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A Survey On Coefficient Estimates For Carathéodory Functions*

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Abstract

The subclasses of univalent functions are closely related to the class of functions with positive real part, also known as the class of Carathéodory functions. Several information about the subclasses of univalent functions can be inferred, just by associating them with a suitable function in the Carathéodory class. The coefficient problems are among many problems which can be easily dealt with the help of the functions in this class. The present article is a survey trying to cover the important results heretofore known about the estimate on coefficients of the functions in the class of Carathéodory functions.

1 Introduction

Let $\mathcal{S}$ denote the class of univalent analytic functions f defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. For analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. It is well-known that the coefficient of the functions in the class $\mathcal{S}$ satisfy $|a_n| \leq n$ with equality in case of the Koebe function $k(z) = z/(1-z)^2$. This result was proposed by Bieberbach [9, 1916] in a footnote, and it took around 68 years before it is settled affirmatively by de Branges [10]. During this period, many researchers tried to prove it which led to the exploration of several subclasses of the class $\mathcal{S}$ introduced by imposing certain geometric conditions on the univalent functions. The class $\mathcal{S}^*$ of starlike functions is a collection of functions $f \in \mathcal{S}$ for which $\mathrm{Re}(zf'(z)/f(z)) > 0$ $(z \in \mathbb{D})$. The class $\mathcal{K}$ of convex functions is a collection of all those functions $f \in \mathcal{S}$ for which $\mathrm{Re}(1 + zf''(z)/f'(z)) > 0$ $(z \in \mathbb{D})$. The subclasses of univalent functions are closely associated with functions having positive real part. The functions with positive real part are useful in claiming some normalized analytic function to be univalent. For example, for some real α, if the function $f \in \mathcal{A}$ satisfies $\mathrm{Re}\left(e^{i\alpha} f'(z)\right) > 0$ for all z in a convex domain D, then f is univalent in D. This beautiful simple sufficient condition was independently established

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by Noshiro [49] and Warschawski [64], see also [17, Theorem 13, p. 88]. A particular case of this result, that is, for $\alpha = 0$ with convex domain $D = \mathbb{D}$, was established by Alexander [1] as early as in 1915, see also [17, Theorem 12, p. 88].

Let $\mathcal{P}$ be the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ with $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This class is known as the Carathéodory class or the class of functions with positive real part [12, 13], pioneered by Carathéodory. The function

$$\hat{p}_0(z) := \frac{1+z}{1-z}$$

belongs to the class $\mathcal{P}$ and plays a vital role similar to the Koebe function for the class $\mathcal{S}$. The function $\hat{p}_0$ maximizes $|p_n|$ in the class $\mathcal{P}$, but if $n \geq 2$, there are infinitely many other functions for which $p_n = 2$ and none of which can be obtained from others by a rotation. For example, if $n = 2$, then the function $(1+z^2)/(1-z^2) = 1+2\sum_{n=1}^{\infty} z^{2n} \in \mathcal{P}$ and $p_2 = 2$. For other values of $n \geq 2$, several examples may be constructed with the help of the result [17, Theorem 3, p. 80] (see also [12]). Since the function $\hat{p}_0$ is univalent in $\mathbb{D}$ and maps the unit disk to the right half-plane, it follows that an analytic function p normalized by $p(0) = 1$ belongs to the class $\mathcal{P}$ if and only if $p \prec \hat{p}_0$ in $\mathbb{D}$. We also note that $\mathcal{P}$ is a convex set and also a compact subset of the set of analytic functions in $\mathbb{D}$, see [19, Graham and Kohr, p. 28]. Now it is a simple matter to conclude that $f \in \mathcal{S}^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$ and also $f \in \mathcal{K}$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$. These implications reveals that information about the classes of starlike and convex functions can be drawn whenever the properties of functions in the class $\mathcal{P}$ are known. For example, suppose a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies $\operatorname{Re} f'(z) > 0$. Then $f'(z) = p(z) \in \mathcal{P}$, for some function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, and so $a_n = p_n/n$. Thus if bound on the coefficients of Carathéodory functions are known, then estimate on $|a_n|$ can be obtained easily. In 1970, Janowski [25] considered the class $\mathcal{P}_M := \{p \in \mathcal{P} : |p(z) - M| < M\}$ and investigated the bounds on $\operatorname{Re} p(z)$ and $\operatorname{Re}(zp'(z)/p(z))$ within the class $\mathcal{P}_M$. Further, these bounds were used by Janowski [25] to investigate the growth and distortion theorems and coefficient estimates for starlike functions associated with the class $\mathcal{P}_M$. Later, we shall see that for such a function, $|a_n| \leq 2/n$ holds (see, p. 8). The function in the class $\mathcal{P}$ need not be univalent as the function $p(z) = 1 + z^n$ is a member of the class $\mathcal{P}$ but if for $n \geq 2$, it is not univalent. The subclass of $\mathcal{P}$ consisting of functions $p \in \mathcal{P}$ satisfying $\operatorname{Re} p(z) > \alpha$ $(0 \leq \alpha < 1)$ is denoted by $\mathcal{P}(\alpha)$. One of the most general forms of the class $\mathcal{P}$ is the class

$$\mathcal{P}[A,B] = \left\{ p \in \mathcal{P} : p(z) \prec \frac{1+Az}{1+Bz} =: p_{A,B}(z), \ -1 \leq B < A \leq 1 \right\}.$$

This class was first introduced and studied by Janowski [26, p. 297], in 1973. In particular, $\mathcal{P}[1-2\alpha, -1] =: \mathcal{P}(\alpha)$ $(0 \leq \alpha < 1)$ and $\mathcal{P}[1,-1] = \mathcal{P}$. He also introduced the subclasses, known as the classes of the Janowski starlike and convex functions, respectively as follows:

$$\mathcal{S}^*[A,B] := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \in \mathcal{P}[A,B] \right\},$$

$$\mathcal{K}[A,B] := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A,B] \right\}.$$

Later, for fixed $n \in \mathbb{N}$, a more general class

$$\mathcal{P}_n[A,B] = \big\{p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots : p(z) \prec p_{A,B}(z)\big\}$$

was introduced by Ravichandran *et al.* [54], in 1997. For convenience, we shall adopt the following notations wherever needed in this manuscript: $\mathcal{P}_1[A,B] = \mathcal{P}[A,B]$, $\mathcal{P}_n(\alpha) := \mathcal{P}_n[1-2\alpha,-1]$ and $\mathcal{P}_n := \mathcal{P}_n[1,-1]$.

In Section 2, several geometric properties such as the growth and distortion theorems, and many other results for the class of Carathéodory functions and its subclasses are covered. In Section 3, special attention is given to recall the results related to the coefficient estimates in sequel. We also mention their use in handling the coefficient problems related to some subclasses of univalent analytic functions. The results covered in this article, with few exceptions, can be found elsewhere (see the cited references), and therefore their proofs are omitted.

