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On The Convergence Of A New Proximal Point Algorithm Of Generalized Nonexpansive Mappings In CAT(0) Spaces*

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Abstract

In this paper, we introduce a new proximal point three step algorithm to establish some strong and Δ-convergence theorems under some suitable conditions and approximate the common fixed point of two finite families of generalized nonexpansive mappings in $CAT(0)$ spaces. Our results generalize and improve several previously known results of the existing literature.

1 Introduction

Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to K$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \quad \forall\ x, y \in K.$$

An element $x \in K$ is said to be a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by by $F(T)$.

Suzuki [40] introduced a generalization of nonexpansive maps and referred them as maps satisfying condition (C) and also established some fixed point theorems for these maps. A mapping $T : K \to K$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),\ \forall\ \ x, y \in K.$$

Every nonexpansive mapping satisfies condition (C) on K. Some of the examples of noncontinuous mappings satisfying condition (C) are mentioned in [40].

Recently, García-Falsat *et al.* [16] defined two new generalizations of condition (C) and termed them as condition (E) and condition (C_λ). They also studied the existence of fixed points and asymptotic behavior under these conditions.

A mapping $T : K \to K$ is said to satisfy condition (C_λ) if for all $x, y \in K$ and $\lambda \in (0, 1)$,

$$\lambda d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

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For $\lambda = \frac{1}{2}$, we recapture the class of mappings satisfying condition (C). Notice that if $0 < \lambda_1 < \lambda_2 < 1$, then condition (C_{λ_1}) implies condition (C_{λ_2}) but the converse is false (see Example 5, [16]).

Now, we recall another generalization of a nonexpansive map named as condition (E). A mapping $T : K \to K$ is said to satisfy condition (E_μ) if for some $\mu \geq 1$ and for all $x, y \in K$,

$$d(x, Ty) \leq \mu\, d(x, Tx) + d(x, y).$$

We say that T satisfies condition (E) on K whenever T satisfies condition (E_μ) for some $\mu \geq 1$. For more details, one can see [36, 38].

In view of the foregoing definitions, we have the following remarks:

REMARK 1.

(i) If $T : K \to K$ is a nonexpansive mapping, then T satisfies condition (E_1). But the converse is not true in general.

(ii) From Lemma 1 in [40], it can be easily seen that if $T : K \to K$ satisfies condition (C), then T satisfies condition (E_3) but the converse is not true in general.

The concept of Δ-convergence in metric spaces was introduced by Lim [28]. Kirk [25] proved the existence of fixed points of nonexpansive mappings in CAT(0) spaces. Kirk and Panyanak [26] specialized this concept to CAT(0) spaces and showed that many results of Banach spaces (involving weak convergence) have precise analogs in this setting. Dhompongsa and Panyanak [14] proved some results by using Mann and Ishikawa iterative process involving a single mapping.

Denote by $\mathbb{N} = \{1, 2, 3, \ldots, m\}$, the indexing set. For approximating a fixed point, Mann [29] and Ishikawa [21] introduced the following iterative schemes for a mapping $T : K \to K$, which are as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n T x_n + (1 - a_n) x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and

$$\begin{cases} x_1 \in K, \\ y_n = b_n T x_n + (1 - b_n) x_n, \\ x_{n+1} = a_n T y_n + (1 - a_n) x_n, \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$. He et al. [20] proved that the sequence $\{x_n\}$ generated by (1) and (2) converges and Δ-converges respectively to a fixed point of T in CAT(k) spaces.

The S-iterative scheme [1] is defined as:

$$\begin{cases} x_1 \in K, \\ y_n = b_n T x_n + (1 - b_n) x_n, \\ x_{n+1} = a_n T y_n + (1 - a_n) T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. For contraction mappings, this iterative scheme has a better convergence rate than those of (1) and (2).

Subsequently, Khan and Abbas [24] studied the approximation of common fixed point by the Ishikawa-type iteration process involving two mappings in CAT(0) spaces. In CAT(0) space, they also modified the process (3) and studied the strong and Δ-convergence of S-iteration as follows:

$$\begin{cases} x_1 \in K, \\ y_n = b_n T x_n \oplus (1-b_n) x_n, \\ x_{n+1} = a_n T y_n \oplus (1-a_n) T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

For solving a fixed point problem of nonlinear mappings in the framework of CAT(0) spaces, one can see [11, 12, 33, 35, 37, 38]. Let (X,d) be a geodesic metric space and $g : X \to (-\infty, \infty)$ a convex function. In optimization theory, one of the major problem is to find $x \in X$ such that

$$g(x) = \min_{y \in X} g(y).$$

Here the set of minimizers of g is denoted by $\operatorname*{argmin}_{y \in X} g(y)$. A successful and powerful tool for solving this problem is the well-known proximal point algorithm (shortly, the PPA) which was introduced by Martinet [30] in 1970. Rockafellar in [34] generally studied the convergence to a solution of the convex minimization problem by the PPA in the framework of Hilbert spaces. Now, we present the PPA in the following manner:

