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The Nullity Conditions For Some Hamiltonian Properties Of Graphs*

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Abstract

The nullity of a graph is the multiplicity of the eigenvalue zero in the spectrum of the graph. Using the nullity of the complement of a graph, we in this note present sufficient conditions for some Hamiltonian properties of the graph.

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V(G), E(G))$, we use n and e to denote its order $|V(G)|$ and size $|E(G)|$, respectively. The complement of G is denoted by G^c. We use sK_1 to denote a graph that consists of s isolated vertices. A clique in a graph G is a subset S of $V(G)$ such that $G[S]$ is complete. The clique number of a graph G, denoted $\omega(G)$, is the number of vertices in a maximum clique of G. For two disjoint graphs G_1 and G_2, we use $G_1 \cup G_2$ and $G_1 \vee G_2$ to denote respectively the union and join of G_1 and G_2. The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ of the adjacency matrix $A(G)$ of a graph G are called the eigenvalues of G. The nullity of a graph G, denoted $\eta(G)$, is defined as the multiplicity of the eigenvalue zero in the spectrum of the graph G. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. It is known that if G is Hamiltonian (resp. traceable) then $c(G[V - S]) \leq |S|$ (resp. $c(G[V - S]) \leq |S| + 1$) for any vertex cut S of G, where $c(G[V - S])$ is the number of components of $G[V - S]$. The purpose of this note is to present the following nullity conditions for Hamiltonian and traceable graphs. The main results are as follows.

THEOREM 1. Let G be a k - connected graph of order n with $k \geq 2$. If $\eta(G^c) \geq n - k - 1$, then G is Hamiltonian or $K_k \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_{k+1}})$.

THEOREM 2. Let G be a k - connected graph of order n with $k \geq 1$. If $\eta(G^c) \geq n - k - 2$, then G is traceable or $K_k \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_{k+2}})$.

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REMARK 1. Let G be a graph obtained by adding one edge to $K_p \vee (p+1)K_1$, where $p \geq 2$. We can verify that G satisfies the conditions in Theorem 1 and therefore we can use Theorem 1 to decide G is a Hamiltonian graph. When $p \geq 3$, G does not satisfy Ore's condition or Dirac's condition (see [1]) . Thus we cannot use Ore's theorem or Dirac's theorem to decide whether G is Hamiltonian when $p \geq 3$.

REMARK 2. Let G be a graph obtained by adding one edge to $K_p \vee (p+2)K_1$, where $p \geq 1$. We can verify that G satisfies the conditions in Theorem 2 and therefore we can use Theorem 2 to decide G is a traceable graph. When $p \geq 2$, G does not satisfy Ore-type condition or Dirac-type condition for the traceability of a graph. Thus we cannot use Ore-type theorem or Dirac-type theorem for the traceability of a graph to decide whether G is traceable when $p \geq 2$.

2 Lemmas

In order to prove Theorems 1 and 2, we need the following results as our lemmas. Lemma 1 below is Corollary 2.5 on Page 62 in [2].

LEMMA 1. Let G be graph on n vertices and G is not isomorphic to nK_1. Then $\eta(G) + \omega(G) \leq n$.

Lemma 2 below is the Interlacing Theorem which can be found in [3] (Theorem 0.10 on Page 19).

LEMMA 2. Let G be a graph of order n with eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, and let H be an induced subgraph of G of order p with eigenvalues $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_p(H)$. Then

$$\lambda_{n-p+i}(G) \leq \lambda_i(H) \leq \lambda_i(G), 1 \leq i \leq p.$$

Lemma 3 below is Theorem 1 in [4, On page 403].

LEMMA 3. A graph has exactly one positive eigenvalue if and only if the non-isolated vertices of the graph form a complete multipartite graph.

3 Proofs

In this section, we prove Theorems 1 and 2.

PROOF OF THEOREM 1. Let G be a graph satisfying the conditions in Theorem 1. If G is complete, then G is Hamiltonian. From now on, we assume that G is not complete. Namely, G^c is not isomorphic to nK_1. Suppose that G is not Hamiltonian. Since $k \geq 2$, G contains a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G) \backslash V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s

between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. We use u_i^+ to denote the successor of u_i along the orientation of C, where $1 \leq i \leq s$. Notice that $x_0 u_i^+ \notin E$ for each i with $1 \leq i \leq s$ otherwise we can easily find a cycle C_1 which is longer than C. Notice also that $u_j^+ u_k^+ \notin E$ for each pair of j and k with $1 \leq j \neq k \leq s$ otherwise we can again find a cycle C_2 which is longer than C. Thus $S := \{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is independent in G. Therefore $\omega(G^c) = \alpha(G) \geq s+1 \geq k+1$. From Lemma 1, we have that $n = n-k-1+k+1 \leq n-k-1+s+1 \leq n-k-1+\alpha(G) \leq \eta(G^c) + \omega(G^c) \leq n$. So $\eta(G^c) = n-k-1$ and $\omega(G^c) = \alpha(G) = s+1 = k+1$.

Let H be a subgraph induced by S in G^c. Then H is a complete graph of order $k+1$. Thus the eigenvalues of H are k and -1 with multiplicity of k. From Lemma 2, we have $\lambda_1(G^c) \geq \lambda_1(H) = k$ and $-1 \geq \lambda_{n-k-1+i}$ for each i with $2 \leq i \leq k+1$. Since $\eta(G^c) = n-k-1$, we must have that $\lambda_j(G^c) = 0$ for each j with $2 \leq j \leq n-k$. Thus G^c is a graph with exactly one positive eigenvalue. From Lemma 3, we have that G^c consists of a complete multipartite graph, denoted $K_{r_1, r_2, ..., r_a}$, and a set, denoted X, of isolated vertices. Notice that $|X| \geq 1$ otherwise $G = (G^c)^c$ would be disconnected.