2 Some Geometric Properties

Let $\mathcal{B}$ be the class of Schwarz functions, that is, $w \in \mathcal{B}$ if and only if w is an analytic function with $w(0) = 0$ and $|w(z)| < 1$ on $\mathbb{D}$. The following correspondence between the classes $\mathcal{B}$ and $\mathcal{P}$ holds [19, p. 28]:

$$p \in \mathcal{P} \text{ if and only if } w(z) = \frac{p(z)-1}{p(z)+1} \in \mathcal{B}. \tag{1}$$

Therefore the properties of the functions in the class $\mathcal{P}$ can be inferred from those of the class $\mathcal{B}$, and vice-versa. Later, in Section 3, we shall make use of this relation to establish the relation between coefficients of functions in the classes $\mathcal{B}$ and $\mathcal{P}$.

The following result gives a sufficient condition for a function to be in the class $\mathcal{P}$.

THEOREM 1 ([17, Theorem 1, p. 79]). Let μ is a non-decreasing function on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 2\pi$. Then the function defined by

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} p_0(ze^{-it})d\mu(t) \quad (z \in \mathbb{D}), \tag{2}$$

is in the class $\mathcal{P}$.

In 1911, Herglotz [23] proved that (2) is necessary as well. That is, for each $p \in \mathcal{P}$, there is an associated non-decreasing function $\mu(t)$ for which $\int_0^{2\pi} d\mu(t) = 2\pi$ and (2) holds. He gave a representation formula in terms of Stieltjes integrals (probability measures) for a function $p \in \mathcal{P}$ as

$$p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t) \quad (z \in \mathbb{D}), \tag{3}$$

where $\mu(t)$ is a non-decreasing function on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$. It gives a one-to-one correspondence between the functions in class $\mathcal{P}$ and the set of probability measures on $\partial\mathbb{D}$. This fundamental result leads to integral representation theorems for several subclasses of $\mathcal{S}$. The symbol μ, defined so, is called the Herglotz measure of the function $p \in \mathcal{P}$.

The following theorem gives a number of transformations under which the class $\mathcal{P}$ is preserved.

THEOREM 2 ([17, Theorem 2, p. 79]). Let the functions h, h_1 and h_2 be in $\mathcal{P}$. Then each of the function H, under the subjected condition noted, is also in $\mathcal{P}$.

1. $H(z) = h(e^{it}z)$ $(t \in \mathbb{R})$,
2. $H(z) = h(z)^t$ or $H(z) = h(tz)$ $(t \in [-1, 1])$,
3. $H(z) = 1/h(z)$,
4. $H(z) = (h_1(z))^{t_1}(h_2(z))^{t_2}$ $(t_1,\ t_2,\ t_1 + t_2 \in [0, 1])$,
5. $H(z) = (1/a)h\left((z + \lambda)/(1 + \bar{\lambda}z)\right) - ib$ $(h(z) = a + ib,\ \lambda \in \mathbb{D})$,
6. $H(z) = (h(z) + ib)/(1 + ibh(z))$ $(b \in \mathbb{R})$.

Likewise there are several other transformations under which the class $\mathcal{P}$ is preserved, see [33]. These transformations are very useful in establishing the estimates on the successive coefficients. The following equivalence was proved by using the Helly selection theorem and can be found in [51].

THEOREM 3. [51, Theorem 2.4, p. 40] Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be analytic in $\mathbb{D}$. Then the following are equivalent:

1. the function $p \in \mathcal{P}$,
2. there exists an increasing function $\mu(t)$ on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$ such that (3) holds,
3. for $m = 1, 2, 3, \cdots$, $\sum_{k=0}^{m} \sum_{l=0}^{m} p_{k-l}\lambda_k\bar{\lambda}_l$ $(\lambda_i \in \mathbb{C},\ i = 0, 1, 2 \cdots m)$ is non-negative, where we have adopted the convention $p_{-k} = \bar{p_k}$ $(k \geq 1)$.

The representation in (3) immediately gives the following growth and distortion estimates, respectively:

$$\frac{1 - |z|}{1 + |z|} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1 + |z|}{1 - |z|} \text{ and } |p'(z)| \leq \frac{2 \operatorname{Re} p(z)}{1 - |z|^2} \leq \frac{2}{(1 - |z|)^2}. \quad (4)$$

In both cases, equality holds in case of the function $\hat{p}_0$. A more general result related to the upper bound on modulus of higher derivatives of the Carathéodory functions is given in [17, Theorem 6, p. 83]:

$$|p^{(k)}(z)| \leq \frac{2(k!)}{(1 - |z|)^{k+1}} \quad (k \geq 0).$$

In particular, for $k = 1$, the above inequality gives the growth estimate for the Carathéodory functions as in (4). It is easy to deduce that for each fixed $z \in \mathbb{D}$, the function $p \in \mathcal{P}$ lies in the disk

$$\left| p(z) - \frac{1+|z|^2}{1-|z|^2} \right| \le \frac{2|z|}{1-|z|^2}. \tag{5}$$

We can obtain the growth estimate from (5). Further, from the growth estimate, we can infer that the class $\mathcal{P}$ is locally uniformly bounded, and thus it is a normal family. In fact this family is compact, see [19, Corollary 2.1.6, p. 32].

The Herglotz representation formula [17, p. 172] for the function $p \in \mathcal{P}(\alpha)$ is given by

$$p(z) = \int_0^{2\pi} \frac{1+(1-2\alpha)ze^{-it}}{1-ze^{-it}} d\mu(t) = \int_0^{2\pi} p_\alpha(ze^{-it}) d\mu(t) \quad (z \in \mathbb{D}), \tag{6}$$

where $\mu(t)$ is a non-decreasing function on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$, and the function $p_\alpha(z) := (1+(1-2\alpha)z)/(1-z)$. The representation formula (6) gives the growth and distortion estimates for function $p \in \mathcal{P}(\alpha)$ as follows:

$$\frac{1-(1-2\alpha)|z|}{1+|z|} \le \operatorname{Re} p(z) \le |p(z)| \le \frac{1+(1-2\alpha)|z|}{1-|z|} \quad \text{and} \quad |p'(z)| \le \frac{2(\operatorname{Re} p(z) - \alpha)}{1-\alpha} \frac{1}{1-|z|^2}.$$

These can also be easily obtained by noting that the function $h(z) = (p(z)-\alpha)/(1-\alpha)$ belongs to the class $\mathcal{P}$, see [17, Problem 43, p. 105].

In 1963, MacGregor [42, Lemma 1, p. 514], proved that if $p \in \mathcal{P}_n$, then

$$\left| \frac{zp'(z)}{p(z)} \right| \le \frac{2n|z|^{n-1}}{1-|z|^{2n}}.$$

He also gave an alternate and short proof of the second inequality in (4) and used these results to discuss radius problems. Later, Shah [62, 1972] gave a generalization to this MacGregor's result as stated in the following Theorem 4. Shah also discussed radius problems for starlike functions of order α and many other classes.

