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Properties And Related Inequalities Of φ-frames In Normed Spaces*

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Abstract

In this paper, we use the properties of sesquilinear forms to introduce a new class of frames, called φ-frames. The notion of continuous φ-frames, its various properties and characterizations in normed spaces are established. Also, some fundamental identities and certain inequalities related to φ-frames are obtained.

1 Notations and Preliminaries

The concept of frame in Hilbert spaces was introduced by Duffin and Schaeffer [14] to study some problems in non-harmonic Fourier series in 1952, reintroduced in 1986 by Daubechies, Grossmann, and Meyer [12] and popularized from then on. Now the theory of frames is widely studied by several authors and they have established a series of results (see [1, 4, 8, 9, 10]). A frame, which is redundant set of vectors in a Hilbert space $\mathcal{H}$ with the property that provides non unique representations of vectors in terms of the frame elements, has been applied in filter bank theory [6], sigma-delta quantization [5], signal and image processing [7] and many other fields. A frame for a complex Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \ \ (f \in \mathcal{H})\,. \tag{1.1}$$

The constants A and B are called the lower and upper frame bounds, respectively. A frame is said to be tight whenever $A = B$ and if we can take $A = B = 1$ it is called a Parseval frame. If the right-hand inequality of (1.1) holds, then we say that $\{f_i\}_{i\in I}$

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is a Bessel sequence for $\mathcal{H}$ with bound B. The analytic operator associated to the frame $\{f_i\}_{i\in I}$ is defined as $T : L^2 \to \mathcal{H}$ by $T\{a_i\} = \sum_{i\in I} a_i f_i$. It is easy to see that $T^* : \mathcal{H} \to L^2$ such that $T^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$. The frame operator for the frame is the positive, self adjoint invertible operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ satisfying

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \ \ (f \in \mathcal{H}).$$

This provides the frame decomposition

$$f = S^{-1}Sf = \sum_{i\in I} \langle f, f_i\rangle \widetilde{f_i} = \sum_{i\in I} \left\langle f, \widetilde{f_i}\right\rangle f_i,$$

where $\widetilde{f_i} = S^{-1} f_i$. The family $\left\{\widetilde{f_i}\right\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the canonical dual frame of $\{f_i\}_{i\in I}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$, for every $J \subset I$ we define the operator S_J by

$$S_J f = \sum_{i\in J} \langle f, f_i\rangle f_i.$$

We refer to [9, 11, 18] for an introduction to the frame theory and its applications. In this section, we recall fundamental definitions, basic properties and notations of sesquilinear forms which are needed for a comprehensive reading of this paper. This background can be found in [13]. Let $\mathscr{E}$ be a vector space then $\varphi : \mathscr{E} \times \mathscr{E} \to \mathbb{C}$ is a sesquilinear form on $\mathscr{E}$ if the following two conditions holds:

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,

(b) $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$

for any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{E}$. Two typical examples of sesquilinear forms are as follows:

(I) Let A and B be operators on an inner product space $\mathscr{E}$. Then $\varphi_1(x, y) = \langle Ax, y\rangle$, $\varphi_2(x, y) = \langle x, By\rangle$, and $\varphi_3(x, y) = \langle Ax, By\rangle$ are sesquilinear forms on $\mathscr{E}$.

(II) Let f and g be linear functionals on a vector space $\mathscr{E}$. Then $\varphi(x, y) = f(x)\overline{g(y)}$ is a sesquilinear form on $\mathscr{E}$.

Let φ be a sesquilinear form on vector space $\mathscr{E}$, then φ is called symmetric if $\varphi(x, y) = \overline{\varphi(y, x)}$ for all $x, y \in \mathscr{E}$. A sesquilinear form φ on vector space $\mathscr{E}$ is said to be positive if $\varphi(x, x) \geq 0$ for all $x \in \mathscr{E}$. Moreover, φ is called Cauchy-Schwarz if $(\varphi(x, y))^2 \leq \varphi(x, x)\varphi(y, y)$ for each $x, y \in \mathscr{E}$. The corresponding quadratic form associated to φ is defined as:

$$\Phi(x) = \varphi(x, x).$$

We remark that, if $\mathscr{E}$ be a normed space and φ is a positive bounded sesquilinear form, then $\sqrt{\Phi(x)}$ defines a semi norm on $\mathscr{E}$ (see [16, p. 52]). Let $\mathcal{B}(\mathscr{E})$ denote the algebra of all bounded linear operators on a complex vector space $\mathscr{E}$. For operator $A \in \mathcal{B}(\mathscr{E})$ there exist $B \in \mathcal{B}(\mathscr{E})$ such that for each x and y in $\mathscr{E}$

$$\varphi(Ax, y) = \varphi(x, By).$$

In this case, B is φ-adjoint of A and it is denoted by A^*. For more information on related ideas and concepts we refer [17, p. 88-90]. The operator A in $\mathcal{B}(\mathscr{E})$ is called φ-positive if for all $x \in \mathscr{E}$, $\varphi(Ax, x) \geq 0$. We note that, $A \geq B$ if $A - B \geq 0$.

