# Quaternionic Approach On Constant Angle Surfaces In $\mathbb{S}^2 \times \mathbb{R}$ *

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**Abstract**

A helix surface or constant angle surface is a surface whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. In this paper we study surfaces in $\mathbb{S}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the $\mathbb{R}$-direction. The main idea is to show that a constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$ can be obtained by a quaternion product and a matrix representation. Also we give some related examples with figures of projections of obtained surfaces

## 1 Introduction

An interesting problem of differential geometry of submanifolds, intensively studied in last years, consists in classification and characterization of surfaces whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. These surfaces are called helix surfaces or constant angle surfaces and they have been studied in all the 3-dimensional geometries. In recent years much work has been done to understand the geometry of the helix surfaces and they have been classsified in all the 3-dimensional Riemannian geometries (see [1, 3, 4, 15, 17] etc.). This kind of surfaces are strictly related to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [17]. The early results were obtained by studying surfaces isometrically immersed in product spaces of type $\mathbb{M}^2 \times \mathbb{R}$, namely taken $\mathbb{M}^2$ to be the unit 2-sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$ respectively in [3, 4]. The angle was considered between the unit normal of the surface $M$ and the tangent direction to $\mathbb{R}$. An interesting classification of surfaces in the 3-dimensional Heisenberg group making a constant angle with the fibers of the Hopf-fibration was obtained in [10]. Moreover, Munteanu and Nistor obtained a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to $\mathbb{R}$ [15]. In [1], it is also classified certain special ruled surfaces in $\mathbb{R}^3$ under the general theorem of characterization of constant angle surfaces. A classification is given of special developable surfaces and some conical surfaces from the point of view the constant angle property in [22]. Also some characterization are

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given for a curve lying on a surface for which the unit normal field along the curve makes a constant angle with a fixed direction [22]. These curves are called isophote curves in literature.

On the other hand, several authors have studied constant angle surfaces in Minkowski 3-space. Lopez and Munteanu investigated spacelike surfaces with the constant timelike direction [18]. By choosing the constant direction as a spacelike vector, Atalay et al. obtained different parametrization for the spacelike constant angle surfaces [6]. Also, the classifications are given for the timelike surfaces whose normal vector field makes a constant angle with a constant direction by Guler et al in [5]. In another recent paper [9] it is defined constant angle spacelike and timelike surfaces in the three-dimensional Heisenberg group and equipped with a 1-parameter family of Lorentzian metrics.

A rotation in $\mathbb{R}^3$ about an axis through the origin can be represented by a $3 \times 3$ orthogonal matrix with determinant 1. However, the matrix representation seems redundant because only four of its nine elements are independent. Also the geometric interpretation of such a matrix is not clear until we carry out several steps of calculation to extract the rotation axis and angle. Furthermore, to compose two rotations, we need to compute the product of the two corresponding matrices, which requires twenty-seven multiplications and eighteen additions.

Quaternions are very efficient for analyzing situations where rotations in $\mathbb{R}^3$ are involved. A quaternion is a 4-tuple, which is a more concise representation than a rotation matrix. Its geometric meaning is also more obvious as the rotation axis and angle can be trivially recovered. The quaternion algebra to be introduced will also allow us to easily compose rotations. This is because quaternion composition takes merely sixteen multiplications and twelve additions [24]. So, quaternionic approach is a very important method for obtaining surfaces. For example in recent years several authors used this method for obtaining canal surfaces and constant slope surfaces [2, 8, 12, 13, 14, 16, 19, 20, 21, 25].

Quaternions are members of a noncommutative division algebra first invented by William Rowan Hamilton. The idea for quaternions occurred to him while he was walking along the Royal Canal on his way to a meeting of the Irish Academy, and Hamilton was so pleased with his discovery that he scratched the fundamental formula of quaternion algebra,

$$i^2 = j^2 = k^2 = ijk = -1$$

into the stone of the bridge.

While the quaternions are not commutative, they are associative, and they form a group known as the quaternion group. Many physical laws in classical, relativistic, and quantum mechanics can be written nicely using them.