Let g be a convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{y \in H} \left(g(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right)$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It is proved that the sequence $\{x_n\}$ converges weakly to a minimizer of g provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. However, the PPA does not necessarily converge strongly in general, as shown by Güler [18]. In 2000, Kamimura-Takahashi [23] combined the PPA with Halpern's algorithm [19] so that the strong convergence is guaranteed.

In 2013, Bačák [4] introduced the PPA in a $CAT(0)$ space (X,d) as follows: $x_1 \in X$ and

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(g(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right)$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. By the concept of the Fejér monotonicity, it has been shown that if g has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see [3]).

Further, in 2014, Bačák [5] employed a split version of the PPA for minimizing a sum of convex functions in complete CAT(0) spaces. For solving optimization problems by the PPA, many convergence results have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [17, 27, 32, 41]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. In 2015, Cholamjiak *et al.* [10] proposed a modified proximal point algorithm by using the S-type iteration process for two nonexpansive mappings in CAT(0) spaces as follows: let $\{T_1\}$ and $\{T_2\}$ be two nonexpansive self-mappings on a $CAT(0)$ space X and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Then the sequence $\{x_n\}$ is generated as: $x_1 \in X$ and

$$\begin{cases} z_n = \underset{y \in X}{\operatorname{argmin}} \left[g(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \\ y_n = b_n x_n \oplus (1 - b_n) T_1 z_n, \\ x_{n+1} = a_n T_1 x_n \oplus (1 - a_n) T_2 y_n, \quad n \in \mathbb{N}, \end{cases}$$

for each $n \in \mathbb{N}$, where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$.

A question arises naturally:

Can we establish strong convergence of the sequence to minimizers of a convex function and to xed points of generalized nonexpansive mappings in CAT(0) spaces?

Motivated by the above work, we introduce a three step proximal point algorithm to establish some convergence theorems and approximate the common fixed point of two finite families of generalized nonexpansive mappings in CAT(0) spaces under some suitable conditions. For more details on generalized nonexpansive mappings one can be referred to [6, 37, 39].

Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : n \in \mathbb{N}\}$ be two finite families of generalized nonexpansive self-mappings on a CAT(0) space X and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. For $x_1 \in X$, we define

$$\begin{cases} z_n = \underset{y \in X}{\operatorname{argmin}} \left[g(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \\ y_n = \beta_{0,n} x_n \oplus (1 - \beta_{0,n}) \sum_{i=1}^{m} \frac{\beta_{i,n}}{(1-\beta_{0,n})} S_i z_n, \\ x_{n+1} = \alpha_{0,n} x_n \oplus (1 - \alpha_{0,n}) \sum_{i=1}^{m} \frac{\alpha_{i,n}}{(1-\alpha_{0,n})} T_i y_n, \end{cases} \tag{4}$$

for each $n \in \mathbb{N}$, and $\{\alpha_{i,n}\}_{i=0}^m$ and $\{\beta_{i,n}\}_{i=0}^m$ are the sequences in $(0, 1)$ satisfying $\sum\limits_{i=0}^{m} \alpha_{i,n} = \sum\limits_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ.

We can approximate all type of finite families of proximal point algorithms (PPA) via S-type iteration process for generalized nonexpansive mappings in CAT(0) spaces. In brief, we can establish strong and Δ-convergence of the sequence to minimizers of a convex function and to fixed points of generalized nonexpansive mappings in CAT(0) spaces.

2 Preliminaries

This section contains preliminary notions, basic definitions and relevant well known results which are required to prove the main results. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$(or more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A metric space (X, d) is said to be a geodesic space if every two points of X are joined by only one geodesic segment. A geodesic triangle $\Delta(x, y, z)$ in a geodesic metric space (X, d) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices. A comparison triangle for the geodesic triangle $\Delta(x, y, z)$ in (X, d) is a triangle $\bar{\Delta}(x, y, z) := \Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{R}^2$ such that $d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{R}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{R}^2}(\bar{z}, \bar{x})$.

DEFINITION 1. A metric space (X, d) is said to be a CAT(0) space if for each geodesic triangle $\Delta(x, y, z)$ in X and its comparison triangle $\bar{\Delta} := \Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(p, q) < d_{\mathbb{R}^2}(\bar{p}, \bar{q})$$

is satisfied for all $p, q \in \Delta$ and comparison points $\bar{p}, \bar{q} \in \bar{\Delta}$.