In this paper, we develop the existing notions of frames on Hilbert spaces by using the definition of sesquilinear form on a normed space $\mathscr{E}$. Section 2 is devoted to some elementary considerations concerning the φ-frames. Some properties and results of such frames are investigated. In Section 3, we derive some characterizations of continuous φ-frames. Finally, in the last section, we give new Parseval type identities and inequalities for φ-frames in normed spaces (see Corollary 4.1 and Proposition 4.1). Our results generalize the remarkable results obtained recently by Găvruţa.

2 φ-frames

The following basic results are essentially known as in [9], but our expression is a little bit different from those in [9]. In fact Hilbert space $\mathcal{H}$ and inner product $\langle \cdot, \cdot \rangle$ are replaced with vector space $\mathscr{E}$ and sesqulinear form φ respectively. Recall that a sequence $\{e_k\}_{k=1}^m$ in a vector space $\mathscr{E}$ is a basis, if the following conditions are satisfied:

(a) $\mathscr{E} = span\{e_k\}_{k=1}^m$;

(b) $\{e_k\}_{k=1}^m$ is linearly independent.

As a consequence of above definition, every $f \in \mathscr{E}$ has a unique representation in terms of the elements in the basis, i.e., there exists unique scalar coefficients $\{c_k\}_{k=1}^m$ such that

$$f = \sum_{k=1}^{m} c_k e_k.$$

If $\{e_k\}_{k=1}^m$ is a φ-orthonormal basis, i.e., a basis for which

$$\varphi(e_k, e_j) = \delta_{k,j} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

then the coefficients $\{c_k\}_{k=1}^m$ are easy to find

$$\varphi(f, e_j) = \varphi\left(\sum_{k=1}^{m} c_k e_k, e_j\right) = \sum_{k=1}^{m} c_k \varphi(e_k, e_j) = c_j.$$

So

$$f = \sum_{k=1}^{m} \varphi(f, e_k) e_k.$$

A sequence $\{f_k\}_{k=1}^{\infty}$ in a vector space $\mathscr{E}$ is called φ-frame if there exist $A, B > 0$ such that

$$A\varphi(f,f) \leq \sum_{k=1}^{n} |\varphi(f,f_k)|^2 \leq B\varphi(f,f), \tag{2.1}$$

for all $f \in \mathscr{E}$. The constants A and B are called φ-frame bounds. If $A = B$, this is a tight φ-frame and if $A = B = 1$ this is a Parseval φ-frame. Consider a vector space $\mathscr{E}$ equipped with a frame $\{f_k\}_{k=1}^{m}$ and define a linear mapping

$$T : \mathbb{C}^m \to \mathscr{E},\ T\{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k f_k.$$

T is called the φ-pre-frame operator. The adjoint operator is given by

$$T^* : \mathscr{E} \to \mathbb{C}^m,\ T^* f = \{\varphi(f, f_k)\}_{k=1}^{m}$$

in fact by the usual inner product on $\mathbb{C}^m$ as the sesquilinear form $\varphi^{'}$ we have

$$\varphi(Tx, y) = \varphi\left(\sum_{k=1}^{m} c_k f_k, y\right) = \sum_{k=1}^{m} c_k \varphi(f_k, y)$$

and

$$\varphi'(x, T^* y) = \varphi'\left(\{c_k\}_{k=1}^{m}, \{\varphi(y, f_k)\}_{k=1}^{m}\right) = \sum_{k=1}^{m} c_k \varphi(f_k, y).$$

In this case, T^* is called the analytic operator and by composing T with its adjoint T^*, we obtain the φ-frame operator

$$S : \mathscr{E} \to \mathscr{E},\ Sf = TT^* f = \sum_{k=1}^{m} \varphi(f, f_k) f_k.$$

Note that in terms of the φ-frame operator,

$$\varphi(Tf, f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2,\ f \in \mathscr{E}.$$

REMARK 2.1. Let φ be a Cauchy-Schwarz bounded sesquilinear form, then

$$\sum_{k=1}^{m} |\varphi(f, f_k)|^2 \leq \sum_{k=1}^{m} \Phi(f_k)\Phi(f). \tag{2.2}$$

PROPOSITION 2.1. Let $\{f_k\}_{k=1}^{m}$ be a sequence in $\mathscr{E}$. Then $\{f_k\}_{k=1}^{m}$ is a φ-frame for span $\{f_k\}_{k=1}^{m}$.

