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Global Nonexistence Of Solutions For A Nonlinear Klein-Gordon Equation With Variable Exponents*

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Abstract

The aim of this work is to study the global nonexistence of solutions for the Klein-Gordon equation with variable exponents with bounded domain.

1 Introduction

In this paper, we consider the initial-boundary value problem for a nonlinear Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

with the inital-boundary conditions

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

and

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{3}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n \geq 1)$.

The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$2 \leq p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ \leq q^* \tag{4}$$

where

$$\begin{aligned} p^- &= ess \inf_{x\in\Omega} p(x), \quad p^+ = ess \sup_{x\in\Omega} p(x), \\ q^- &= ess \inf_{x\in\Omega} q(x), \quad q^+ = ess \sup_{x\in\Omega} q(x), \end{aligned}$$

and

$$q^* = \begin{cases} \infty, \text{ if } n = 1, 2, \\ \frac{2n}{n-2}, \text{ if } n \geq 3. \end{cases}$$

The Klein-Gordon equation arises in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. The Klein-Gordon equation is the

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first relativistic equation in quantum mechanics for the wave function of a particle with zero spin.

When $p(x)$ and $q(x)$ are constants, (1) become the following the Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{5}$$

There have been extensive researches on the existence, the asymptotic behaviour and the blow up for Eq. (5) (see [2, 8, 9]).

In the absence of the m^2 term ($m = 0$) the problem (1) reduces to the following form

$$u_{tt} - \triangle u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{6}$$

Messaoudi et al. [6] studied the local existence and blow up of the solutions of the equation (6).

Motivated by the above results, in this paper, we prove the global nonexistence of the solution (1) under some conditions.

The outline of this paper is as follows. In section 2, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In section 3, the blow up results will be proved.

2 Preliminaries

In this section, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (see [1, 3, 5, 7]). Also, $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a bounded domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to R,\ u \text{ is measurable and } \rho_{p(.)}(\lambda u) \ < \infty, \ \text{ for some } \lambda > 0 \right\}$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Also endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Variable exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Let the variable exponents $p(.)$ and $q(.)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{7}$$

where $A > 0$ and $0 < \delta < 1$.

Next, we state the local existence theorem of problem (1), that can be obtained by combining arguments in [4, 6].

THEOREM 1 (Local existence). Assume that (7) holds, and that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then there exists a unique solution u of (1) satisfying

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p(\cdot)}(\Omega \times (0,T)).$$

3 Blow Up of Solutions

In this section, we are going to consider the blow up of the solution for problem (1). Firstly, we give following lemma.

LEMMA 2. [6] If $q : \Omega \to [1, \infty)$ is a measurable function and

$$2 \leq q^- \leq q(x) \leq q^+ < \frac{2n}{n-2}; \quad n \geq 3 \tag{8}$$

holds. Then, we have following inequalities:

i)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(\|\nabla u\|^2 + \rho_{q(.)}(u)\right), \tag{9}$$

ii)

$$\|u\|_{q^-}^s \leq c\left(\|\nabla u\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{10}$$

iii)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(|H(t)| + \|u_t\|^2 + \rho_{q(.)}(u)\right), \tag{11}$$

iv)

$$\|u\|_{q^-}^s \leq c\left(|H(t)| + \|u_t\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{12}$$

v)

$$c\left\|u\right\|_{q^-}^{q^-} \leq \rho_{q(.)}\left(u\right) \tag{13}$$

for any $u \in H_0^1\left(\Omega\right)$ and $2 \leq s \leq q^-$. Where $c > 1$ a positive constant and $H\left(t\right) = -E\left(t\right)$ will be specified later.

Now, we state and prove our blow up result.

THEOREM 3. Under the assumptions of Theorem 1**,** and the initial energy $E\left(0\right) < 0$. Then the solution (1) blows up in finite time T^*, and

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}\left(0\right)},$$

where $0 < \xi < 1$ constant, also $\Psi\left(t\right)$ and σ are given in (17) and (18) respectively.