LEMMA 1 ([8]). Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining x to y, that is,

$$[x, y] = \{(1 - t)x \oplus ty \; : \; t \in [0, 1]\}.$$

A subset K of a CAT(0) space is said to be convex if $[x, y] \subset K$ for all $x, y \in K$. For more details, one can see [7].

LEMMA 2 ([8]). A geodesic space X is a CAT(0) space if and only if

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y) \tag{5}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular, if x, y, z are points in X and $t \in [0, 1]$, then we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z). \tag{6}$$

Let $\{v_1, v_2, \dots, v_n\} \subset X$ and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset (0,1)$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$. For $n = 2$, in view of Lemma 1, we have

$$\sum_{i=1}^{2} \oplus \lambda_i v_i = \frac{\lambda_1}{\lambda_1 + \lambda_2} v_1 \oplus \frac{\lambda_2}{\lambda_1 + \lambda_2} v_2.$$

For $n = 3$, we have to find $\sum\limits_{i=1}^{2} \oplus \lambda_i v_i$ with $\sum\limits_{i=1}^{3} \lambda_i = 1$. As we have

$$\sum_{i=1}^{2} \oplus \lambda_i v_i = \frac{\lambda_1}{\lambda_1 + \lambda_2} v_1 \oplus \frac{\lambda_2}{\lambda_1 + \lambda_2} v_2 = \frac{\lambda_1}{1 - \lambda_3} v_1 \oplus \frac{\lambda_2}{1 - \lambda_3} v_2.$$

Again in view of Lemma 1, we have

$$\sum_{i=1}^{3} \oplus \lambda_i v_i = (1 - \lambda_3) \left(\frac{\lambda_1}{1 - \lambda_3} v_1 \oplus \frac{\lambda_2}{1 - \lambda_3} v_2 \right) \oplus \lambda_3 v_3.$$

By induction, we can write

$$\sum_{i=1}^{n} \oplus \lambda_i v_i := (1 - \lambda_n) \left(\frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \ldots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1} \right) \oplus \lambda_n v_n.$$

In view of (6), we have the following:

LEMMA 3 ([15]). Let (X, d) be a CAT(0) space and $\{v_1, v_2, \dots, v_n\} \subset X$ and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset (0,1)$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$. Then

$$d\left(\sum_{i=1}^{n} \oplus \lambda_i v_i, x \right) \leq \sum_{i=1}^{n} \lambda_i d(v_i, x) \quad \text{for each } x \in X.$$

In view of Lemma 2, we have the following:

LEMMA 4 ([9]). Let (X, d) be a CAT(0) space and $\{v_1, v_2, \dots, v_n\} \subset X$ and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset (0,1)$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$. Then

$$d\left(\sum_{i=1}^{n} \oplus \lambda_i v_i, x \right)^2 \leq \sum_{i=1}^{n} \lambda_i d(v_i, x)^2 - \sum_{i,j=1, i \neq j}^{n} \lambda_i \lambda_j d(v_i, v_j)^2$$

for each $x \in X$.

The following are some examples of CAT(0) spaces:

1. Any convex subset of a Euclidean space $\mathbb{R}^n$, when endowed with the induced metric is a CAT(0) space.
2. Every pre-Hilbert space is a CAT(0) space.
3. The Hilbert ball with the hyperbolic metric is a CAT(0) space.
4. Simply connected Riemannian manifolds of non-positive sectional curvature are CAT(0) spaces.
5. If X_1 and X_2 are CAT(0) spaces, then $X_1 \times X_2$ is also a CAT(0) space.

Now, we recall some more definitions.

Let K be a closed and convex subset of a CAT(0) space (X, d) and $\{x_n\}$ a bounded sequence in K. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(\{x_n\}) = r(x, \{x_n\})\}.$$

It is well known that in CAT(0) spaces, $A(\{x_n\})$ consists of exactly one point [13].

DEFINITION 2. A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

We write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ where x is called the Δ-limit of $\{x_n\}$. We denote $w_\Delta(x_n) := \cup\{A(\{u_n\})\}$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

DEFINITION 3. A family $\{A, B, C\}$ of mappings is said to satisfy the condition (I) if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0,\ g(r) > 0$ for all $r \in [0, \infty)$ such that $d(x, Ax) \geq g(d(x, \mathcal{F}))$ or $d(x, Bx) \geq g(d(x, \mathcal{F}))$ or $d(x, Cx) \geq g(d(x, \mathcal{F}))$ for all $x \in X$, where $\mathcal{F} = F(A) \cap F(B) \cap F(C)$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is well known that every bounded sequence in X has a Δ-convergent subsequence [26].

