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A Note On Domination In Intersecting Linear Systems*

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Abstract

A linear system is a pair $(P, \mathcal{L})$ where $\mathcal{L}$ is a family of subsets on a ground finite set P such that $|l \cap l'| \leq 1$, for every $l, l' \in \mathcal{L}$. The elements of P and $\mathcal{L}$ are called points and lines, respectively, and the linear system is called intersecting if any pair of lines intersect in exactly one point. A subset D of points of a linear system $(P, \mathcal{L})$ is a dominating set of $(P, \mathcal{L})$ if for every $u \in P \setminus D$ there exists $v \in D$ such that $u, v \in l$, for some $l \in \mathcal{L}$. The cardinality of a minimum dominating set of a linear system $(P, \mathcal{L})$ is called domination number of $(P, \mathcal{L})$, denoted by $\gamma(P, \mathcal{L})$. On the other hand, a subset R of lines of a linear system $(P, \mathcal{L})$ is a 2-packing if any three elements of R do not have a common point (are triplewise disjoint). The cardinality of a maximum 2-packing of a linear system $(P, \mathcal{L})$ is called 2-packing number of $(P, \mathcal{L})$, denoted by $\nu_2(P, \mathcal{L})$.

It is known that for intersecting linear systems $(P, \mathcal{L})$ of rank r it satisfies $\gamma(P, \mathcal{L}) \leq r - 1$. In this note, we prove if q is an even prime power and $(P, \mathcal{L})$ is an intersecting linear system of rank $q + 2$ satisfying $\gamma(P, \mathcal{L}) = q + 1$, then this linear system can be constructed from a spanning $(q + 1)$-uniform intersecting linear subsystem $(P', \mathcal{L}')$ of the projective plane of order q satisfying $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1 = q + 1$.

1 Introduction

A *set system* is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a finite family of subsets on a ground set X. A set system can be also thought of as a hypergraph, where the elements of X and $\mathcal{F}$ are called *vertices* and *hyperedges*, respectively. The set system $(X, \mathcal{F})$ is *intersecting* if $E \cap F \neq \emptyset$, for for every pair of distinct subsets $E, F \in \mathcal{F}$. On the other hand, the set system $(X, \mathcal{F})$ is a *linear system* if it satisfies $|E \cap F| \leq 1$, for every pair of distinct subsets $E, F \in \mathcal{F}$; and it is denoted by $(P, \mathcal{L})$. The elements of P and $\mathcal{L}$ are called *points* and *lines* respectively. For the remainder of this work we will only consider linear systems, and most of the following definitions can be generalized for set systems.

The *rank* of a linear system is the maximum size of a line. An r-*uniform* linear system is a linear system such that all lines contains exactly r points. In this context, a

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simple graph is a 2-uniform linear system. Throughout this paper, we will only consider linear systems of rank $r \geq 2$.

Let $(P, \mathcal{L})$ be a linear system and $p \in P$ be a point. The *degree* of p is the number of lines containing p, denoted by $deg(p)$, the maximum degree overall points of the linear systems is denoted by $\Delta(P, \mathcal{L})$. A point of degrees 2 and 3 is called *double point* and *triple point* respectively. A point of degree zero is called an *isolated point.* Two points $p, q \in P$ are *adjacent* if there is a line $l \in \mathcal{L}$ such that $p, q \in l$.