The main idea in this paper is to show that constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ can be obtained by quaternion product and the matrix representations with similar methods of the paper [12]. Finally, some examples of these surfaces are given with their projections of figures by using the Mathematica Programme.

## 2    Preliminary

In this section we introduce the notion of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and give some first characterizations. Let $\mathbb{S}^2 \times \mathbb{R}$ be the Riemannian product of the 2-sphere $\mathbb{S}^2(1)$ and $\mathbb{R}$ with the standard metric $\langle , \rangle$ and Levi-Civita connection $\widetilde{\nabla}$. We denote by $\frac{\partial}{\partial t}$ a unit vector field in the tangent bundle $T\left(\mathbb{S}^2 \times \mathbb{R}\right)$ that is tangent to the $\mathbb{R}-$direction.

Now consider a surface $M$ in $\mathbb{S}^2 \times \mathbb{R}$. Let us denote by $\xi$ a unit normal to $M$. Then we can decompose $\frac{\partial}{\partial t}$ as

$$\frac{\partial}{\partial t} = T + \cos\theta \xi$$

where $T$ is the projection of $\frac{\partial}{\partial t}$ on the tangent space of $M$ and is the angle function defined by

$$\cos\theta(p) = \left\langle \frac{\partial}{\partial t}, \xi \right\rangle$$

for every point $p \in M$.

By a constant angle surface $M$ in $\mathbb{S}^2 \times \mathbb{R}$, we mean a surface for which the angle function $\theta$ is constant on $M$. There are two trivial cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The condition $\theta = 0$ means that $\frac{\partial}{\partial t}$ is always normal, so we get a $\mathbb{S}^2 \times \{t_0\}$. In the second case $\frac{\partial}{\partial t}$ is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{S}^2$ and $\mathbb{R}$.

The characterization of constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$ was given in [3], where look at $\mathbb{S}^2 \times \mathbb{R}$ as a hypersurface in $E^4$ and denote by $\frac{\partial}{\partial t} = (0,0,0,1)$. The main result is the following:

THEOREM 1. A surface $M$ in $\mathbb{S}^2 \times \mathbb{R}$ is a constant angle surface if and only if the immersion $F$ is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) locally given by

$$F : M \longrightarrow \mathbb{S}^2 \times \mathbb{R} : (u,v) \to F(u,v),$$

where

$$F(u,v) = \left(\cos\left(\xi\right) f(v) + \sin\left(\xi\right) f(v)\Lambda f'(v), u\sin\theta\right),$$

$f : I \to \mathbb{S}^2$ is a unit speed curve in $\mathbb{S}^2$ and $\theta \in \left[0, \frac{\pi}{2}\right]$ is the constant angle, $\xi = \xi(u) = u\cos\theta$, and $\Lambda$ denotes the cross product in $\mathbb{R}^3$ [3].

Now let us give some basic concepts about the real quaternions. Let

$$Q = \{q = d + ai + bj + ck : a, b, c, d \in R\}$$

denotes the set of all real quaternions. A real quaternion is defined by $q = d + ai + bj + ck$ where $a, b, c, d$ are real numbers and $i$, $j$, $k$ are orthogonal unit spatial vectors in three dimensional space such that $i^2 = j^2 = k^2 = ijk = -1$. Moreover, $i$, $j$ and $k$ are orthogonal unit spatial vectors of $\mathbb{R}^3$ and the quaternion product $\times$ of spatial vectors is the same as the cross product of the vectors which satisfy following multiplication rules: $i \times j = k$, $j \times k = i$, $k \times i = j$. Although quaternion algebra is associative,

it is not commutative. For this reason, extra care has to be taken when performing arbitrary multiplications in $Q$. Standard orthonormal basis of quaternions is $\{1, i, j, k\}$ and identity element of $Q$ is 1. A quaternion can be written as $q = d + ai + bj + ck$ or $q = Sq + Vq$, where $Sq = d \in \mathbb{R}$ is the scalar component of $q$ and $Vq = ai + bj + ck$ is the vectorial component of $q$. We also write following four-tuple notation to represent a quaternion:

$$q = (a, b, c, d) = (w, d),$$

where $Sq = d \in \mathbb{R}$ and $Vq = w \in \mathbb{R}^3$. If $Sq = 0$, the quaternion is called pure quaternion. Addition of two quaternions, multiplication of a quaternion with a scalar $\lambda \in \mathbb{R}$ and conjugate of a quaternion, $\bar{q}$, can be given in $Q$ as follows:

$$q + p = (Sq + Sp) + (Vq + Vp),$$

$$\lambda q = \lambda Sq + \lambda Vq,$$

$$\bar{q} = Sq - Vq.$$

By using dot and cross-product we can give the quaternion product of two quaternions $p$ and $q$ as:

$$q \times p = SqSp - \langle Vq, Vp \rangle + SqVp + SpVq + Vq \wedge Vp,$$

where $\times$ is the quaternion product. Given $q = d + ai + bj + ck \in Q$, the norm of $q$ is denoted by $Nq = d^2 + a^2 + b^2 + c^2$ and we can note the following relationship between $q$ and its conjugate $\bar{q}$

$$Nq = q \times \bar{q} = \bar{q} \times q.$$

If $Nq = 1$, it is called a unit quaternion. So if $q \neq 0$, then we get the equation $q \times \frac{\bar{q}}{Nq} = \frac{\bar{q}}{Nq} \times q = 1$, which gives us that the inverse of $q$ can be given as

$$q^{-1} = \frac{\bar{q}}{Nq}$$

Moreover, a unit quaternion can be written as $q = \cos\theta + \sin\theta \mathcal{S}$, where $\mathcal{S} \in \mathbb{R}^3$ and $\|\mathcal{S}\| = 1$, [7, 11, 23].

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping and $\Phi = q \times v \times q^{-1}$, where $q$ is a unit quaternion and $v$ is a pure quaternion (that is, a vector in $\mathbb{R}^3$). So, for every unit quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, we can give matrix representation $M_q$ of $\Phi$ by using pure quaternion basis elements of $Q$ as:

$$M_q = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 - a_1^2 + a_2^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix}$$

where $M_q$ is orthogonal because $M_q M_q^T = I_3$ and $\det M_q = 1$. Thus, we can say that the linear mapping $\Phi$ is a rotation in 3-dimensional space.

# 3   A New Approach On Constant Angle Surface in $\mathbb{S}^2 \times \mathbb{R}$ with Quaternions

In this section we consider $Q(u,v) = \cos \xi(u) - \sin \xi(u) f'(v)$ defines a 2-dimensional surface in $\mathbb{S}^3 \subset \mathbb{R}^4$, where $f'(v) = (f'_1(v), f'_2(v), f'_3(v))$ and $\|f'(v)\| = 1$. As we gave earlier, for the unit quaternion $Q(u,v)$, the matrix represantation of the map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$
M_Q = \left[ \begin{array}{ccc} \cos^2 \xi + (f'^2_1 - f'^2_2 - f'^2_3) \sin^2 \xi & 2\,(f'_3 \cos \xi + f'_1 f'_2 \sin \xi) \sin \xi & 2\,(-f'_2 \cos \xi + f'_1 f'_3 \sin \xi) \sin \xi \\ 2\,(-f'_3 \cos \xi + f'_1 f'_2 \sin \xi) \sin \xi & \cos^2 \xi + (-f'^2_1 + f'^2_2 - f'^2_3) \sin^2 \xi & 2\,(f'_1 \cos \xi + f'_2 f'_3 \sin \xi) \sin \xi \\ 2\,(f'_2 \cos \xi + f'_1 f'_3 \sin \xi) \sin \xi & 2\,(-f'_1 \cos \xi + f'_2 f'_3 \sin \xi) \sin \xi & \cos^2 \xi + (-f'^2_1 - f'^2_2 + f'^2_3) \sin^2 \xi \end{array} \right].
$$

We are now ready to show main result of this paper:

THEOREM 2.   Let $F : M \longrightarrow \mathbb{S}^2 \times \mathbb{R} : (u,v) \to F(u,v)$ be an immersion (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$). Then the constant angle surface $M$ can be reparametrized by

$$F(u,v) = u \sin \theta + Q(u,v) \times Q_1(u,v), \qquad (1)$$

where "$\times$" is the quaternion product, $Q_1(u,v) = f(v)$ is a unit speed curve in $\mathbb{S}^2$ and a pure quaternion and $\theta \in \left[0, \frac{\pi}{2}\right]$ is the constant angle.