LEMMA 5. Let K be a closed and convex subset of a complete CAT(0) space X and $T : K \to K$ satisfies condition (E). Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$. Then $x = Tx$.

LEMMA 6 ([14]). Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. If the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

DEFINITION 4. A function $g : K \to (-\infty, \infty)$ defined on a convex subset K of a CAT(0) space is convex if for any geodesic $\gamma : [a, b] \to K$, the function $go\gamma$ is convex.

One can see the examples in [7].

DEFINITION 5. A function g defined on K is lower semi-continuous at a point $x \in K$ if

$$g(x) \leq \liminf_{k\to\infty} g(x_n)$$

for each sequence $x_n \to x$. A function g is said to be lower semi-continuous on K if it is lower semi-continuous at any point in K.

DEFINITION 6. A mapping $T : K \to K$ is said to be semi-compact if any sequence $\{x_n\}$ in K satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ has a convergent subsequence.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of g in complete CAT(0) spaces as

$$J_\lambda(x) = \operatorname*{argmin}_{y\in X} \left\{ g(y) + \frac{1}{2\lambda} d^2(y, x) \right\} \tag{7}$$

for all $x \in X$. The mapping J_λ is well defined for all $\lambda > 0$ (see [22, 31]).

Let $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. It is shown in [3] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with g coincides with the set $\operatorname*{argmin}_{y\in X} g(y)$ of minimizers of g.

LEMMA 7 ([22]). Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ a convex and lower semi-continuous function. For any $\lambda > 0$, the resolvent J_λ of g is nonexpansive.

LEMMA 8 ([2]). Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ a convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + g(J_\lambda x) \leq g(y).$$

PROPOSITION 1 ([22, 31]). (The resolvent identity) Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ a convex and lower semi-continuous function. Then the following identity holds:

$$J_\lambda x = J_\mu\left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x\right)$$

for all $x \in X$ and $\lambda > \mu > 0$.

3 Main Results

In this section, we approximate the fixed points of generalized nonexpansive mappings in complete CAT(0) spaces and establish some strong and Δ-convergence theorems. In the sequel $\mathcal{F}$ denotes the following:

$$\mathcal{F} = \cap_{i\in\mathbb{N}}(F(T_i)\cap F(S_i))\cap \underset{y\in X}{\operatorname{argmin}}\ g(y),$$

where $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of generalized nonexpansive self-mappings on a CAT(0) space X and $g : X \to (-\infty, \infty)$ is a convex and lower semi-continuous function.

THEOREM 1. Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of mappings on X satisfying condition (C_λ) and condition (E) such that $\mathcal{F} \neq \emptyset$. Assume that $\{\alpha_{i,n}\}_{i=0}^m$ and $\{\beta_{i,n}\}_{i=0}^m$ are the sequences such that $0 < a \leq \alpha_{i,n},\ \beta_{i,n} \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum\limits_{i=0}^{m} \alpha_{i,n} = \sum\limits_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be a sequence as generated by (4). Then $\{x_n\}$ Δ-converges to an element of $\mathcal{F}$.

PROOF. We will prove the following:

1. $\lim\limits_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$;
2. $\lim\limits_{n\to\infty} d(x_n, z_n) = 0$;
3. $\lim\limits_{n\to\infty} d(x_n, T_i x_n) = 0;\ \lim\limits_{n\to\infty} d(x_n, S_i x_n) = 0$ for $i \in \mathbb{N}$.

Since T_i and S_i satisfy condition (C_λ), we have

$$\lambda d(p, S_i p) = 0 \leq d(z_n, p) \Rightarrow d(S_i z_n, p) \leq d(z_n, p) \tag{8}$$

and

$$\lambda d(p, T_i p) = 0 \leq d(y_n, p) \Rightarrow d(T_i y_n, p) \leq d(y_n, p). \tag{9}$$

Let $p \in \mathcal{F}$. Then $p = T_i p = S_i p$ and $g(p) \leq g(y)\ \ \forall\ y \in X$. Therefore

$$g(p) + \frac{1}{2\lambda_n} d^2(q, q) \leq g(y) + \frac{1}{2\lambda_n} d^2(y, q)\ \ \forall\ y \in X,$$

and hence $p = J_{\lambda_n} p\ \ \forall\ n \in \mathbb{N}$. Now, we show that $\lim\limits_{n\to\infty} d(x_n, p)$ exists. Since $z_n = J_{\lambda_n} x_n\ \ \forall\ n \in \mathbb{N}$, by Lemma 7, we have