PROOF. Assume that none of the f_k's are zeros. From, Remark 2.1, the upper φ-frame condition is satisfied with $B = \sum_{k=1}^{m} \Phi(f_k)$. Now let

$$W = span\{f_k\}_{k=1}^{m}$$

and consider the continuous mapping

$$\psi : W \to \mathbb{R},\ \psi(f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2.$$

The unit ball in W is compact since, W is finite dimensional. So the function ψ takes its infimum on the unit ball W. We can find $g \in W$ with $\sqrt{\Phi(g)} = 1$ such that

$$A = \sum_{k=1}^{m} |\varphi(g, f_k)|^2 = \inf\left\{\sum_{k=1}^{m} |\varphi(f, f_k)|^2 :\ f \in W,\ \sqrt{\Phi(f)} = 1\right\}.$$

It is clear that $A > 0$. Now for $f \in W$, $f \neq 0$, we have

$$\sum_{k=1}^{m} |\varphi(f, f_k)|^2 = \sum_{k=1}^{m} \varphi\left(\frac{f}{\sqrt{\Phi(f)}}, f_k\right)^2 |\Phi(f)| \geq A\,|\Phi(f)|\,.$$

COROLLARY 2.1. A family of elements $\{f_k\}_{k=1}^{m}$ in $\mathscr{E}$ is a φ-frame for $\mathscr{E}$ if and only if $span\{f_k\}_{k=1}^{m} = \mathscr{E}$.

THEOREM 2.1. Let $\{f_k\}_{k=1}^{m}$ be a φ-frame for $\mathscr{E}$ with φ-frame operator S. Then

(a) S is invertible and self adjoint.

(b) Every $f \in \mathscr{E}$ can be represented as

$$f = \sum_{k=1}^{m} \varphi\left(f, S^{-1} f_k\right) f_k = \sum_{k=1}^{m} \varphi(f, f_k) S^{-1} f_k. \tag{2.3}$$

PROOF. Since $S = TT^*$, it is clear that S is a self adjoint. We have to prove that S is injective. Let $f \in \mathscr{E}$ and assume that $Sf = 0$. Then

$$0 = \varphi(Sf, f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2,$$

by the φ-frame condition $f = 0$. S is injective implies that S is surjective, but let us give direct proof. By Corollary 2.1, the φ-frame condition implies that $span\{f_k\}_{k=1}^{m} = \mathscr{E}$, so the φ-pre frame operator T is surjective. For $f \in \mathscr{E}$ we can find $g \in \mathscr{E}$ such that

$Tg = f$. We can choose $g \in N_T^{\perp} = R_{T^*}$, so it follows that $R_S = R_{TT^*} = \mathscr{E}$. Thus S is surjective. Each $f \in \mathscr{E}$ has the representation

$$f = SS^{-1}f = TT^*S^{-1}f = \sum_{k=1}^{m} \varphi\left(S^{-1}f, f_k\right) f_k.$$

Since S is self adjoint, we get

$$f = \sum_{k=1}^{m} \varphi\left(f, S^{-1}f_k\right) f_k.$$

The second representation in (2.3) is obtained in the same way, hence $f = S^{-1}Sf$.

THEOREM 2.2. Let $\{f_k\}_{k=1}^m$ be a φ-frame for $\mathscr{E}$ with φ-frame operator T. Then If $f \in \mathscr{E}$ also has the representation $f = \sum\limits_{k=1}^{m} c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^m$, then

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} \left|\varphi\left(f, T^{-1}f_k\right)\right|^2 + \sum_{k=1}^{m} \left|c_k + \varphi\left(f, T^{-1}f_k\right)\right|^2. \tag{2.4}$$

PROOF. Suppose that $f = \sum\limits_{k=1}^{m} c_k f_k$. We can write

$$\{c_k\}_{k=1}^m = \{c_k\}_{k=1}^m - \left\{\varphi\left(f, T^{-1}f_k\right)\right\}_{k=1}^m + \left\{\varphi\left(f, T^{-1}f_k\right)\right\}_{k=1}^m.$$