THEOREM 4 ([62, Lemma 2, p. 239]). If $p \in \mathcal{P}_n(\alpha)$, then

$$\left| \frac{zp'(z)}{p(z)} \right| \le \frac{2(1-\alpha)n|z|^n}{(1-|z|^n)(1+(1-2\alpha)|z|^n)}.$$

THEOREM 5 ([62, Lemma 3, p. 240]). If $p \in \mathcal{P}_n(\alpha)$, then

$$\operatorname{Re} p(z) \ge \frac{1+(2\alpha-1)|z|^n}{1+|z|^n}.$$

In 1973, Janowski [26, Lemma 1, p. 298] proved that a function p in the class $\mathcal{P}[A, B]$ also belongs to the class $\mathcal{P}$ if and only if

$$p(z) = \frac{(1+A)l(z)+1-A}{(1+B)l(z)+1-B}$$

for some $l \in \mathcal{P}$. An equivalent Herglotz representation formula for function $p \in \mathcal{P}[A,B]$ is given by

$$p(z)=\int_0^{2\pi}\frac{1+Aze^{-it}}{1+Bze^{-it}}d\mu(t)=\int_0^{2\pi}p_{A,B}(ze^{-it})d\mu(t)\quad (z\in\mathbb{D}).$$

Janowski also deduced a general version of (5):

$$\left|p(z)-\frac{1-AB|z|^2}{1-B^2|z|^2}\right|\le\frac{(A-B)|z|}{1-B^2|z|^2}. \tag{7}$$

Janowski used the variational method to obtain the bounds on $\mathrm{Re}(p(z)+zp'(z)/p(z))$, $\mathrm{Re}(zp'(z)/p(z))$ and $\mathrm{Re}\,p(z)$ within the class $\mathcal{P}[A,B]$. These results then are utilized by him to discuss the radius of convexity of Janowski starlike functions. Geometrically, $p\in\mathcal{P}[A,B]$ if and only if $p(0)=1$ and $p(\mathbb{D})$ is inside the open disk centered on the real axis with diameter end points $d_1 := (1-A)/(1-B) = p_{A,B}(-1)$ and $d_2 := (1+A)/(1+B) = p_{A,B}(1)$. In other words, given any pair d_1, d_2 with $0 < d_1 < 1 < d_2$, there are real numbers A and B with $-1 \le B < A \le 1$ such that $p(\mathbb{D})$ is in the open disk with d_1 and d_2 as diameter end points. For special choices of A and B, the class $\mathcal{P}[A,B]$ reduces to several familiar subclasses, see [18, p. 111–112]. The class for which the derivative of a normalized analytic function belongs to the class $\mathcal{P}[A,B]$ was studied by Juneja and Mogra [27], in 1979. For more results related to the class $\mathcal{P}[A,B]$, the reader can refer to [26]. Observing closely the Herglotz representation of the Carathéodory functions, Kaczmarski [28] introduced the class $\tilde{\mathcal{P}}[A,B]$ as the set of functions p of the form

$$p(z)=\int_0^{2\pi}\frac{1+Aze^{it}}{1+Bze^{it}}d\mu(t)\quad (z\in\mathbb{D}),$$

where μ is a non-decreasing function on $[0,2\pi]$ with $\mu(2\pi)-\mu(0)=1$ and $-1\le B < A \le 1$. Kaczmarski also discussed the radius of convexity problem for the set $\tilde{\mathcal{P}}[A,B]$ and proved that if $|B|<1$, then $\tilde{\mathcal{P}}[A,B]\neq\mathcal{P}[A,B]$. Later, in 1993, Ma and Owa [45] gave another generalization of (5) as follows:

THEOREM 6 ([45]). (see also, [6, Lemma 7, p. 489]) Let $p\in\mathcal{P}$. Then, for $0<\gamma\le 1$,

$$\left|\frac{1+|z|^2}{1-|z|^2}-p^{\gamma}(z)\right|\le\frac{1+|z|^2}{1-|z|^2}-\left(\frac{1-|z|}{1+|z|}\right)^{\gamma}.$$

The result is sharp.

In 1997, Ravichandran *et al.* [54], proved the ensuing result which is a generalization of (7).

THEOREM 7 ([54, Lemma 2.1, p. 267]). Let $p\in\mathcal{P}_n[A,B]$. Then

$$\left|p(z)-\frac{1-AB|z|^{2n}}{1-B^2|z|^{2n}}\right|\le\frac{(A-B)|z|^n}{1-B^2|z|^{2n}}.$$

In particular, for $p \in \mathcal{P}_n(\alpha)$, we have

$$\left|p(z) - \frac{1+(1-2\alpha)|z|^{2n}}{1-|z|^{2n}}\right| \leq \frac{2(1-\alpha)|z|^n}{1-|z|^{2n}}.$$

They dealt with the properties of convexity and starlikeness of positive order, uniform convexity and a special type of starlikeness that is in a natural way corresponds to uniform convexity. In 2011, for any complex numbers A and B with $A \neq B$ and $|B| \leq 1$, Theorem 7 was further generalized by Ali *et al.* [3, p. 256].

In 2015, for $-1 \leq B < A \leq 1, 0 \leq \alpha < 1$ and $0 < \beta \leq 1$, Arif *et al.* [7] introduced a more general class $\mathcal{P}_\beta[A,B,\alpha]$ consisting of analytic functions of the form p with $p(0) = 1$ and satisfying

$$p(z) \prec \alpha + (1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^\beta =: p^{\alpha,\beta}_{A,B}(z).$$

Clearly $p^{0,1}_{A,B}(z) =: p_{A,B}(z)$ and $\mathcal{P}_1[A,B,0] =: \mathcal{P}[A,B]$. Let $\mathcal{P}_\beta[A,B,0] =: \mathcal{P}_\beta[A,B]$. The Herglotz representation for functions $p \in \mathcal{P}_\beta[A,B,\alpha]$ is given by (see, [7, Eqn. (1.4), p. 62])

$$p(z) = \alpha + \frac{1-\alpha}{2\pi}\int_0^{2\pi}\left(\frac{1+Aze^{-it}}{1+Bze^{-it}}\right)^\beta d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} p^{\alpha,\beta}_{A,B}(ze^{-it})d\mu(t) \quad (z \in \mathbb{D}),$$

where μ is a non-decreasing function on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 2\pi$. In addition to these, they also established the growth estimate.

THEOREM 8 ([7, Lemma 2.1, p. 63]). Let $p \in \mathcal{P}_\beta[A,B]$. Then

$$\left(\frac{1-A|z|}{1-B|z|}\right)^\beta \leq \operatorname{Re} p(z) \leq |p(z)| \leq \left(\frac{1+A|z|}{1+B|z|}\right)^\beta.$$

The result is sharp with extremal function $p^{0,\beta}_{A,B}(z)$.

THEOREM 9 ([7, Lemma 2.3, p. 63]). Let $p(z) = 1+p_1z+p_2z^2+\cdots \in \mathcal{P}_\beta[A,B,\alpha]$. Then

$$|p_n| \leq \beta(1-\alpha)(A-B).$$

The result is sharp with extremal function $p^{\alpha,\beta}_{A,B}(z)$.