Now $G = G[X] \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_a})$, where $G[X]$ is complete in G. Choose one vertex $w_i \in V(K_{r_i})$ for each i with $1 \leq i \leq a$ to form a set $W := \{w_1, w_2, ..., w_a\}$. Then W is independent in G. Thus $a = |W| \leq \alpha(G) = k+1$. Since $|S| = |\{x_0, u_1^+, u_2^+, ..., u_s^+\}| = \alpha(G)$, S is an independent set of G with the maximum cardinality. Since each vertex in X is adjacent to all the other vertices in G, $S \cap X = \emptyset$. Notice also that $|S \cap V(K_{r_i})| \leq 1$ for each i with $1 \leq i \leq a$. Therefore

$$\begin{aligned} k+1 &= |S| = |S \cap V(G)| = |S \cap (X \cup V(K_{r_1}) \cup V(K_{r_2}) \cup \cdots V(K_{r_a}))| \\ &= |S \cap X| + |S \cap V(K_{r_1})| + |S \cap V(K_{r_2})| + \cdots + |S \cap V(K_{r_a})| \leq a. \end{aligned}$$

Hence $a = k+1$. Since G is k - connected and $G[V(G) - X]$ is disconnected, $|X| \geq k$. If $|X| \geq k+1$, then it is easy to see that G is Hamiltonian, leading to a contradiction. Thus $|X| = k$ and $G[X] = K_k$. Notice that G is not Hamiltonian since G has a vertex cut X such that $c(G[V(G)-X]) = |X|+1$. Therefore G is $K_k \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_{k+1}})$.

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Let G be a graph satisfying the conditions in Theorem 2. If G is complete, then G is traceable. From now on, we assume that G is not complete. Namely, G^c is not isomorphic to nK_1. Suppose that G is not traceable. Choose a longest path P in G and give an orientation on P. Let y and z be the two end vertices of P. Since G is not traceable, there exists a vertex $x_0 \in V(G) \backslash V(P)$. By Menger's theorem, we can find s $(s \geq k)$ pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(P)$. Let u_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Since P is a longest path in G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use u_i^+ to denote the successor of u_i along the orientation of P, where $1 \leq i \leq s$. Then similar proofs as the ones in the proof of Theorem 1 show that $S := \{x_0, y, u_1^+, u_2^+, ..., u_s^+\}$ is independent (otherwise G would have paths which are longer than P). Thus $\omega(G^c) = \alpha(G) \geq s+2 \geq k+2$. From Lemma 1, we have that $n = n-k-2+k+2 \leq n-k-2+s+2 \leq n-k-2+\alpha(G) \leq \eta(G^c) + \omega(G^c) \leq n$. Therefore $\eta(G^c) = n-k-2$ and $\omega(G^c) = \alpha(G) = s+2 = k+2$.

Let H be a subgraph induced by S in G^c. Then H is a complete graph of order $k+2$. Thus the eigenvalues of H are $k+1$ and -1 with multiplicity of $k+1$. From

Lemma 2, we have $\lambda_1(G^c) \geq \lambda_1(H) = k+1$ and $-1 \geq \lambda_{n-k-2+i}$ for each i with $2 \leq i \leq k+2$. Since $\eta(G^c) = n-k-2$, we must have that $\lambda_j(G^c) = 0$ for each j with $2 \leq j \leq n-k-1$. Thus G^c is a graph with exactly one positive eigenvalue. From Lemma 3, we have that G^c consists of a complete multipartite graph, denoted $K_{r_1, r_2, \ldots, r_b}$, and a set, denoted X, of isolated vertices. Notice that $|X| \geq 1$ otherwise $G = (G^c)^c$ would be disconnected.

Now $G = G[X] \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_b})$, where $G[X]$ is complete in G. Choose one vertex $w_i \in V(K_{r_i})$ for each i with $1 \leq i \leq b$ to form a set $W := \{w_1, w_2, \ldots, w_b\}$. Then W is independent in G. Thus $b = |W| \leq \alpha(G) = k+2$. Since

$$|S| = \left|\{x_0, y, u_1^+, u_2^+, ..., u_s^+\}\right| = \alpha(G),$$

S is an independent set of G with the maximum cardinality. Since each vertex in X is adjacent to all the other vertices in G, $S \cap X = \emptyset$. Notice also that $|S \cap V(K_{r_i})| \leq 1$ for each i with $1 \leq i \leq b$. Therefore

$$\begin{aligned} k+2 &= |S| = |S \cap V(G)| = |S \cap (X \cup V(K_{r_1}) \cup V(K_{r_2}) \cup \cdots V(K_{r_b}))| \\ &= |S \cap X| + |S \cap V(K_{r_1})| + |S \cap V(K_{r_2})| + \cdots + |S \cap V(K_{r_b})| \leq b. \end{aligned}$$

Hence $b = k+2$. Since G is k - connected and $G[V(G) - X]$ is disconnected, $|X| \geq k$. If $|X| \geq k+1$, then it is easy to see that G is traceable, leading to a contradiction. Thus $|X| = k$ and $G[X] = K_k$. Notice that G is not traceable since G has a vertex cut X such that $c(G[V(G)-X]) = |X|+2$. Therefore G is $K_k \vee (K_{r_1} \cup K_{r_2} \cup \cdots \cup K_{r_{k+2}})$.

This completes the proof of Theorem 2.

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