By the choice of $\{c_k\}_{k=1}^m$ we have

$$\sum_{k=1}^{m} \left(c_k - \varphi\left(f, T^{-1}f_k\right)\right) f_k = 0$$

i.e.,

$$\{c_k\}_{k=1}^m - \left\{\varphi\left(f, T^{-1}f_k\right)\right\}_{k=1}^m \in N_S = R_{S^*}^{\perp},$$

since

$$\left\{\varphi\left(f, T^{-1}f_k\right)\right\}_{k=1}^m = \left\{\varphi\left(T^{-1}f, f_k\right)\right\}_{k=1}^m \in R_{S^*}$$

we obtain (2.4).

REMARK 2.2. If $\{f_k\}_{k=1}^m$ is a φ-frame but not a basis, there exist non zero sequences $\{d_k\}_{k=1}^m$ such that $\sum\limits_{k=1}^{m} d_k f_k = 0$. Therefore $f \in \mathscr{E}$ can be written

$$f = \sum_{k=1}^{m} \varphi\left(f, T^{-1}f_k\right) f_k + \sum_{k=1}^{m} d_k f_k$$

and

$$= \sum_{k=1}^{m} \left(\varphi\left(f, T^{-1}f_k\right) + d_k\right) f_k$$

showing that f has many representations as superpositions of the φ-frame elements.

PROPOSITION 2.2. Let $\{f_k\}_{k=1}^m$ be a basis for $\mathscr{E}$. Then there exists a unique family $\{g_k\}_{k=1}^m$ in $\mathscr{E}$ such that

$$f = \sum_{k=1}^{m} \varphi(f, g_k) f_k, \quad (\forall f \in \mathscr{E}). \tag{2.5}$$

PROOF. The existence of a family $\{g_k\}_{k=1}^m$ satisfying (2.5) follows from Theorem 2.1, also the uniqueness part is direct.

REMARK 2.3. Applying (2.5) on a fixed element f_j and since $\{f_k\}_{k=1}^m$ is a basis, we get $\varphi(f_j, g_k) = \delta_{j,k}$ for all $k = 1, 2, ..., m$.

THEOREM 2.3. Let $\{f_k\}_{k=1}^m$ be a φ-frame for subspace F of the vector space $\mathscr{E}$. Then the φ-orthogonal projection of $\mathscr{E}$ onto F is given by

$$Pf = \sum_{k=1}^{m} \varphi\left(f, T^{-1} f_k\right) f_k. \tag{2.6}$$

PROOF. It is enough to prove that if we define P by (2.6), then

$$Pf = f \text{ for } f \in F \text{ and } Pf = 0 \text{ for } f \in F^{\perp}.$$

The first equation follows by Theorem 2.1, and the second by the fact that the range of T^{-1} equals F because T is a bijection on F.

3 Continuous φ-frames

In this section, we introduce the concept of continuous φ-frames, which is a partial extension of continuous frames. To prove our main result related to continuous φ-frames, we need the following essential definitions. Let I be a locally compact group, and $\mathscr{E}$ be a vector space, and φ be a sesquilinear form on $\mathscr{E}$. A function

$$f : I \to \mathscr{E}$$

is called a continuous φ-frame in $\mathscr{E}$, if there are positive numbers A, B, such that for all x in $\mathscr{E}$

$$A\varphi(x, x) \leq \int_I |\varphi(x, f_i)|^2 di \leq B\varphi(x, x), \tag{3.1}$$

where di is a Haar measure on I. The constants A and B are called the frame bounds. In this case, we define the corresponding frame operator as $S : I \to I$ such that

$$S(x) = \int_I \varphi(x, f_i)\, di. \tag{3.2}$$

Moreover, we can define the analysis operator as this $T : \mathscr{E} \to L^2(I)$ such that

$$x \to (\varphi(x, f_i))_{i\in I}. \tag{3.3}$$

The notation $(\varphi(x, f_i))_{i\in I}$ in (3.3) denotes the function in $L^2(I)$

$$i \to (\varphi(x, f_i))_{i\in I}.$$

It easy to prove that $T^* : L^2(I) \to \mathscr{E}$ which

$$g \to \int_I f_i g_i di,$$

and it implies that

$$S = T^*T.$$

THEOREM 3.1. Let I be a locally compact group, φ be a symmetric sesquilinear form on a vector space $\mathscr{E}$, and let $f : I \to \mathscr{E}$ be a φ-frame in $\mathscr{E}$, with frame bounds A and B. Then the operator S is a positive, self adjoint, invertible operator on $\mathscr{E}$, moreover