It should be noted that the above two theorems, in particular cases, give the growth theorem and coefficient bounds for functions in the class $\mathcal{P}[A,B]$. Besides these results, Arif *et al.* [7] also investigated the properties of certain classes related to the class of generalized Janowski functions and the class of functions with bounded boundary rotation. For several other results on functions with positive part, the reader may refer to the works by Robertson [57, 56, 58], Bernardi [8], Ruscheweyh and Singh [60] and the related references cited therein.

In the next section we shall focus on coefficient estimates for the Carathéodory functions.

3 Coefficient Estimates

The class $\mathcal{P}$ of functions with positive real part plays a significant role in Geometric Function Theory. Its importance can be seen from the fact that most of the subclasses of the class of univalent functions are associated with this class $\mathcal{P}$. In this section, several coefficient estimates for functions with positive real part are discussed. As mentioned earlier, since the proof of the results covered here can be found in the cited references, their proofs are omitted.

For brevity, let us assume that the function $w \in \mathcal{B}$ has the form $w(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots$ and from now on, unless otherwise stated specifically, we shall assume that a function $p \in \mathcal{P}$ is of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. In view of the interconnection in (1), we see that the coefficients of w and p are related by the following relations, see [17, Theorem 8, p. 58-59].

$$p_n := \begin{vmatrix} b_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ b_2 & b_1 & -1 & 0 & \cdots & 0 & 0 \\ b_3 & b_2 & b_1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_1 & -1 \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_2 & b_1 \end{vmatrix}$$

and

$$b_n := \frac{(-1)^{n+1}}{2^n} \begin{vmatrix} p_1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ p_2 & p_1 & 2 & 0 & \cdots & 0 & 0 \\ p_3 & p_2 & p_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & p_{n-4} & \cdots & p_1 & 2 \\ p_n & p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_2 & p_1 \end{vmatrix}.$$

The following are some special cases of the above:

$$p_1 = 2b_1,\ p_2 = 2b_2 + 2b_1^2,\ p_3 = 2b_3 + 4b_1 b_2 + 2b_1^3,\ p_4 = 2b_4 + 4b_1 b_3 + 2b_2^2 + 6b_1^2 b_2 + 2b_1^4 \quad (8)$$

and

$$b_1 = \frac{p_1}{2},\ b_2 = \frac{2p_2 - p_1^2}{4},\ b_3 = \frac{4p_3 - 4p_1 p_2 + p_1^3}{8},\ b_4 = \frac{8p_4 - 8p_1 p_3 - 4p_2^2 + 6p_1^2 p_2 - p_1^4}{16}. \quad (9)$$

Thus estimate on the coefficients of the Schwarz functions and the Carathéodory functions can be obtained if the bound of any of them are known. Now by applying the triangle inequality and using the well-known fact $|b_i| \le 1$ $(i \in \mathbb{N})$, it follows from (9) that

$$|p_1| \le 2,\ |2p_2 - p_1^2| \le 4,\ |4p_3 - 4p_1 p_3 + p_1^3| \le 8 \text{ and } |8p_4 - 8p_1 p_3 - 4p_2^2 + 6p_1^2 p_2 - p_1^4| \le 16.$$

We now relook the Herglotz representation formula (3). It at once gives

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu(t),$$

which shows that $|p_n| \leq 2$. This sharp result was proved by Carathéodory [12] (see also [17, Theorem 3, p. 80]). Moreover, if $\eta = e^{2\pi i/n_0}$ and

$$F(z) = \sum_{k=1}^{n_0} \mu_k \frac{1+\eta^k z}{1-\eta^k z} = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

where $\mu_k \geq 0$, for $k = 1, 2, 3, \ldots n_0$ and $\sum_{k=1}^{n_0} \mu_k = 1$, then $F \in \mathcal{P}$ and $p_{n_0} = 2$. This result is popularly known as the Carathéodory theorem. There are many proofs available of this result in literature, see [17, 24, 55]. Another useful inequality (see [19]) for Carathéodory functions is

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{10}$$

Now we can use the estimate $|p_n| \leq 2$ to derive the following from (8):

$$|b_1| \leq 1,\ |b_2 + b_1^2| \leq 1,\ |b_3 + 2b_1b_2 + b_1^3| \leq 1 \text{ and } |b_4 + 2b_1b_3 + b_2^2 + 3b_1^2b_2 + b_1^4| \leq 1.$$

Moreover, for normalized analytic functions satisfying $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{D})$, we have $|a_n| = |p_n|/n \leq 2/n$.

In 1973, by using the Grunsky-Nehari's [47] inequalities and their extensions by Schiffer and Tammi [61], Leutwiler and Schober [35] investigated many results related to the functions with positive real part. In particular, they obtained the conditions that Toeplitz [63] gave as an algebraic characterization of Carathéodory's [12] geometric description of the coefficient region for functions with positive real part. They proved the following results:

THEOREM 10 ([35, Theorem 1, p. 131]). Let $p \in \mathcal{P}$. Then

$$\left| \sum_{m,n=0}^{n_0} p_{m+n} x_m x_n \right| \leq \sum_{m,n=0}^{n_0} p_{m-n} x_m \bar{x}_n \quad (x_0, x_1, \ldots x_{n_0} \in \mathbb{C}),$$

where by definition, $p_0 = 2$ and $p_{-n} = \bar{p}_n$ $(n = 1, 2, 3, \ldots)$.

By taking the extreme value of the inequality in the above theorem with respect to x_0, we get the following result.

THEOREM 11 ([35, Theorem 2, p. 132]). Let $p \in \mathcal{P}$. Then

$$\left| \sum_{m,n=1}^{n_0} \left(p_{m+n} - \frac{p_m p_n}{2} \right) x_m x_n \right| \leq \sum_{m,n=1}^{n_0} \left(p_{m-n} - \frac{p_m \bar{p}_n}{2} \right) x_m \bar{x}_n \quad (x_1, \ldots x_{n_0} \in \mathbb{C}).$$

By choosing certain special values for x_i, we can obtain some classical results for functions with positive real part. In particular, the choice $x_k = 1$ and $x_n = 0$ $(n \neq k)$ gives the following result.