PROOF. Multiplying u_t on two sides of the problem (1), and integrating by part, we get

$$\frac{d}{dt}\left[\frac{1}{2}\left\|u_t\right\|^2 + \frac{1}{2}\left\|\nabla u\right\|^2 + \frac{m^2}{2}\left\|u\right\|^2 - \int_\Omega \frac{1}{q\left(x\right)}\left|u\right|^{q(x)} dx\right] = -\int_\Omega \left|u_t\right|^{p(x)} dx,$$

$$E'\left(t\right) = -\int_\Omega \left|u_t\right|^{p(x)} dx, \tag{14}$$

where

$$E\left(t\right) = \frac{1}{2}\left\|u_t\right\|^2 + \frac{1}{2}\left\|\nabla u\right\|^2 + \frac{m^2}{2}\left\|u\right\|^2 - \int_\Omega \frac{1}{q\left(x\right)}\left|u\right|^{q(x)} dx \tag{15}$$

Set

$$H\left(t\right) = -E\left(t\right),$$

then $E\left(0\right) < 0$ and (14) gives $H\left(t\right) \geq H\left(0\right) > 0$. Also, by the definition $H\left(t\right)$, we have

$$\begin{aligned} H\left(t\right) &= -\frac{1}{2}\left\|u_t\right\|^2 - \frac{1}{2}\left\|\nabla u\right\|^2 - \frac{m^2}{2}\left\|u\right\|^2 + \int_\Omega \frac{1}{q\left(x\right)}\left|u\right|^{q(x)} dx \\ &\leq \int_\Omega \frac{1}{q\left(x\right)}\left|u\right|^{q(x)} dx \\ &\leq \frac{1}{q^-}\rho_{q(.)}\left(u\right). \end{aligned} \tag{16}$$

We define

$$\Psi\left(t\right) = H^{1-\sigma}\left(t\right) + \varepsilon\int_\Omega u u_t dx, \tag{17}$$

where ε small to be chosen later and

$$0<\sigma\leq\min\left\{\frac{q^{-}-p^{+}}{\left(p^{+}-1\right)q^{-}},\frac{q^{-}-2}{2q^{-}}\right\}. \tag{18}$$

By taking a derivative of (17) and using Eq. (1), we obtain

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\int_{\Omega}\left(u_t^2+uu_{tt}\right)dx\\ &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\|u_t\|^2-\varepsilon\|\nabla u\|^2-\varepsilon m^2\|u\|^2\\ &\quad+\varepsilon\int_{\Omega}|u|^{q(.)}dx-\varepsilon\int_{\Omega}uu_t|u_t|^{p(.)-2}dx.\end{aligned} \tag{19}$$

By using the definition of the $H(t)$, it follows that

$$\begin{aligned}-\varepsilon q^{-}(1-\xi)H(t) &= \frac{\varepsilon q^{-}(1-\xi)}{2}\|u_t\|^2+\frac{\varepsilon q^{-}(1-\xi)}{2}\|\nabla u\|^2\\ &\quad+\frac{\varepsilon q^{-}(1-\xi)m^2}{2}\|u\|^2-\varepsilon q^{-}(1-\xi)\int_{\Omega}\frac{1}{q(x)}|u|^{q(.)}dx,\end{aligned} \tag{20}$$

where $0<\xi<1$.

Adding and subtracting (20) into (19), we obtain

$$\begin{aligned}\Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon q^{-}(1-\xi)H(t)\\ &\quad+\varepsilon\left(\frac{q^{-}(1-\xi)}{2}+1\right)\|u_t\|^2+\varepsilon\left(\frac{q^{-}(1-\xi)}{2}-1\right)\|\nabla u\|^2\\ &\quad+\varepsilon m^2\left(\frac{q^{-}(1-\xi)}{2}-1\right)\|u\|^2\\ &\quad+\varepsilon\xi\int_{\Omega}|u|^{q(.)}dx-\varepsilon\int_{\Omega}uu_t|u_t|^{p(.)-2}dx.\end{aligned} \tag{21}$$

Then, for ξ small enough, we get

$$\begin{aligned}\Psi'(t) &\geq \varepsilon\beta\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+m^2\|u\|^2+\rho_{q(.)}(u)\right]\\ &\quad+(1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\int_{\Omega}uu_t|u_t|^{p(.)-2}dx.\end{aligned} \tag{22}$$

where

$$\beta=\min\left\{q^{-}(1-\xi),\ \varepsilon\xi,\ \frac{q^{-}(1-\xi)}{2}-1,\ \frac{q^{-}(1-\xi)}{2}+1\right\}>0$$