A *linear subsystem* $(P', \mathcal{L}')$ of a linear system $(P, \mathcal{L})$ satisfies that for any line $l' \in \mathcal{L}'$ there exists a line $l \in \mathcal{L}$ such that $l' = l \cap P'$. The *linear subsystem induced* by a set of lines $\mathcal{L}' \subseteq \mathcal{L}$ is the linear subsystem $(P', \mathcal{L}')$ where $P' = \bigcup_{l \in \mathcal{L}'} l$. The linear subsystem $(P', \mathcal{L}')$ of $(P, \mathcal{L})$ is called *spanning linear subsystem* if $P' = P$. Given a linear system $(P, \mathcal{L})$, and a point $p \in P$, the linear system obtained from $(P, \mathcal{L})$ by *deleting point* p is the linear system $(P', \mathcal{L}')$ induced by $\mathcal{L}' = \{l \setminus \{p\} : l \in \mathcal{L}\}$. Given a linear system $(P, \mathcal{L})$ and a line $l \in \mathcal{L}$, the linear system obtained from $(P, \mathcal{L})$ by *deleting the line* l is the linear system $(P', \mathcal{L}')$ induced by $\mathcal{L}' = \mathcal{L} \setminus \{l\}$. Let $(P', \mathcal{L}')$ and $(P, \mathcal{L})$ be two linear systems. $(P', \mathcal{L}')$ and $(P, \mathcal{L})$ are isomorphic, denoted by $(P', \mathcal{L}') \simeq (P, \mathcal{L})$, if after deleting points of degree 1 or 0 from both, the systems $(P', \mathcal{L}')$ and $(P, \mathcal{L})$ are isomorphic as hypergraphs, see [4].

A subset D of points of a linear system $(P, \mathcal{L})$ is a *dominating set* of $(P, \mathcal{L})$ if for every $u \in P \setminus D$ there exists $v \in D$ such that u and v are adjacent. The minimum cardinality of a dominating set of a linear system $(P, \mathcal{L})$ is called *domination number*, and it is denoted by $\gamma(P, \mathcal{L})$. Domination in set systems was introduced by Acharya [1] and studied further in [2, 5, 8, 9, 10]. A subset T of points of a linear system $(P, \mathcal{L})$ is a *transversal* of $(P, \mathcal{L})$ (also called *vertex cover* or *hitting set*) if $T \cap l \neq \emptyset$, for every line $l \in \mathcal{L}$. The minimum cardinality of a transversal of a linear system $(P, \mathcal{L})$ is called *transversal number*, and it is denoted by $\tau(P, \mathcal{L})$. On the other hand, a subset R of lines of a linear system $(P, \mathcal{L})$ is a *2-packing* of $(P, \mathcal{L})$ if the elements of R are triplewise disjoint, that is, if three elements are chosen in R then they are not incidents in a common point. The *2-packing number* of $(P, \mathcal{L})$ is the maximum cardinality of a 2-packing of $(P, \mathcal{L})$ and it is denoted by $\nu_2(P, \mathcal{L})$. Transversals and 2-packings in linear systems was studied in [3, 4], while domination and 2-packing in linear systems was studied in [13].

In [11] it was proved that, if $(P, \mathcal{L})$ is an intersecting linear system of rank $r \geq 2$, then $\gamma(P, \mathcal{L}) \leq r - 1$. In [12] a characterization of set systems $(X, \mathcal{F})$ holding the equality when $r = 3$ was given. On the other hand, in [9] it was shown that all intersecting linear systems $(P, \mathcal{L})$ of rank 4 satisfying $\gamma(P, \mathcal{L}) = 3$ can be constructed by the Fano plane. In this note, we prove if q is an even prime power and $(P, \mathcal{L})$ is an intersecting linear system of rank $(q + 2)$ satisfying $\gamma(P, \mathcal{L}) = q + 1$, then this linear system can be constructed from a spanning $(q + 1)$-uniform intersecting linear subsystem $(P', \mathcal{L}')$ of the projective plane of order q satisfying $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1 = q + 1$. This result generalizes the main result given in [9].

2 Previous Results

Let $\mathcal{I}_r$ be the family of linear systems $(P, \mathcal{L})$ of rank r with $\gamma(P, \mathcal{L}) = r - 1$. To better understand the main result, we need the following:

LEMMA 1 ([9]). For every linear system $(P, \mathcal{L}) \in \mathcal{I}_r$, there exists an r-uniform spanning linear subsystem $(P^*, \mathcal{L}^*)$ of $(P, \mathcal{L})$ such that every line in $\mathcal{L}^*$ contains one point of degree one.