THEOREM 12 ([35, Corollary 1, p. 133]). Let $p \in \mathcal{P}$. Then

$$\left|p_{2k} - \frac{p_k^2}{2}\right| \le 2 - \frac{|p_k^2|}{2}.$$

The equality is attained in case of the function

$$p(z) = \frac{2 + (p_k + \eta \bar{p}_k)z^k + 2\eta z^{2k}}{2 - (p_k - \eta \bar{p}_k)z^k - 2\eta z^{2k}}, \quad |\eta| \le 1.$$

We can observe that within the class $\mathcal{B}$, the functionals $|b_4 - 2b_3b_1 + b_2b_1^2|$ and $|b_4+2b_3b_1+b_2b_1^2|$ have the same upper bound because if $w \in \mathcal{B}$, then $w_1(z) = -w(-z) \in \mathcal{B}$. Further computation using (9) shows that $2(b_4 + 2b_3b_1 + b_2b_1^2) = p_4 - p_2^2/2$. From Theorem 12, since $|p_2| \le 2$, we have $|p_4 - p_2^2/2| \le 2 - |p_2|^2/2 \le 2$. Thus, we have $|b_4 + 2b_3b_1 + b_2b_1^2| \le 1$. Dorff and Szynal [14] used this result to obtain the sharp bound for the first three consecutive Schwarzian derivatives of higher order. Another important results in this direction related to the estimate on the functional $\Psi(\mu, \nu, w) := |b_3 + \mu b_1b_2 + \nu b_1^3|$, for any real numbers μ and ν and Schwarz function w with complex coefficients b_i, was investigated by Prokhorov and Szynal [52]. For details, we refer the readers to the concerned cite papers [14, 52]. Later, in 2001, Kiepiela *et al.* [30] investigated the sharp bound on $\Psi(\mu, \nu, w)$, by assuming the Schwarz function w with real coefficients b_i. Later, the result investigated in [52, Lemma 2, p. 128] was employed by Ali *et al.* [4] to discuss the sharp coefficient estimate for certain classes of $p-$valent analytic functions.

To discuss the bound on the second Hankel determinant for $p-$valent starlike functions of order α, Hayami and Owa gave the following result which generalizes the bound on the nth coefficient of the Carathéodory functions.

THEOREM 13 ([21, Lemma 2, p. 33]). Let $p(z) = b_0 + p_1z + p_2z^2 + \cdots$ with $\operatorname{Re} p(z) > \alpha$. Then for $\alpha \in [0, b_0)$, we have $|p_n| \le 2(b_0 - \alpha)$ with equality in case of the function p defined by

$$p(z) = \frac{b_0 + (b_0 - 2\alpha)z}{1 - z}.$$

PROOF. The result follows immediately by noting that the transformation

$$q_{b_0,\alpha}(z) := \frac{p(z) - \alpha}{b_0 - \alpha} = 1 + \sum_{k=1}^{\infty} \frac{p_k}{b_0 - \alpha} \tag{11}$$

is in $\mathcal{P}$ and the fact that the modulus of the coefficients of Carathéodory functions are bounded by 2.

Another generalization of the estimate $|p_n| \le 2$ was given by Peng, in 2010.

THEOREM 14. [50, Lemma 1, p. 1450] Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \prec (1 + Az)/(1 + Bz)$. Then $|p_n| \le B - A$ $(-1 \le A < B \le 1)$. The inequality is sharp.

From (11), it can be noted that the function $q_{1,\alpha}$ belongs to the class $\mathcal{P}$, where

$$q_{1,\alpha}(z) = \frac{p(z)-\alpha}{1-\alpha} = 1 + \sum_{k=1}^{\infty} \frac{p_k}{1-\alpha}.$$

Noting this, recently Liu *et al.* [38] (see also [39]) established the following result for the class $\mathcal{P}(\alpha)$.

THEOREM 15 ([38, Lemma 2.2]). Let $p \in \mathcal{P}(\alpha)$. Then we have

$$\left| p_2 - \frac{p_1^2}{2(1-\alpha)} \right| \le 2(1-\alpha) - \frac{|p_1|^2}{2(1-\alpha)},$$

and

$$\left| p_3 - \frac{p_1 p_2}{1-\alpha} + \frac{p_1^3}{4(1-\alpha)^2} \right| \le 2(1-\alpha) - \frac{|p_1|^2}{2(1-\alpha)}.$$

Furthermore, $|p_1| \le 2(1-\alpha)$ and

$$\left| p_2 - \frac{p_1^2}{1-\alpha} \right| \le 2(1-\alpha).$$

In 1969, Livingston proved the following result and used a particular case of this to investigate the sharp bound on the coefficients of a certain subclass of multivalent close-to-convex functions.

THEOREM 16 ([41, Lemma 1, p. 546]). Let $p(z) = b_0 + p_1 z + p_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > 0$. Then for $1 \le s \le n-1$

$$\left| \frac{p_n}{b_0} - \frac{p_s p_{n-s}}{b_0^2} \right| \le 2 \left| \frac{\operatorname{Re} b_0}{b_0} \right| \le 2.$$

These inequalities are sharp for all n and s, equality being attained in case of the function

$$p(z) = (\operatorname{Re} b_0)\frac{1+z}{1-z} + i \operatorname{Im} b_0 \ \ (\operatorname{Re} b_0 > 0).$$

This inequality was later used by Libera and Zlotkiewicz [37] in their study of the coefficient problem for the inverse of convex functions. In particular, if $b_0 = 1$, then the above result takes the following form which is popularly known as Livingston inequality.

THEOREM 17. Let $p \in \mathcal{P}$. Then for $1 \le s \le n-1$

$$|p_n - p_s p_{n-s}| \le 2. \tag{12}$$

These inequalities are sharp for all n and s, equality being attained in case of the function

$$p(z) = \frac{1+z}{1-z}.$$

REMARK 1. Writing $p_n - p_s^2 p_{n-2s} = p_n - p_s p_{n-s} + p_s(p_{n-s} - p_s p_{n-2s})$ and using the triangle inequality, we have $|p_n - p_s^2 p_{n-2s}| \leq |p_n - p_s p_{n-s}| + |p_s(p_{n-s} - p_s p_{n-2s})|$. Now by using the inequality in 12 and $|p_n| \leq 2$, we have $|p_n - p_s^2 p_{n-2s}| \leq 6$, see [65, Lemma 1.4, p. 3].

Theorem 16 has another general form due to Hayami and Owa [22] in 2010. They proved that:

THEOREM 18 ([22, Lemma 2.4, p. 2576]). Let $p(z) = b_0 + p_1 z + p_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > 0$. Then for $1 \leq s \leq n-1$

$$\left|\nu \frac{p_n}{b_0} - \frac{p_s p_{n-s}}{b_0^2}\right| \leq \begin{cases} 2\frac{\operatorname{Re} b_0}{|b_0|}\sqrt{\nu^2 + 4\left(\frac{\Re(b_0)}{|b_0|}\right)^2(1-\nu)} \leq 2(2-\nu), & \nu \leq 1; \\ 2\nu\frac{\operatorname{Re} b_0}{|b_0|} \leq 2\nu, & \nu \geq 1. \end{cases}$$

Equality is attained for

$$p(z) = (\operatorname{Re} b_0)\frac{1+z^d}{1-z^d} + i\operatorname{Im} b_0 \;\; (\mu \leq 1) \;\text{ and }\; p(z) = (\operatorname{Re} b_0)\frac{1+z^l}{1-z^l} + i\operatorname{Im} b_0 \quad (\mu \geq 1),$$

Set $h(z) = p(z) - \alpha = 1 - \alpha + \sum_{k=1}^{\infty} p_k z^k$ $(0 \leq \alpha < 1)$. Then $\operatorname{Re} h(z) > 0$ and $b_0 = 1 - \alpha > 0$; and thus the following result can be deduced:

COROLLARY 1 ([22, Corollary 2.5, p. 2577]). Let $p \in \mathcal{P}(\alpha)$. Then for $1 \leq s \leq n-1$

$$|(1-\alpha)\mu p_n - p_s p_{n-s}| \leq \begin{cases} 2(1-\alpha)^2(2-\mu), & \mu \leq 1; \\ 2(1-\alpha)^2\mu, & \mu \geq 1. \end{cases}$$

Equality is attained for

$$p(z) = \frac{1+(1-2\alpha)z^d}{1-z^d} \quad (\mu \leq 1) \;\text{ and }\; p(z) = \frac{1+(1-2\alpha)z^l}{1-z^l} \quad (\mu \geq 1),$$

Hayami and Owa [22] further used these results to obtain the sharp estimate for the generalized Hankel determinant.