and

$$\rho_{q(.)}(u)=\int_{\Omega}|u|^{q(.)}dx.$$

In order to estimate the last term in (22), we make use of the following Young inequality

$$XY\leq\frac{\delta^k X^k}{k}+\frac{\delta^{-l}Y^l}{l},$$

where $X, Y \geq 0,\ \delta > 0,\ k, l \in R^{+}$ such that $\frac{1}{k} + \frac{1}{l} = 1$. Consequently, applying the previous we have

$$\begin{aligned}\int_{\Omega} u\left|u_t\right|^{p(x)-1} dx &\leq \int_{\Omega} \frac{1}{p(x)} \delta^{p(x)} |u|^{p(x)} dx + \int_{\Omega} \frac{p(x)-1}{p(x)} \delta^{-\frac{p(x)}{p(x)-1}} \left|u_t\right|^{p(x)} dx \\ &\leq \frac{1}{p^{-}} \int_{\Omega} \delta^{p(x)} |u|^{p(x)} dx + \frac{p^{+}-1}{p^{+}} \int_{\Omega} \delta^{-\frac{p(x)}{p(x)-1}} \left|u_t\right|^{p(x)} dx, \quad (23)\end{aligned}$$

where δ specified later. Inserting estimates (23) into (22), we get

$$\begin{aligned}\Psi'(t) \geq\ & \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + m^2\|u\|^2 + \rho_{q(.)}(u)\right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{1}{p^{-}} \int_{\Omega} \delta^{p(x)} |u|^{p(x)} dx - \varepsilon \frac{p^{+}-1}{p^{+}} \int_{\Omega} \delta^{-\frac{p(x)}{p(x)-1}} \left|u_t\right|^{p(x)} dx \qquad (24)\end{aligned}$$

Let us choose δ, so that $\delta^{-\frac{p(x)}{p(x)-1}} = kH^{-\sigma}(t)$, where $k > 0$ is specified later, we obtain

$$\begin{aligned}\Psi'(t) \geq\ & \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + m^2\|u\|^2 + \rho_{q(.)}(u)\right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{1}{p^{-}} \int_{\Omega} k^{1-p(x)} H^{\sigma(p(x)-1)}(t) |u|^{p(x)} dx - \varepsilon \frac{p^{+}-1}{p^{+}} \int_{\Omega} kH^{-\sigma}(t) \left|u_t\right|^{p(x)} dx \\ \geq\ & \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + m^2\|u\|^2 + \rho_{q(.)}(u)\right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{k^{1-p^{-}}}{p^{-}} H^{\sigma\left(p^{+}-1\right)}(t) \int_{\Omega} |u|^{p(x)} dx - \varepsilon\left(\frac{p^{+}-1}{p^{+}}\right) kH^{-\sigma}(t) \int_{\Omega} \left|u_t\right|^{p(x)} dx \\ \geq\ & \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + m^2\|u\|^2 + \rho_{q(.)}(u)\right] \\ & + \left[(1-\sigma) - \varepsilon\left(\frac{p^{+}-1}{p^{+}}\right) k\right] H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{k^{1-p^{-}}}{p^{-}} H^{\sigma\left(p^{+}-1\right)}(t) \int_{\Omega} |u|^{p(x)} dx. \qquad (25)\end{aligned}$$

By using (13) and (16), we get

$$
\begin{aligned}
& H^{\sigma\left(p^{+}-1\right)}(t) \int\limits_{\Omega} |u|^{p(x)}\, dx \\
\leq\ & H^{\sigma\left(p^{+}-1\right)}(t) \left[\int\limits_{\Omega_{-}} |u|^{p^{-}}\, dx + \int\limits_{\Omega_{+}} |u|^{p^{+}}\, dx \right] \\
\leq\ & H^{\sigma\left(p^{+}-1\right)}(t)\, c \left[\left(\int\limits_{\Omega_{-}} |u|^{q^{-}}\, dx \right)^{\frac{p^{-}}{q^{-}}} + \left(\int\limits_{\Omega_{+}} |u|^{q^{-}}\, dx \right)^{\frac{p^{+}}{q^{-}}} \right] \\
\leq\ & H^{\sigma\left(p^{+}-1\right)}(t)\, c \left[\|u\|_{q^{-}}^{p^{-}} + \|u\|_{q^{-}}^{p^{+}} \right] \\
\leq\ & c \left(\frac{1}{q^{-}} \rho_{q(.)}(u) \right)^{\sigma\left(p^{+}-1\right)} \left[\left(\rho_{q(.)}(u)\right)^{\frac{p^{-}}{q^{-}}} + \left(\rho_{q(.)}(u)\right)^{\frac{p^{+}}{q^{-}}} \right] \\
=\ & c_1 \left[\left(\rho_{q(.)}(u)\right)^{\frac{p^{-}}{q^{-}}+\sigma\left(p^{+}-1\right)} + \left(\rho_{q(.)}(u)\right)^{\frac{p^{+}}{q^{-}}+\sigma\left(p^{+}-1\right)} \right] \qquad (26)
\end{aligned}
$$

