In this paper we have examined the relation between Sion's minimax theorem for a continuous function and a Nash equilibrium in an asymmetric three-players zero-sum game with two groups. We want to extend this result to more general multi-players zero-sum game.

In [7], we have studied the choice of strategic variables under relative profit maximization in a three-firms asymmetric oligopoly. Here we generalize the model of [7] as an asymmetric three-players zero-sum game and consider the relation between minimax theorem and existence of Nash equilibrium.

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