In 2016, Efraimidis [16] generalized Theorem 17 and gave a much simpler proof than that of Livingston [41]. Before presenting his results we need some definitions and notations. Let $n \in \mathbb{N}$ and denote by $U_n = \{e^{2k\pi i/n} : k = 1, 2, 3, \ldots, n\}$, the set of n-th root of unity. For $n = 0$ we write $U_0 = \partial\mathbb{D} = \mathbf{T}$. Let $(X, \mathbf{T})$ be a topological space. The support of a measure μ is defined as the set of all points $x \in X$ for which every open neighbourhood N_x of x has a positive measure. The symbol μ denote the Herglotz measure of p and we write $supp(\mu)$ for its support which is given by

$$supp(\mu) := \{x \in X : x \in N_x \;\text{ with }\; \mu(N_x) > 0\}.$$

He proved that the following result holds for all ν such that $|1 - 2\nu| \leq 1$.

THEOREM 19 ([16, Theorem 1, p. 370]). Let $p \in \mathcal{P}$. Then for $\nu \in \mathbb{C}$ and for all integers n and s with $1 \le s \le n-1$

$$|p_n - \nu p_s p_{n-s}| \le 2 \max\{1, |1-2\nu|\}.$$

Let ν be the Herglotz measure of p. In the case $|1-2\nu| < 1$, equality holds if and only if $p_k = 0$ and $supp(\mu) \subseteq e^{i\varphi} U_n$, for some $\varphi \in [0, 2\pi)$. In the case $|1-2\nu| > 1$, equality holds if and only if $supp(\mu) \subseteq e^{i\theta} U_k \cap e^{i\varphi} U_n$ for some $\theta, \varphi \in [0, 2\pi)$. In the case $|1-2\nu| = 1$, equality holds if $supp(\mu)$ consists of one point.

For $\nu \in \mathbb{C}$ and $p \in \mathcal{P}$, Efraimidis [16] defined the determinant

$$A_{k,n}(\nu) := \begin{vmatrix} p_{n+k} & p_{n+k-1} & p_{n+k-1} & \cdots & p_{n+1} & p_n \\ \nu p_1 & 1 & 0 & \cdots & 0 & 0 \\ \nu p_2 & \nu p_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu p_{k-1} & \nu p_{k-2} & \nu p_{k-3} & \cdots & 1 & 0 \\ \nu p_k & \nu p_{k-1} & \nu p_{k-2} & \cdots & \nu p_1 & 0 \end{vmatrix}$$

and proved the following:

THEOREM 20 ([16, Theorem 2, p. 370]). If $p \in \mathcal{P}$, then for all $\nu \in \mathbb{C}$, $k \ge 0$ and $n \ge 1$, we have the following sharp inequality

$$|A_{k,n}(\nu)| \le 2 \max\left\{1; |1-2\nu|^k\right\}.$$

Let μ be the Herglotz measure of p. In the case $|1-2\nu| < 1$, equality holds if and only if $p_k = 0$ and $supp(\mu) \subseteq e^{i\varphi} U_{n+k}$, for some $\varphi \in [0, 2\pi)$ and $p_1 = p_2 = p_3 = \cdots = p_k = 0$. In the case $|1-2\nu| \ge 1$, if $supp(\mu)$ consists of one point, then equality holds.

Both Theorems 19 and 20 have a similar version for non-normalized functions $p(z) = b_0 + \sum_{n=1}^{\infty} p_n z^n$ with positive real part. For such a function p, let $b_0 = x + iy$ $(x > 0)$ and $q(z) = (p(z) - iy)/x$, which is obviously a function in $\mathcal{P}$. To this q, having coefficients $q_0 = 1$, $q_n = p_n/x$ $(n \in \mathbb{N})$, we can apply Theorems 19 and 20. Then multiply both inequalities by $x/|b_0|$ and set $\nu x/b_0$ in place of ν. This gives

$$\left|\frac{p_n}{b_0} - \nu \frac{p_k p_{n-k}}{b_0^2}\right| \le 2 \frac{\operatorname{Re} b_0}{|b_0|} \max\left\{1; \left|1 - 2\nu \frac{\operatorname{Re} b_0}{b_0}\right|\right\}$$

and

$$\left|A^*_{k,n}(\nu)\right| \le 2 \frac{\operatorname{Re} b_0}{|b_0|} \max\left\{1; \left|1 - 2\nu \frac{\operatorname{Re} b_0}{b_0}\right|^k\right\}.$$

Here $A^*_{k,n}(\nu)$ is the modified form of $A_{k,n}(\nu)$ with p_j being replaced by p_j/b_0 for all j.

One of the most popular results in this direction is due to Ma and Minda [44]. This result is very useful in deriving the Fekete-Szegö coefficient inequality.

THEOREM 21. [44, Lemma 1, p. 162] If $p \in \mathcal{P}$, then

$$|p_2 - \nu p_1^2| \le \begin{cases} -4\nu + 2 & (\nu \le 0), \\ 2 & (0 \le \nu \le 1), \\ 4\nu - 2 & (\nu \ge 1). \end{cases}$$

When $\nu < 0$ or $\nu > 1$, equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, then equality holds if and only if

$$p(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \quad (0 \le \gamma \le 1, z \in \mathbb{D}) \tag{13}$$

or one of its rotations. For $\nu = 1$, equality holds if and only if p is the reciprocal of one of the functions such that equality holds in case of $\nu = 0$. Also for $0 < \nu < 1$, the following improved estimate holds:

$$|p_2 - \nu p_1^2| + \nu |p_1|^2 \le 2 \quad (0 < \nu \le 1/2)$$

and

$$|p_2 - \nu p_1^2| + (1-\nu)|p_1|^2 \le 2 \quad (1/2 \le \nu < 1).$$

For any complex number ν, the above inequality was proved by Koegh and Merkes [29]:

THEOREM 22 ([29]). (see also [53]) If $p \in \mathcal{P}$, then for any complex number ν,

$$|p_2 - \nu p_1^2| \le 2 \max\{1; |2\nu - 1|\}$$

and the equality holds for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Another estimate on the functional $|p_2 - \nu p_1^2|$, for any real number ν, was given by Mishra and Gochhayat [46], see also [15].