It is denoted by $(P^*, \mathcal{L}^*)$ to be the r-uniform intersecting spanning linear subsystem of a linear system of $(P, \mathcal{L}) \in \mathcal{I}_r$ obtained from the Lemma 2. Further, let $(P', \mathcal{L}')$ be the $(r-1)$-uniform intersecting linear subsystem obtained from $(P^*, \mathcal{L}^*)$ by deleting the point of degree one of each line of $\mathcal{L}^*$, see [9].

LEMMA 2 ([9]). For every linear system $(P, \mathcal{L}) \in \mathcal{I}_r$ it satisfies

$$\gamma(P, \mathcal{L}) = \gamma(P^*, \mathcal{L}^*) = \tau(P^*, \mathcal{L}^*) = \tau(P', \mathcal{L}') = r - 1.$$

LEMMA 3 ([9]). Let $(P, \mathcal{L}) \in \mathcal{I}_r$ $(r \geq 3)$ then every line of $(P', \mathcal{L}')$ has at most one point of degree 2 and $\Delta(P', \mathcal{L}') = r - 1$.

LEMMA 4 ([9]). Let $(P, \mathcal{L}) \in \mathcal{I}_r$ $(r \geq 3)$ then

$$3(r-2) \leq |\mathcal{L}'| \leq (r-1)^2 - (r-1) + 1 \quad \text{and} \quad |P'| = (r-1)^2 - (r-1) + 1,$$

and so $\gamma(P', \mathcal{L}') = 1$.

THEOREM 1 ([3]). Let $(P, \mathcal{L})$ be a linear system and $p, q \in P$ be two points such that $deg(p) = \Delta(P, \mathcal{L})$ and $deg(q) = \max\{deg(x) : x \in P \setminus \{p\}\}$. If $|\mathcal{L}| \leq deg(p) + deg(q) + \nu_2(P, \mathcal{L}) - 3$, then $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L}) - 1$.

3 Main Result

In this section we prove, if q is an even prime power and $(P, \mathcal{L})$ is an intersecting linear system of rank $(q+2)$ satisfying $\gamma(P, \mathcal{L}) = q + 1$, then this linear system can be constructed from a spanning $(q+1)$-uniform intersecting linear subsystem $(P', \mathcal{L}')$ of the projective plane of order q satisfying $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1 = q + 1$.

Recall that, a *finite projective plane* (or merely *projective plane*) is an uniform linear system satisfying that any pair of points have a common line, any pair of lines have a common point and there exist four points in general position (there are not three collinear points). It is well known that if $(P, \mathcal{L})$ is a projective plane then there exists a number $q \in \mathbb{N}$, called *order of projective plane*, such that every point (line, respectively) of $(P, \mathcal{L})$ is incident to exactly $q + 1$ lines (points, respectively), and $(P, \mathcal{L})$ contains exactly $q^2 + q + 1$ points (lines, respectively). In addition to this, it is well known that projective planes of order q, denoted by Π_q, exist when q is a prime power. For more information about the existence and the unicity of projective planes see, for instance, [6, 7]. In [4] it was proved, if q is an even prime power then $\tau(\Pi_q) = \nu_2(\Pi_q) - 1 = q+1$, however, if q is an odd prime power then $\tau(\Pi_q) = \nu_2(\Pi_q) = q+1$.

To prove the following lemma (Lemma 3), we need a result shown in [13].

LEMMA 5 ([13]). Let $(P, \mathcal{L})$ be an r-uniform intersecting linear system with $r \geq 2$ be an even integer. If $\nu_2(P, \mathcal{L}) = r+1$ then $\tau(P, \mathcal{L}) = \frac{r+2}{2}$.