$$AI_E \leq S \leq BI_{\mathscr{E}}.$$

PROOF. By definition, we can write

$$\begin{aligned}\varphi(Sx, x) = \varphi\left(\int_I \varphi(x, f_i) f_i di, x\right) &= \int_I \varphi(\varphi(x, f_i) f_i, x) di \\ &= \int_I \varphi(x, f_i) \varphi(f_i, x) di \\ &= \int_I \varphi(x, f_i) \overline{\varphi(x, f_i)} di \\ &= \int_I |\varphi(x, f_i)|^2 di.\end{aligned}$$

Therefore from definition of frame bounds, we conclude that

$$A\varphi(x, x) \leq \varphi(Sx, x) \leq B\varphi(x, x)$$

which is equivalent to

$$AI_{\mathscr{E}} \leq S \leq BI_{\mathscr{E}}.$$

EXAMPLE 3.1. Let I be the positive real number, and $\mathscr{E}$ be $L^2(R)$. Define $f : R^+ \to L^2(R)$ which

$$\alpha \to f_\alpha$$

where

$$f_\alpha(x) = e^{2\pi i\alpha x}.$$

Then it easy to show that the frame operator corresponding to the inner product of $L^2(R)$ is the identity on $\mathscr{E}$. In other words, for any function f

$$f = \int_0^{+\infty} \varphi(f, f_\alpha) f_\alpha d\alpha$$

or equivalently

$$f(x) = \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} f(x)\overline{f_\alpha(x)}dx \right) f_\alpha(x)\, d\alpha$$

or

$$f(x) = \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i\alpha x}dx \right) e^{2\pi i\alpha x}d\alpha.$$

This is the Fourier integral for the function f.

EXAMPLE 3.2. In the previous, let I be the set of all positive integers, then we have

$$f = \sum_0^{\infty} \varphi(f, f_n) f_n$$

or

$$f(x) = \sum_0^{\infty} \left(\int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i\alpha x}dx \right) e^{2\pi i\alpha x}d\alpha.$$

which is the Fourier series for the function f.

Example 3.2 shows that the Fourier system is a continuous φ-frame, which has a discrete sub frame, but not in a same measure.

REMARK 3.1. In general, it is not necessary for I to be a group, it is enough that I is a subset of a locally compact group with a suitable measure. As we see in the examples, it is important to define an integral or summation on I.

4 Applications

As an application of previous sections, we prove the following inequalities and by using the model technique of Balan et al. [2, 3] and Gavruta [15], we obtain an analogue, called Parseval's identity of φ-frames in normed spaces.

THEOREM 4.1. Let $\{f_i\}_{i\in I}$ be a φ-frame for a vector space $\mathscr{E}$ with frame bounds A, B. Let $J \subset I$, so that $\{f_i\}_{i\in J}$ has Bessel bound $B(J) < A$. Then $\{f_i\}_{i\in J^c}$ is a φ-frame for $\mathscr{E}$.

PROOF. Since $\{f_i\}_{i\in J^c}$ has B as a Bessel bound, we only need to check its lower frame bound. For this just compute for any $f\in\mathscr{E}$

$$\begin{aligned}\sum_{i\in J^c}|\varphi(f,f_i)|^2&=\sum_{i\in I}|\varphi(f,f_i)|^2-\sum_{i\in J}|\varphi(f,f_i)|^2\\&\geq A\Phi(f)-B(J)\Phi(f)=(A-B(J))\Phi(f).\end{aligned}$$

Since $A-B(J)>0$, we deduce the desired result.

COROLLARY 4.1. Let $\{f_i\}_{i\in I}$ be a Parseval φ-frame for $\mathscr{E}$ and $J\subset I$. In order for $\{f_i\}_{i\in J}$ to be a φ-frame for $\mathscr{E}$ is necessary and sufficient that $B(J^c)<1$. In this case, the optimal lower frame bound for $\{f_i\}_{i\in J}$ is $1-B(J^c)$.