$$d(z_n, p) = d(J_{\lambda_n} x_n, J_{\lambda_n} p) \leq d(x_n, p). \tag{10}$$

Now, by Lemma 3, equation (4) and inequalities (8) & (10), we have

$$\begin{aligned} d(y_n,p) &= d\Big(\beta_{0,n}x_n \oplus (1-\beta_{0,n})\sum_{i=1}^{m}\frac{\beta_{i,n}}{(1-\beta_{0,n})}S_i z_n, p\Big) \\ &\le \beta_{0,n}d(x_n,p) + \sum_{i=1}^{m}\beta_{i,n}d(S_i z_n,p) \\ &\le \beta_{0,n}d(x_n,p) + \sum_{i=1}^{m}\beta_{i,n}d(z_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{11}$$

Again by using Lemma 3, equation (4) and inequalities (9) & (11), we have

$$\begin{aligned} d(x_{n+1},p) &= d\Big(\alpha_{0,n}x_n \oplus (1-\alpha_{0,n})\sum_{i=1}^{m}\frac{\alpha_{i,n}}{(1-\alpha_{0,n})}T_i y_n, p\Big) \\ &\le \alpha_{0,n}d(x_n,p) + \sum_{i=1}^{m}\alpha_{i,n}d(T_i y_n,p) \\ &\le \alpha_{0,n}d(x_n,p) + \sum_{i=1}^{m}\alpha_{i,n}d(y_n,p) \\ &\le d(x_n,p), \end{aligned} \tag{12}$$

which shows that $\lim_{n\to\infty} d(x_n,p)$ exists. Thus $\lim_{n\to\infty} d(x_n,p) = l$ for some l.

Next, we show that $\lim_{n\to\infty} d(x_n,z_n) = 0$. Now, by Lemma 8, we have

$$\frac{1}{2\lambda_n}d^2(z_n,p) - \frac{1}{2\lambda_n}d^2(x_n,p) + \frac{1}{2\lambda_n}d^2(x_n,z_n) \le g(p) - g(z_n).$$

As $g(p) \le g(z_n)$ for all $n \in \mathbb{N}$, we have

$$d^2(x_n,z_n) \le d^2(x_n,p) - d^2(z_n,p). \tag{13}$$

Now, for showing $\lim_{n\to\infty} d(x_n,z_n) = 0$, it suffices to show that $\lim_{n\to\infty} d(z_n,p) = l$. From (12), we have

$$d(x_{n+1},p) \le \alpha_{0,n}d(x_n,p) + \sum_{i=1}^{m}\alpha_{i,n}d(y_n,p)$$

which is equivalent to

$$\sum_{i=1}^{m}\alpha_{i,n}d(x_n,p) \le d(x_n,p) - d(x_{n+1},p) + \sum_{i=1}^{m}\alpha_{i,n}d(y_n,p)$$

$$\begin{aligned} d(x_n,p) &\le \frac{1}{\sum_{i=1}^{m}\alpha_{i,n}}\big[d(x_n,p) - d(x_{n+1},p)\big] + d(y_n,p) \\ &\le \frac{1}{a}\big[d(x_n,p) - d(x_{n+1},p)\big] + d(y_n,p). \end{aligned}$$

As $d(x_{n+1},p) \le d(x_n,p)$ and $0 < a \le \alpha_n \le \sum\limits_{i=1}^{m} \alpha_{i,n}$ for all $n \in \mathbb{N}$, we have

$$l = \liminf_{n\to\infty} d(x_n,p) \le \liminf_{n\to\infty} d(y_n,p).$$

On the other hand, from (11) we have

$$\limsup_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(x_n,p) = l.$$

Thus, we have $\lim\limits_{n\to\infty} d(y_n,p) = l$. Also, from (11) we have

$$d(y_n,p) \le \beta_{0,n} d(x_n,p) + \sum_{i=1}^{m} \beta_{i,n} d(z_n,p)$$

which is equivalent to

$$\begin{aligned} d(x_n,p) &\le \frac{1}{\sum\limits_{i=1}^{m} \beta_{i,n}} \big[d(x_n,p) - d(y_n,p)\big] + d(z_n,p) \\ &\le \frac{1}{a}\big[d(x_n,p) - d(y_n,p)\big] + d(z_n,p), \end{aligned}$$

which yields that

$$l = \liminf_{n\to\infty} d(x_n,p) \le \liminf_{n\to\infty} d(z_n,p).$$

Therefore, from (10) we have

$$\lim_{n\to\infty} d(z_n,p) = l,$$

which form (13) yields that

$$\lim_{n\to\infty} d(x_n,z_n) = 0. \tag{14}$$

Now, we show that $\lim\limits_{n\to\infty} d(x_n,T_i x_n) = 0$ and $\lim\limits_{n\to\infty} d(x_n,S_i x_n) = 0$. From Lemma 4, equation (4) and inequalities (8) and (10), we have