THEOREM 23 ([46, Theorem 1.2, p. 2817]) If $p \in \mathcal{P}$, then, for any real number v,

$$|p_2 - \nu p_1^2| \le \begin{cases} 2 + (\nu - 1)|p_1|^2 & (\nu > 1/2), \\ 2 - \frac{1}{2}|p_1|^2 & (\nu = 1/2), \\ 2 - \nu |p_1|^2 & (\nu \le 1/2). \end{cases}$$

A consequence of the Schwarz lemma is that $p \in \mathcal{P}$ implies $|p_n| \le 2$ with equality if and only if $p(z) = (1+xz)/(1-xz)$ for some $|x| = 1$. This includes the uniqueness statement that

$$p \in \mathcal{P}, \;\; p_1 = 2x \; (|x| = 1) \Rightarrow p(z) = (1+xz)/(1-xz).$$

Koepf [31], in 1994, gave a generalization of this statement. The proof is a consequence of the Carathéodory-Toeplitz-Fejér theory on positive harmonic functions, in particular, of the following result due to Carathéodory:

THEOREM 24 ([13, Theorem 2]).(see also [31])The power series of $p \in \mathcal{P}$ converges in $\mathcal{P}$ if and only if the Teoplitz determinants

$$D_n := \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \\ p_{n-1} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}$$

and $p_{-k} = \bar{p_k}$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k (1 + e^{it_k} z)/(1 - e^{it_k} z)$, $\rho_k > 0$ real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$, $n \in \mathbb{N}$.

By applying Theorem 24, Libera and Zlotkiewicz, in 1982, proved the following result which gives an alternate representation for the coefficients p_2 and p_3 in terms of p_1. These representations have been abundantly used to find the bound for the initial coefficients and of the Hankel determinants.

THEOREM 25 ([37, p. 228]). If $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y$$

for some x and y such that $|x| \leq 1$ and $|y| \leq 1$.

In 2009, a similar representations for the Carathéodory functions of order α, were derived by Hayami and Owa as stated in the following result.

THEOREM 26 ([21, Lemma 5, p. 35]). Let $p(z) = b_0 + p_1 z + p_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > \alpha$. Then for $\alpha \in [0, b_0)$, we have

$$2(b_0 - \alpha)p_2 = p_1^2 + x(4(b_0 - \alpha)^2 - p_1^2)$$

and

$$\begin{aligned} 4(b_0 - \alpha)^2 p_3 &= p_1^3 + 2p_1(4(b_0 - \alpha)^2 - p_1^2)x - p_1(4(b_0 - \alpha)^2 - p_1^2)x^2 \\ &\quad + 2(b_0 - \alpha)(4(b_0 - \alpha)^2 - p_1^2)(1 - |x|^2)y, \end{aligned}$$

for some complex numbers x and y such that $|x| \leq 1$ and $|y| \leq 1$.

The following theorem is due to Carathéodory and appeared in [20]:

THEOREM 27 ([20]).(see also [37, Lemma 3, p. 227]) Let $p \in \mathcal{P}$. Then the following expressions are all bounded by 2, and all are sharp:

1. $|p_1^2 - p_2|$,
2. $|p_1^3 - 2p_1p_2 + p_3|$,
3. $|p_1^4 + 2p_1p_3 + p_2^2 - 3p_1^2p_2 - p_4|$,
4. $|p_1^5 + 3p_1p_2^2 + 3p_1^2p_3 - 4p_1^3p_2 - 2p_1p_4 - 2p_2p_3 + p_5|$,
5. $|p_1^6 + 6p_1^2p_2^2 + 4p_1^3p_3 + 2p_1p_5 + 2p_2p_4 + p_2^3 - p_2^3 - 5p_1^4p_2 - 3p_1^2p_4 - 6p_1p_2p_3 - p_6|$.

The following inequalities can also be obtained in the proof of a result in [37, p. 227-228]

1. $|2p_1^2 - p_2| \leq 6$,
2. $|-6p_1^3 + 7p_1p_2 - 2p_3| \leq 24$,
3. $|24p_1^4 - 46p_1^2p_2 + 22p_1p_3 + 7p_2^2 - 6p_4| \leq 120$,
4. $|-120p_1^5 + 96p_4p_1 + 50p_2p_3 + 326p_1^3p_2 - 202p_1^2p_3 - 127p_1p_2^2 - 24p_5| \leq 720$.

In 1985, Livingston [40] obtained sharp bounds on the modulus of certain determinants, whose entries are the coefficients of a function of positive real part. He used these inequalities to solve coefficient problems for a certain subclass of multivalent functions. In the same paper, Livingston generalized Theorem 16 as follows:

THEOREM 28 ([40, Lemma 1, p. 140]). Let $\operatorname{Re} b_0 > 0$ and define the function p by

$$p(z) = (\operatorname{Re} b_0) \sum_{j=1}^{m} \lambda_j \frac{1 + ze^{it_j}}{1 - ze^{it_j}} + i \operatorname{Im} b_0 = b_0 + \sum_{n=1}^{\infty} p_n z^n,$$

where t_j and λ_j are real with $\lambda_j \geq 0$ for all $j = 1, 2, 3, \ldots$ satisfying $\sum_{j=1}^{m} \lambda_j = 1$. For fixed n and natural numbers s and k, define the determinant $Q_k^{(s)}$ by

$$Q_k^{(s)} := \begin{vmatrix} e^{i(n+s-2)t_k} & p_{n+s-2}/b_0 & p_{n+s-3}/b_0 & \cdots & p_n/b_0 \\ 1 & 1 & 0 & \cdots & 0 \\ e^{it_k} & p_1/b_0 & 1 & \cdots & 0 \\ e^{2it_k} & p_2/b_0 & p_1/b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{i(s-2)t_k} & p_{s-2}/b_0 & p_{s-3}/b_0 & \cdots & 1 \end{vmatrix}.$$

Then, for all integers $n \geq 2$ and $s \geq 1$, we have

$$\sum_{k=1}^{m} \lambda_k \left| Q_k^{(s)} \right|^2 = 1.$$

By using the concept of the uniform limits on compact subsets of the unit disk $\mathbb{D}$, for function $p(z) = b_0 + \sum_{k=1}^{\infty} p_n z^n$ with positive real part and $\operatorname{Re} b_0 > 0$, and Theorem 28, Livingston established the following.

THEOREM 29 ([40, Corollary 1, p. 142]). Let $p(z) = b_0 + \sum_{k=1}^{\infty} p_n z^n$ satisfies $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$ and $\operatorname{Re} b_0 > 0$. For all natural number s, define the determinant $A_k^{(s)}$ by

$$A_k^{(s)} := \begin{vmatrix} p_{n+s}/b_0 & p_{n+s-1}/b_0 & p_{n+s-2}/b_0 & \cdots & p_n/b_0 \\ p_1/b_0 & 1 & 0 & \cdots & 0 \\ p_2/b_0 & p_1/b_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_s/b_0 & p_{s-1}/b_0 & p_{s-2}/b_0 & \cdots & 1 \end{vmatrix}.$$

Then for all integers $n \geq 1$ and $s \geq 1$,

$$\left| A_k^{(s)} \right| \leq 2 \left| \frac{\operatorname{Re} b_0}{b_0} \right| \leq 2.$$

Equality is attained for the function $\hat{p}_0$.