PROOF. For any $f\in\mathscr{E}$ we have

$$\begin{aligned}\sum_{i\in J}|\varphi(f,f_i)|^2&=\sum_{i\in I}|\varphi(f,f_i)|^2-\sum_{i\in J^c}|\varphi(f,f_i)|^2\\&\geq\Phi(f)-B(J^c)\Phi(f)=(1-B(J^c))\Phi(f).\end{aligned}$$

It is easy to see that the inequality above is optimal, hence the proof.

The following result can be stated as well.

THEOREM 4.2. Assume that φ is a bounded positive sesquilinear form. If $U,V\in\mathcal{L}(\mathscr{E})$ are φ-self adjoint operators satisfying $U+V=1_{\mathscr{E}}$, then for all $f\in\mathscr{E}$ we have

$$\varphi(Uf,f)+\Phi(Vf)=\varphi(Vf,f)+\Phi(Vf)\geq\frac{3}{4}\Phi(f).$$

PROOF. We have

$$\begin{aligned}\varphi(Uf,f)+\Phi(Vf)&=\varphi(Uf,f)+\varphi(Vf,Vf)\\&=\varphi((I_{\mathscr{E}}-V)f,f)+\varphi\left(V^2f,f\right)\\&=\varphi\left(\left(V^2-V+I_{\mathscr{E}}\right)f,f\right)\\&=\varphi(Vf,f)+\varphi(Uf,Uf)+\varphi\left((I_{\mathscr{E}}-V)^2f,f\right)\\&=\varphi\left(\left(V^2f-V+I_{\mathscr{E}}\right)f,f\right)\\&=\varphi\left(\left(\left(V-\frac{1}{2}I_{\mathscr{E}}\right)^2+\frac{3}{4}I_{\mathscr{E}}\right)f,f\right)\\&\geq\frac{3}{4}\Phi(f).\end{aligned}$$

This completes the proof of Theorem 4.2.

REMARK 4.1. We consider now $\{f_i\}_{i\in I}$, a φ-frame for $\mathscr{E}$ with S its frame operator and $\left\{\widetilde{f_i}\right\}_{i\in I}$ its canonical dual frame and $J\subset I$. We have

$$S_J+S_{J^c}=S,$$

hence

$$S^{-\frac{1}{2}}S_J S^{-\frac{1}{2}} + S^{-\frac{1}{2}}S_{J^c}S^{-\frac{1}{2}} = 1_{\mathscr{E}}.$$

PROOF. If in the Theorem 4.2 we take $U = S^{-\frac{1}{2}}S_J S^{-\frac{1}{2}}$, $V = S^{-\frac{1}{2}}S_{J^c}S^{-\frac{1}{2}}$ and $S^{\frac{1}{2}}f$ instead of f, we get

$$\begin{aligned}\varphi\left(S^{-\frac{1}{2}}S_J f, S^{\frac{1}{2}}f\right) + \Phi\left(S^{-\frac{1}{2}}S_{J^c}f\right) &= \varphi\left(S^{-\frac{1}{2}}S_J f, S^{\frac{1}{2}}f\right) + \Phi\left(S^{-\frac{1}{2}}S_J f\right) \\ &\geq \frac{3}{4}\Phi\left(S^{\frac{1}{2}}f\right),\end{aligned}$$

or

$$\begin{aligned}\varphi\left(S_J f, f\right) + \varphi\left(S^{-\frac{1}{2}}S_{J^c}f, S^{-\frac{1}{2}}S_{J^c}f\right) &= \varphi\left(S_{J^c}f, f\right) + \varphi\left(S^{-1}S_J f, S_J f\right) \\ &\geq \frac{3}{4}\varphi\left(Sf, f\right).\end{aligned}$$

The following result also holds (see [15, Theorem 3.2] for the case of Hilbert space).

THEOREM 4.3. Let $\{f_i\}_{i\in I}$ be a φ-frame for $\mathscr{E}$ and $\{g_i\}_{i\in I}$ be an alternative dual of $\{f_i\}_{i\in I}$. Then for all $J \subset I$ and all $f \in \mathscr{E}$, we have

$$\begin{aligned} & \mathrm{Re}\sum_{i\in J}\varphi(f,g_i)\overline{\varphi(f,f_i)} + \Phi\left(\sum_{i\in J^c}\varphi(f,g_i)f_i\right) \\ = \; & \mathrm{Re}\sum_{i\in J}\varphi(f,g_i)\overline{\varphi(f,f_i)} + \Phi\left(\sum_{i\in J}\varphi(f,g_i)f_i\right) \\ \geq \; & \frac{3}{4}\Phi\left(f\right).\end{aligned}$$