$$\begin{aligned} d^2(y_n,p) &= d^2\left(\beta_{0,n} x_n \oplus (1-\beta_{0,n}) \sum_{i=1}^{m} \frac{\beta_{i,n}}{(1-\beta_{0,n})} S_i z_n, p\right) \\ &\le \beta_{0,n} d^2(x_n,p) + \sum_{i=1}^{m} \beta_{i,n} d^2(S_i z_n,p) - \beta_{0,n} \sum_{i=1}^{m} \beta_{i,n} d^2(x_n,S_i z_n) \\ &\le \beta_{0,n} d^2(x_n,p) + \sum_{i=1}^{m} \beta_{i,n} d^2(z_n,p) - \beta_{0,n}(1-\beta_{0,n}) d^2(x_n,S_i z_n) \\ &\le \beta_{0,n} d^2(x_n,p) + \sum_{i=1}^{m} \beta_{i,n} d^2(x_n,p) - a(1-b) d^2(x_n,S_i z_n) \\ &= d^2(x_n,p) - a(1-b) d^2(x_n,S_i z_n), \end{aligned}$$

which implies that

$$d^2(x_n, S_i z_n) \leq \frac{1}{a(1-b)}[d^2(x_n, p) - d^2(y_n, p)] \to 0 \ \text{ as } n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty} d(x_n, S_i z_n) = 0. \tag{15}$$

Since S_i satisfy condition (E), therefore from (14) and (15), we have

$$\begin{aligned} d(x_n, S_i x_n) &\leq d(x_n, z_n) + d(z_n, S_i x_n) \\ &\leq d(z_n, x_n) + \mu d(z_n, S_i z_n) + d(z_n, x_n) \\ &\leq 2d(z_n, x_n) + \mu[d(z_n, x_n) + d(x_n, S_i z_n)] \\ &= (2+\mu) d(z_n, x_n) + d(x_n, S_i z_n) \end{aligned}$$

which tends to zero as $n \to \infty$, that is

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0. \tag{16}$$

Further, since S_i and T_i satisfy condition (C_λ), therefore by using Lemma 4, equation (4) and inequality (11), we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2\left(\alpha_{0,n} x_n \oplus (1-\alpha_{0,n}) \sum_{i=1}^{m} \frac{\alpha_{i,n}}{(1-\alpha_{0,n})} T_i y_n, p\right) \\ &\leq \alpha_{0,n} d^2(x_n, p) + \sum_{i=1}^{m} \alpha_{i,n} d^2(T_i y_n, p) - \alpha_{0,n} \sum_{i=1}^{m} \alpha_{i,n} d^2(x_n, T_i y_n) \\ &\leq \alpha_{0,n} d^2(x_n, p) + \sum_{i=1}^{m} \alpha_{i,n} d^2(y_n, p) - \alpha_{0,n}(1-\alpha_{0,n}) d^2(x_n, T_i y_n) \\ &\leq \alpha_{0,n} d^2(x_n, p) + \sum_{i=1}^{m} \alpha_{i,n} d^2(x_n, p) - a(1-b) d^2(x_n, T_i y_n) \end{aligned}$$

which implies that

$$d^2(x_n, T_i y_n) \leq \frac{1}{a(1-b)}[d^2(x_n, p) - d^2(x_{n+1}, p)]$$

which tends to zero as $n \to \infty$, that is

$$\lim_{n\to\infty} d(x_n, T_i y_n) = 0. \tag{17}$$

Now, by using (15), we have

$$d(y_n, x_n) = \sum_{i=1}^{m} \beta_{i,n} d(x_n, S_i z_n)$$

which tends to zero as $n \to \infty$, that is

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{18}$$

Since S_i and T_i satisfy condition (E), therefore from (16), (17) and (18), we have