PROOF.   Since $Q(u,v) = \cos \xi(u) - \sin \xi(u) f'(v)$ and $Q_1(u,v) = f(v)$, we obtain

$$
\begin{aligned}
Q(u,v) \times Q_1(u,v) &= (\cos \xi(u) - \sin \xi(u) f'(v)) \times f(v) \\
&= \cos \xi(u) f(v) - \sin \xi(u) f'(v) \times f(v).
\end{aligned}
$$

By using the quaternion product, we get

$$f'(v) \times f(v) = -\langle f'(v), f(v) \rangle + f'(v) \Lambda f(v).$$

We know that $\langle f'(v), f(v) \rangle = 0$ since $f$ is a unit speed curve in $\mathbb{S}^2$. Thus

$$f'(v) \times f(v) = f'(v) \Lambda f(v) = -f(v) \Lambda f'(v).$$

So we find that $Q(u,v) \times Q_1(u,v)$ is given by

$$Q(u,v) \times Q_1(u,v) = \cos \xi(u) f(v) + \sin \xi(u) f(v) \Lambda f'(v).$$

Then the immersion $F : M \to \mathbb{S}^2 \times \mathbb{R}$ is given by

$$
\begin{aligned}
F(u,v) &= u \sin \theta + \cos (u \cos \theta) f(v) + \sin (u \cos \theta) f(v) \Lambda f'(v) \\
&= (\cos (u \cos \theta) f(v) + \sin (u \cos \theta) f(v) \Lambda f'(v), u \sin \theta),
\end{aligned}
$$

which conclude the proof.

REMARK 1.   Theorem 2 says that a unit speed curve $f(v)$ in $\mathbb{S}^2$ is rotated by $Q(u,v)$ through the angle $\xi(u)$ about the axis $Sp\{f'(v)\}$ to obtain a constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$.

As a consequence of this theorem, we get the following corollary.

COROLLARY 1. Let $M_Q$ be the matrix representation of the map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ for the unitary quaternion $Q(u,v)$. Then, for the pure quaternion $Q_1(u,v)$, we get the constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$ as

$$F(u,v) = u\sin\theta + M_Q Q_1(u,v).$$

Remark also that the two trivial cases are included in the parametrization (1).

(i) If $\theta = 0$, then $\xi(u) = u$, $Q(u,v) = \cos u - \sin u f'(v)$, (1) becomes

$$F(u,v) = Q(u,v) \times Q_1(u,v)$$

which gives us $\mathbb{S}^2 \times \{0\}$.

(ii) If $\theta = \frac{\pi}{2}$, then $\xi(u) = 0$, $Q(u,v) = 1$, (1) becomes

$$F(u,v) = u + Q_1(u,v)$$

This clearly gives the Riemannian product of a curve in $\mathbb{S}^2$ and $\mathbb{R}$.

We now want to give some examples about the constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

EXAMPLE 1. Let $f(v) = \left( \frac{\sqrt{2}}{2} \sin v, \cos v, \frac{\sqrt{2}}{2} \sin v \right)$ is a unit speed curve in $\mathbb{S}^2$ and $\theta = 0$. Then the constant angle surface $M$ can be parametrized by

$$
\begin{aligned}
F(u,v) &= Q(u,v) \times Q_1(u,v) \\
&= \cos u \left( \frac{\sqrt{2}}{2} \sin v, \cos v, \frac{\sqrt{2}}{2} \sin v \right) + \sin u \left( \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2} \right) \\
&= \left( \frac{\sqrt{2}}{2} (\cos u \sin v + \sin u), \cos u \cos v, \frac{\sqrt{2}}{2} (\cos u \sin v - \sin u), 0 \right),
\end{aligned}
$$

see Figure 1.