PROOF. For every $J \subset I$ we define the operator L_J by

$$L_J f = \sum_{i\in J}\varphi\left(f, g_i\right)f_i.$$

By the Cauchy-Schwarz inequality it follows that this series converges unconditionally and $L_J \in \mathcal{L}\left(\mathscr{E}\right)$. Since $L_J + L_{J^c} = I_{\mathbb{E}}$,

$$\begin{aligned}\varphi\left(\left(L_J^*L_J\right)f,f\right) + \frac{1}{2}\varphi\left(\left(L_{J^c}^*L_{J^c}\right)f,f\right) &= \varphi\left(\left(L_{J^c}^*L_{J^c}\right)f,f\right) + \frac{1}{2}\varphi\left(\left(L_J^* + L_J^*\right)f,f\right) \\ &\geq \frac{3}{4}\Phi\left(f\right),\end{aligned}$$

or

$$\begin{aligned}&\Phi\left(\sum_{i\in J}\varphi\left(f,g_i\right)f_i\right) + \frac{1}{2}\left(\overline{\varphi\left(L_{J^c}f,f\right)} + \varphi\left(L_{J^c}f,f\right)\right) \\ &= \Phi\left(\sum_{i\in J^c}\varphi\left(f,g_i\right)f_i\right) + \frac{1}{2}\left(\overline{\varphi\left(L_J f,f\right)} + \varphi\left(L_J f,f\right)\right) \\ &\geq \frac{3}{4}\Phi\left(f\right).\end{aligned}$$

To prove Theorem 4.4, we need the following lemma.

LEMMA 4.1. If S, T are operators on $\mathscr{E}$ satisfying $S+T=I$, then $S-T=S^2-T^2$.

PROOF. Easy computation and simplification yield

$$S-T=S-(I-S)=2S-I=S^2-\left(I-2S+S^2\right)=S^2-(I-S)^2=S^2-T^2.$$

THEOREM 4.4. Let $\{f_i\}_{i\in I}$ be a φ-frame for $\mathscr{E}$ with canonical frame $\left\{\widetilde{f}_i\right\}_{i\in I}$. Then for all $J\subset I$ and for all $f\in\mathscr{E}$ we have

$$\sum_{i\in J}\left|\varphi\left(f,f_i\right)\right|^2-\sum_{i\in I}\left|\varphi\left(S_Jf,\widetilde{f}_i\right)\right|^2=\sum_{i\in J^c}\left|\varphi\left(f,f_i\right)\right|^2-\sum_{i\in I}\left|\varphi\left(S_{J^c}f,\widetilde{f}_i\right)\right|^2.$$

PROOF. Let S denote the frame operator for $\{f_i\}_{i\in I}$. Since $S=S_J+S_{J^c}$, it follows that $I=S^{-1}S_J+S^{-1}S_{J^c}$. Applying Lemma 4.1 to the two operators $S^{-1}S_J$ and $S^{-1}S_{J^c}$ yields

$$S^{-1}S_J-S^{-1}S_JS^{-1}S_J=S^{-1}S_{J^c}-S^{-1}S_{J^c}S^{-1}S_{J^c}. \tag{4.1}$$

Further, for every $f,g\in\mathscr{E}$ we obtain

$$\varphi\left(S^{-1}S_Jf,g\right)-\varphi\left(S^{-1}S_JS^{-1}S_Jf,g\right)=\varphi\left(S_Jf,S^{-1}g\right)-\varphi\left(S^{-1}S_Jf,S_JS^{-1}g\right). \tag{4.2}$$

Now, we choose g to be $g=Sf$. Then we can continue the equality (4.2) in the following as

$$\varphi\left(S_Jf,f\right)-\varphi\left(S^{-1}S_Jf,S_Jf\right)=\sum_{i\in J}\left|\varphi\left(f,f_i\right)\right|^2-\sum_{i\in I}\left|\varphi\left(S_Jf,\widetilde{f}_i\right)\right|^2.$$

Setting equality (4.2) equal to the corresponding equality for J^c and using (4.1), we finally get

$$\sum_{i\in J}\left|\varphi\left(f,f_i\right)\right|^2-\sum_{i\in I}\left|\varphi\left(S_Jf,\widetilde{f}_i\right)\right|^2=\sum_{i\in J^c}\left|\varphi\left(f,f_i\right)\right|^2-\sum_{i\in I}\left|\varphi\left(S_{J^c}f,\widetilde{f}_i\right)\right|^2.$$