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, y_n) + d(y_n, T_i x_n) \\ &\leq d(x_n, y_n) + \mu d(y_n, T_i y_n) + d(y_n, x_n) \\ &\leq 2d(x_n, y_n) + \mu[d(y_n, x_n) + d(x_n, T_i y_n)] \\ &\leq (2+\mu) d(x_n, y_n) + \mu d(x_n, T_i y_n) \end{aligned}$$

which tends to zero as $n \to \infty$, that is

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0. \tag{19}$$

Now, as $\lambda_n \geq \lambda > 0$, by Proposition 1 and (14), we have

$$\begin{aligned} d(J_\lambda x_n, J_{\lambda_n} x_n) &= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right)\right) \\ &\leq d\left(x_n, \left(1 - \frac{\lambda}{\lambda_n}\right) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right) \\ &= \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, z_n) \end{aligned}$$

which tends to zero as $n \to \infty$, that is

$$\lim_{n\to\infty} d(J_\lambda x_n, J_{\lambda_n} x_n) = 0$$

or,

$$\lim_{n\to\infty} d(z_n, J_{\lambda_n} x_n) = 0.$$

Thus, we have

$$d(x_n, J_\lambda x_n) \leq d(x_n, z_n) + d(z_n, J_\lambda x_n)$$

which tends to zero as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0. \tag{20}$$

From above, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$ and also (16) and (19) imply that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, and $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$.

Next, we prove that $w_\Delta(x_n) \subset \mathcal{F}$. Let $s \in w_\Delta(x_n)$. Then there exists a subsequence $\{s_n\}$ of $\{x_n\}$ such that $A(\{s_n\}) = \{s\}$. Now, from Lemma 5, there exists a subsequence $\{t_n\}$ of $\{s_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} t_n = t$ for some $t \in \mathcal{F}$. Hence by Lemma 6, we get that $s = t$. This shows that $w_\Delta(x_n) \subset \mathcal{F}$.

Now, we show that $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$. For this, it suffices to show that $w_\Delta(x_n)$ consists of exactly one point. Let $\{s_n\}$ be a subsequence of $\{x_n\}$ with

$A(\{s_n\}) = \{s\}$ and let $A(\{x_n\}) = \{x\}$. Since $s \in w_\Delta(x_n) \subset \mathcal{F}$ and $\{d(x_n, s)\}$ converges, by Lemma 6, we have $x = s$. Hence $w_\Delta(x_n) = \{x\}$. This completes the proof.

Putting $S_i = S$ and $T_i = T$ in Theorem 1, we obtain the following Corollary:

COROLLARY 1. Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let T and S be two self-mappings on X satisfying condition (C_λ) and condition (E) such that $\mathcal{F}' = F(T) \cap F(S) \cap \operatorname*{argmin}_{y \in X} g(y)$ is nonempty. Assume that $\{\alpha_{i,n}\}_{i=0}^m$ and $\{\beta_{i,n}\}_{i=0}^m$ are the sequences such that $0 < a \le \alpha_{i,n},\ \beta_{i,n} \le b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum_{i=0}^{m} \alpha_{i,n} = \sum_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be a sequence generated as $x_1 \in X$, and

$$\begin{cases} z_n = \operatorname*{argmin}_{y \in X} \left[g(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \\ y_n = \beta_{0,n} x_n \oplus (1 - \beta_{0,n}) \sum_{i=1}^m \frac{\beta_{i,n}}{(1-\beta_{0,n})} S z_n, \\ x_{n+1} = \alpha_{0,n} x_n \oplus (1 - \alpha_{0,n}) \sum_{i=1}^m \frac{\alpha_{i,n}}{(1-\alpha_{0,n})} T y_n, \end{cases}$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to an element of $\mathcal{F}'$.

COROLLARY 2. In addition to the hypotheses of Corollary 1, suppose that $S = T$. Then $\{x_n\}$ Δ-converges to an element of $\mathcal{F}'$.

Since every Hilbert space is a complete CAT(0) space, we obtain the following Corollary directly:

COROLLARY 3. Let H be a Hilbert space and $g : H \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of mappings on H satisfying condition (C_λ) and condition (E) such that $\mathcal{F} \neq \emptyset$. Assume that $\{\alpha_{i,n}\}_{i=0}^m$ and $\{\beta_{i,n}\}_{i=0}^m$ are the sequences such that $0 < a \le \alpha_{i,n},\ \beta_{i,n} \le b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum_{i=0}^{m} \alpha_{i,n} = \sum_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be a sequence generated as $x_1 \in H$, and

$$\begin{cases} z_n = \operatorname*{argmin}_{y \in H} \left[g(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right], \\ y_n = \beta_{0,n} x_n \oplus (1 - \beta_{0,n}) \sum_{i=1}^m \frac{\beta_{i,n}}{(1-\beta_{0,n})} S_i z_n, \\ x_{n+1} = \alpha_{0,n} x_n \oplus (1 - \alpha_{0,n}) \sum_{i=1}^m \frac{\alpha_{i,n}}{(1-\alpha_{0,n})} T_i y_n, \end{cases}$$

for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ weakly converges to an element of $\mathcal{F}$.