EXAMPLE 2. Let $f(v) = \left( \frac{\sqrt{3}}{2} \cos v, \sin v, \frac{1}{2} \cos v \right)$ is a unit speed curve in $\mathbb{S}^2$ and $\theta = \frac{\pi}{2}$. Then the constant angle surface $M$ can be parametrized by

$$
\begin{aligned}
F(u,v) &= u + Q_1(u,v) \\
&= \left( \frac{\sqrt{3}}{2} \cos v, \sin v, \frac{1}{2} \cos v, u \right),
\end{aligned}
$$

see Figure 2.

EXAMPLE 3. Let $Q_1(u,v) = f(v) = (\cos v, \sin v, 0)$ is a great circle in $\mathbb{S}^2$, $\xi(u) = u\cos\theta$ and $Q(u,v) = \cos\xi(u) - \sin\xi(u)(-\sin v, \cos v, 0)$. Then the constant angle surface $M$ can be parametrized by

$$
\begin{aligned}
F(u,v) &= u\sin\theta + Q(u,v) \times Q_1(u,v) \\
&= u\sin\theta + \cos\xi(u)f(v) + \sin\xi(u)f(v) \times f'(v) \\
&= u\sin\theta + \cos\xi(u)(\cos v, \sin v, 0) + \sin\xi(u)(0,0,1) \\
&= u\sin\theta + (\cos\xi(u)\cos v, \cos\xi(u)\sin v, \sin\xi(u)).
\end{aligned}
$$

Up to parametrization we get

$$ F(u,v) = (\cos u\cos v, \cos u\sin v, \sin u, u\tan\theta), $$

where $\theta \in \left(0, \frac{\pi}{2}\right)$, see Figure 3.

# 4   Visualization

Geometric modeling of the $3D$-surfaces is very important step at the surface modeling systems. We visualize the surfaces with the parametrization

$$ F(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v)) $$

in $\mathbb{R}^4$ by Mathematica Programme. We plot the graph of the surface with plotting command

ParametricPlot3D[{x(u,v),y(u,v),z(u,v)+w(u,v)},{u,a,b},{v,c,d}]

We construct the geometric model of the constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ defined in Example 1 (See Figure 1), Example 2 (See Figure 2) and Example 3 (See Figure 3).
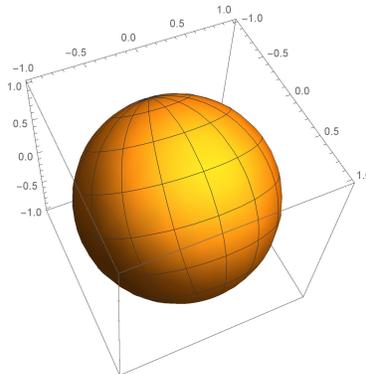


Figure 1: The projections of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$, obtained for $u \in [-\pi, \pi], v \in [-\pi, \pi]$.
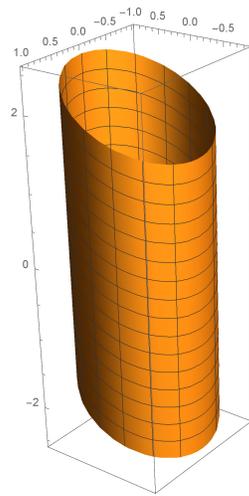
Figure 2: The projections of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$, obtained for $u \in [-\pi, \pi]$, $v \in [-\pi, \pi]$.
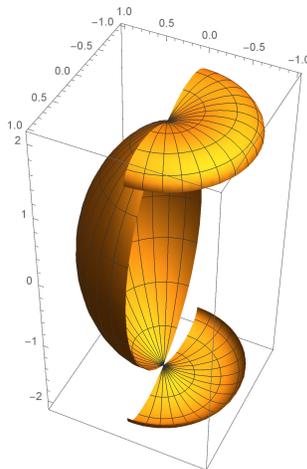


Figure 3: The projections of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$, obtained for $u \in [-\pi, \pi]$, $v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\theta = \frac{\pi}{6}$.

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