where $\Omega_{-} = \{x \in \Omega : |u| < 1\}$ and $\Omega_{+} = \{x \in \Omega : |u| \geq 1\}$.

We then use (9) in Lemma 2 and (18), for

$$s = p^{-} + \sigma q^{-} \left(p^{+} - 1\right) \leq q^{-}$$

and

$$s = p^{+} + \sigma q^{-} \left(p^{+} - 1\right) \leq q^{-},$$

to deduce, from (26),

$$H^{\sigma\left(p^{+}-1\right)}(t) \int\limits_{\Omega} |u|^{p(x)}\, dx \leq c_1 \left[\|\nabla u\|^2 + \rho_{q(.)}(u) \right]. \qquad (27)$$

Thus, inserting estimate (27) into (25), we have

$$
\begin{aligned}
\Psi'(t) \ \geq\ & \varepsilon \left(\beta - \frac{k^{1-p^{-}}}{p^{-}} c_1 \right) \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + m^2 \|u\|^2 + \rho_{q(.)}(u) \right] \\
& + \left[(1-\sigma) - \varepsilon \left(\frac{p^{+}-1}{p^{+}} \right) k \right] H^{-\sigma}(t)\, H'(t). \qquad (28)
\end{aligned}
$$

Let us choose k large enough so that $\gamma = \beta - \frac{k^{1-p^{-}}}{p^{-}} c_1 > 0$, and picking ε small enough such that $(1-\sigma) - \varepsilon \left(\frac{p^{+}-1}{p^{+}} \right) k \geq 0$ and

$$\Psi(t) \geq \Psi(0) = H^{1-\sigma}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx > 0, \ \forall t \geq 0.$$

Consequently, (28) yields

$$\begin{aligned}\Psi'(t) &\geq \varepsilon\gamma\left[H(t)+\|u_t\|^2+\|\nabla u\|^2 + m^2\|u\|^2+\rho_{q(.)}(u)\right] \\ &\geq \varepsilon\gamma\left[H(t)+\|u_t\|^2+\|\nabla u\|^2 + m^2\|u\|^2+\|u\|_{q^-}^{q^-}\right], \qquad (29)\end{aligned}$$

due to (13).

On the other hand, applying Hölder inequality, we obtain

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} &\leq \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u\|_{q^-}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}\right).\end{aligned}$$

Young inequality gives

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_{q^-}^{\frac{\mu}{1-\sigma}}+\|u_t\|^{\frac{\theta}{1-\sigma}}\right), \qquad (30)$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta=2(1-\sigma)$, to obtain $s=\frac{\mu}{1-\sigma}=\frac{2}{1-2\sigma}\leq q^-$ by (18). Therefore, (30) becomes

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2+\|u\|_{q^-}^{s}\right),$$

where $\frac{2}{1-2\sigma}\leq q^-$. By using (12), we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2+\|u\|_{q^-}^{q^-}+H(t)\right).$$

Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t)+\varepsilon\int_\Omega uu_t dx\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t)+\varepsilon^{\frac{1}{1-\sigma}}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\left(\|u_t\|^2+\|u\|_{q^-}^{q^-}+H(t)\right) \\ &\leq C\left(H(t)+\|u_t\|^2+\|\nabla u\|^2 + m^2\|u\|^2+\|u\|_{q^-}^{q^-}\right) \qquad (31)\end{aligned}$$

where

$$(a+b)^p \leq 2^{p-1}(a^p+b^p)$$

is used. By combining of (29) and (31), we arrive at

$$\Psi'(t)\geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \qquad (32)$$

where ξ is a positive constant.

A simple integration of (32) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

This completes the proof of the theorem.

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