Now, we prove strong convergence theorems:

THEOREM 2. Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of mappings on X satisfying condition (C_λ) and condition (E) such that $\mathcal{F} \neq \emptyset$. Assume that $\{\alpha_{i,n}\}_{i=0}^{m}$ and $\{\beta_{i,n}\}_{i=0}^{m}$ are the sequences such that $0 < a \leq \alpha_{i,n},\ \beta_{i,n} \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum_{i=0}^{m} \alpha_{i,n} = \sum_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be a sequence as generated by (4). Then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F}) = \inf\{d(x,p)\ :\ p \in \mathcal{F}\}$.

PROOF. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. From (12), we have

$$d(x_{n+1}, p) \leq d(x_n, p) \ \ \forall\ p \in \mathcal{F},$$

which follows that

$$d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F}).$$

Therefore $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists and $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Proceeding on the same lines of the proof of Theorem 2 of [24], this can be shown that $\{x_n\}$ is a Cauchy sequence in X. This implies that $\{x_n\}$ converges to a point x^* in X and hence $d(x^*, \mathcal{F}) = 0$. As $\mathcal{F}$ is closed, we get that $x^* \in \mathcal{F}$. This completes the proof.

THEOREM 3. Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of mappings on X satisfying condition (C_λ) and condition (E) such that $\mathcal{F} \neq \emptyset$. Assume that $\{\alpha_{i,n}\}_{i=0}^{m}$ and $\{\beta_{i,n}\}_{i=0}^{m}$ are the sequences such that $0 < a \leq \alpha_{i,n},\ \beta_{i,n} \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum_{i=0}^{m} \alpha_{i,n} = \sum_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. If T_i, S_i and J_λ satisfy condition (I), then the sequence $\{x_n\}$ generated by (4) converges strongly to an element of $\mathcal{F}$.

PROOF. From Theorem 1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$. This implies that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Now, from inequalities (16), (19) and (20), we have

$$\lim_{n\to\infty} g(d(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, S_i x_n) = 0,$$

or,

$$\lim_{n\to\infty} g(d(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, T_i x_n) = 0,$$

or,

$$\lim_{n\to\infty} g(d(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} g(d(x_n, \mathcal{F})) = 0.$$

Now, by using the property of g, it follows that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Proceeding on the lines of the proof of Theorem 2, we get the result. This completes the proof.

THEOREM 4. Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty)$ be a convex and lower semi-continuous function. Let $\{T_i : i \in \mathbb{N}\}$ and $\{S_i : i \in \mathbb{N}\}$ be two finite families of mappings on X satisfying condition (C_λ) and condition (E) such that $\mathcal{F} \neq \emptyset$. Assume that $\{\alpha_{i,n}\}_{i=0}^m$ and $\{\beta_{i,n}\}_{i=0}^m$ are the sequences such that $0 < a \leq \alpha_{i,n},\ \beta_{i,n} \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b satisfying $\sum_{i=0}^{m} \alpha_{i,n} = \sum_{i=0}^{m} \beta_{i,n} = 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. If T_i or S_i or J_λ is semi-compact, then the sequence $\{x_n\}$ generated by (4) converges strongly to an element of $\mathcal{F}$.

PROOF. Suppose that T_i is semi-compact. Then from Theorem 3, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0.$$

Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in X$. As

$$\lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0,$$

$\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ and also $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$ implies that $d(x^*, J_\lambda x^*) = 0$, $d(x^*, T_i x^*) = 0$ and also $d(x^*, S_i x^*) = 0$. This shows that $x^* \in \mathcal{F}$. This completes the proof.

EXAMPLE 1. The essentials of hypotheses in our results are natural in view of following observations: let $X = [0, 3]$. Define mappings $T_i,\ S_i : X \to X$ by

$$T_i(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ \frac{i}{i+1}, & \text{if } x = 3, \end{cases}$$

and

$$S_i(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ \frac{i+2}{i+3}, & \text{if } x = 3, \end{cases}$$

(i) Then S_i and T_i satisfy condition C and hence satisfy condition (E), from Lemma 7 in [40].

(ii) $\cap_{i\in\mathbb{N}}(F(T_i) \cap F(S_i)) = \{0\}$.

REMARK 2. In this paper, we present a new three step proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of generalized nonexpansive mappings in CAT(0) spaces. Therefore we give the following observations:

1. Our main results generalize Theorem 1, Theorem 2 and Theorem 3 of Khan and Abbas [24] and Theorem 1, Theorem 2 and Theorem 3 of Cholamjiak *et al.* [10] from one and two nonexpansive mappings respectively to infinite families of generalized nonexpansive mappings which involve convex and lower semi-continuous function in CAT(0) spaces.

2. Our main result (Theorem 1) extends that of Bačák [4] in CAT(0) spaces.

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