LEMMA 6. Let $r \geq 2$ be an even integer . For every $(P, \mathcal{L}) \in \mathcal{I}_{r+2}$ it satisfies $\nu_2(P', \mathcal{L}') = r+2$. Hence $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1$.

PROOF. Since $(P', \mathcal{L}')$ is an intersecting $(r+1)$-uniform linear system, then $|l'| \geq \nu_2(P', \mathcal{L}') - 1$, for any line $l' \in \mathcal{L}'$. Hence, $\nu_2(P', \mathcal{L}') \leq r+2$. If $r+2$ is odd integer, then by Lemma 3 it satisfies $\tau(P', \mathcal{L}') = \frac{r+3}{2}$, which is a contradiction, since $\tau(P', \mathcal{L}') = r+1$.

Let $p \in P'$ such that $deg(p) = \Delta(P', \mathcal{L}')$, and let $\Delta'(P', \mathcal{L}') = \max\{deg(x) : x \in P' \setminus \{p\}\}$. By Theorem 2 if $|\mathcal{L}'| \leq \Delta(P', \mathcal{L}') + \Delta'(P', \mathcal{L}') + \nu_2(P, \mathcal{L}) - 3 \leq 3r+1$ (see Lemma 2) then $\tau(P', \mathcal{L}') \leq \nu_2(P', \mathcal{L}') - 1$, which implies $\nu_2(P', \mathcal{L}') \geq r+2$, and the equality $\nu_2(P', \mathcal{L}') = r+2$ holds. Hence, by Lemma 2, if $|\mathcal{L}'| = 3r$ then $\nu_2(P', \mathcal{L}') = r+2$, which implies if $(P, \mathcal{L}) \in \mathcal{I}_{r+2}$ then $\nu_2(P', \mathcal{L}') = r+2$, since if $(\hat{P}, \hat{\mathcal{L}})$ is a spanning linear subsystem of $(P', \mathcal{L}')$ then $\nu_2(\hat{P}, \hat{\mathcal{L}}) \leq \nu_2(P', \mathcal{L}')$.

THEOREM 2. Let q be an even prime power. For every $(P, \mathcal{L}) \in \mathcal{I}_{q+2}$ the linear system $(P', \mathcal{L}')$ is a spanning $(q+1)$-uniform linear subsystem of Π_q such that $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1 = q+1$ with $|\mathcal{L}'| \geq 3q$.

PROOF. Let $(P, \mathcal{L}) \in \mathcal{I}_{q+2}$. Then $(P', \mathcal{L}')$ is an $(q+1)$-uniform intersecting linear system with $|P'| = q^2+q+1$ (by Lemma 2). Furthermore, if $|\mathcal{L}'| = q^2+q+1$ then all points of $(P', \mathcal{L}')$ have degree $q+1$ (it is a consequence of Lemma 2, see [9]). Since projective planes are dual systems, the 2-packing number coincides with the cardinality of an oval, which is the maximum number of points in general position (no three of them collinear), and it is equal to $q+2 = \nu_2(P', \mathcal{L}')$ (Lemma 3), when q is even, see for example [7]. Hence, the linear system $(P', \mathcal{L}')$ is a projective plane of order q, Π_q. Therefore, if $(P, \mathcal{L}) \in \mathcal{I}_{q+2}$ then $(P', \mathcal{L}')$ is a spanning linear subsystem of Π_q satisfying $\tau(P', \mathcal{L}') = \nu_2(P', \mathcal{L}') - 1 = q+1$, with $|\mathcal{L}'| \geq 3q$ (see Lemma 2).

The following is a straightforward consequence of Theorem 3 which is the main result given in [9].

COROLLARY 1. If $(P, \mathcal{L}) \in \mathcal{I}_4$ then either $(P', \mathcal{L}')$ is the Fano plane, Π_2, or $(P', \mathcal{L}')$ is obtained from Π_2 by deleting any line.

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