PROPOSITION 4.1. Let $\{f_i\}_{i\in I}$ be a Parseval φ-frame for $\mathscr{E}$. For every subset $J\subset I$ and every $f\in\mathscr{E}$, we have

$$\sum_{i\in J}\left|\varphi\left(f,f_i\right)\right|^2-\Phi\left(\varphi\left(f,f_i\right)f_i\right)=\sum_{i\in J^c}\left|\varphi\left(f,f_i\right)\right|^2-\Phi\left(\sum_{i\in J^c}\varphi\left(f,f_i\right)f_i\right).$$

PROOF. Let $\left\{\widetilde{f}_i\right\}_{i\in I}$ denote the dual frame of $\{f_i\}_{i\in I}$. Since $\{f_i\}_{i\in I}$ is a Parseval φ-frame, its frame operator equal identity operator and hence $\widetilde{f}_i = f_i$ for all $i \in I$. Employing Theorem 4.4 and the fact that $\{f_i\}_{i\in I}$ is a Parseval φ-frame yields

$$\begin{aligned}
\sum_{i\in J} |\varphi(f,f_i)|^2 - \Phi\left(\sum_{i\in J}\varphi(f,f_i)\,f_i\right) &= \sum_{i\in J} |\varphi(f,f_i)|^2 - \Phi(S_J f) \\
&= \sum_{i\in J} |\varphi(f,f_i)|^2 - \sum_{i\in I} |\varphi(S_J f, f_i)|^2 \\
&= \sum_{i\in J} |\varphi(f,f_i)|^2 - \sum_{i\in I} \left|\varphi\left(S_J f, \widetilde{f}_i\right)\right|^2 \\
&= \sum_{i\in J^c} |\varphi(f,f_i)|^2 - \sum_{i\in I} \left|\varphi\left(S_{J^c} f, \widetilde{f}_i\right)\right|^2 \\
&= \sum_{i\in J^c} |\varphi(f,f_i)|^2 - \Phi(S_{J^c} f) \\
&= \sum_{i\in J^c} |\varphi(f,f_i)|^2 - \Phi\left(\sum_{i\in J^c}\varphi(f,f_i)\,f_i\right).
\end{aligned}$$

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References

[1] P. Anielloa, G. Cassinellib, E. De Vitoc and A. Levrerod, Frames from imprimitivity systems. J. Math. Phys., 40(1999), 5184–5202.

[2] R. Balan, P. Casazza, D. Edidin and G. Kutyniok, A new identity for Parseval frames, Proc. Amer. Math. Soc., 135(2007), 1007–1015.

[3] R. Balan, P. Casazza, D. Edidin, G. Kutyniok, Decompositions of Frames and a New Frame Identity, Optics & Photonics 2005. International Society for Optics and Photonics, 2005.

[4] F. Bagarello, Intertwining operators between different Hilbert spaces: connection with frames, J. Math. Phys., 50(2009), 043509.

[5] J. Benedetto, A. Powell and O. Yilmaz, Sigma-delta ($\sum \Delta$) quantization and finite frames, IEEE Trans. Inform. Theory., 52(2006), 1990–2005.

[6] H. Bolcskei, F. Hlawatsch and H. G. Feichtinger, Frame-theoretic analysis of oversampled filter banks, IEEE Trans. Signal Process., 46(1998), 3256–3268.

[7] E. J. Candé and D. L. Donoho, New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities, Comm. Pure and Applied Mathematics., 57(2004), 219–266.

[8] P. G. Casazza, The art of frame theory, Taiwanese J. Math, 4(2000), 129–201.

[9] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[10] O. Christensen, Frames, Riesz bases, and discrete Gabor/wavelet expansions, Bull. Amer. Math. Soc., 38(2001), 273–291.

[11] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[12] I. Daubechies, A. Grossmann and Y. Meyer, Painless non-orthogonal expansions, J. Math. Phys., 27(1986), 1271–1283.

[13] L. Debnath and P. Mikusiski, Hilbert Spaces with Applications, Academic Press, 2005.

[14] R. J. Duffin and A. C. Schaeffer, A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., (1952), 341–366.

[15] P. Găvruţa, On some identities and inequalities for frames in Hilbert spaces, J. Math. Anal. Appl., 321(2006), 469–478.

[16] J. G. Murphy, Operator Theory and C^*-Algebras, Academic Press, San Diego, 1990.

[17] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.

[18] R. Young, An Introduction to Non-Harmonic Fourier Series, Academic Press, New York, 1980.