THEOREM 30 ([40, Theorem 1, p. 143, Livingston]). Let $p(z) = b_0 + \sum_{k=1}^{\infty} p_n z^n$ satisfies $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$ and $\operatorname{Re} b_0 > 0$. Let $1/p(z) = \sum_{k=0}^{\infty} q_n z^n$. Then for all integers $m \geq 1$, $t \geq 0$ and $n \geq m$

$$\left| \sum_{k=t}^{m} q_{k-t} p_{n-k} \right| \leq 2 \left| \frac{\operatorname{Re} b_0}{b_0} \right| \leq 2.$$

Earlier in 1956, Nehari and Netanyahu [48] proved the following results:

THEOREM 31 ([48, Lemma I, p. 17]). Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ and $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$ be functions in the class $\mathcal{P}$. Then the function $r(z) = 1 + \sum_{k=1}^{\infty} (p_k q_k/2) z^k$ is also a member of $\mathcal{P}$.

THEOREM 32 ([48, Lemma II, p. 17]). Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k = 1 + G(z)$ and $h(z) = 1 + \sum_{k=1}^{\infty} \beta_k z^k$ be functions in the class $\mathcal{P}$. If $A_n's$ are defined by

$$\sum_{k=1}^{\infty} (-1)^{k+1} \gamma_{k-1} G^k(z) = \sum_{k=1}^{\infty} A_k z^k,$$

where

$$\gamma_k = \frac{1}{2^k} \left(1 + \frac{1}{2} \sum_{m=1}^{k} \binom{k}{m} \beta_m \right) \text{ with } \gamma_0 = 1,$$

then $|A_n| \leq 2$.

By using Theorems 31 and 32, Nehari and Netanyahu [48, Theorem I, p. 16] obtained the sharp bound of the initial coefficients of meromorphic starlike functions.

3.1 Successive Coefficients

This subsection is devoted to recollecting the results on the successive coefficients related to the Carathéodory functions.

There are several transformations which preserve the class $\mathcal{P}$, see [33] and Theorem 2. In particular, if $p \in \mathcal{P}$, then the function q defined by the transformation

$$q(z) = \frac{1-z^2}{z} - \frac{(1-z)^2}{z} p(z)$$

satisfies $\operatorname{Re} q(z) > 0$ $(z \in \mathbb{D})$. Using this fact, Robertson [59], in 1981, proved the following estimate on the successive coefficients difference of functions with positive real part.

THEOREM 33 ([59, Theorem 10, p. 341]). *Let $p \in \mathcal{P}$. Then, for $n \geq 3$,*

$$|p_{n+1} - p_n| \leq (2n+1)|2 - p_1|$$

and

$$|\,|p_{n+1}| - |p_n|\,| \leq (2n+1)|2 - p_1|.$$

The factor $2n+1$ cannot be replaced by a smaller one. Equality occurs in case of the function p defined by

$$p(z) = \frac{1-z^2}{1 - 2z\cos\phi + z^2}.$$

By using this result, Robertson also derived the sharp estimate on the difference of successive coefficients for the starlike and convex functions. Goodman [17, Problem 35, p. 104], raised a question about the sharp bound on $|p_{n+1} - p_n|$ for fixed p_1. A partial answer to this question may be obtained from Theorem 17 by setting $b_0 = 1, p_1 = 1$ and $s = n-1$. This gives $|p_{n+1} - p_n| \leq 2$. If we set $b_0 = 1, p_1 = 2$ and $s = n-1$, then $|p_{n+1} - 2p_n| \leq 2$.

For functions $f(z) = 1 + 2\sum_{k=1}^{\infty} a_k z^k$ and $g(z) = 1 + 2\sum_{k=1}^{\infty} b_k z^k \in \mathcal{P}$, consider the weighted Hadamard product defined by

$$(f * g)(z) = 1 + 2\sum_{k=1}^{\infty} a_k b_k z^k.$$

Komatu [32, Theorem 1, p. 141], in 1958, proved that if $f, g \in \mathcal{P}$, then $f * g \in \mathcal{P}$. Using this result, Brown [11], in 2010, proved several results related to the power of successive coefficients of functions with the positive real part.

THEOREM 34 ([11, Theorem 2.1, p. 2492]). *Let $p \in \mathcal{P}$. Then for $m, n \in \mathbb{N}$ and $\nu \in \mathbb{R}$,*

$$\left|e^{i\nu} p_{n+m} - p_n\right| \leq 2\sqrt{2 - \operatorname{Re}(e^{i\nu} p_m)}.$$

The result is sharp.

Recently, in 2016, Efraimidis [16] also provided an alternate and easy proof of Theorem 34. Setting $m = 1$ and $\nu = 0$ in Theorem 34, we get the following result:

THEOREM 35 ([11, Corollary 2.2, p. 2493]). Let $p \in \mathcal{P}$ and $n \in \mathbb{N}$. Then

$$|p_{n+1} - p_n| \leq 2\sqrt{2 - \operatorname{Re}(p_1)}.$$

The result is sharp. Equality holds for the function $p(z) = (1 + e^{i\alpha}z)/(1 - e^{i\alpha}z)$, where $\alpha = \arccos(b/2)$ and $\operatorname{Re} p_1 = 2b$.

Theorem 34 can also be generalized to the powers of successive coefficients as follows:

THEOREM 36 ([11, Theorem 2.3, p. 2493]). Let $p \in \mathcal{P}$ and $\nu \in \mathbb{R}$. Then, for $m, n, N \in \mathbb{N}$,

$$\left|e^{i\nu} p_{n+m}^N - p_n^N\right| \leq 2^N \sqrt{2^N - 2^{1-N} \operatorname{Re}(e^{i\nu} p_m^N)}.$$

The result is sharp.

THEOREM 37 ([11, Theorem 4.1, p. 2497]). Let $p \in \mathcal{P}$. Then, for $n \geq 3$,

$$|(p_{n+1} - p_n) - (p_{n-1} - p_{n-2})| \leq 2\sqrt{(\operatorname{Re}(2 - p_1))(\operatorname{Re}(2 - p_2))}.$$

The result is sharp.

Using the fact that the functions in the class $\mathcal{P}$ retain their properties under certain transformations, Lecko [33], in 2000, proved the following result:

THEOREM 38 ([33, Theorem 2.1, p. 62]). For fixed $\alpha \in [0, 1)$ and $\zeta \in \bar{\mathbb{D}}$, let $p \in \mathcal{P}(\alpha)$. Then, for $n \geq 2$,

1. $|\zeta p_2 - (1 + |\zeta|^2)p_1 + 2(1 - \alpha)\bar{\zeta}| \leq 2\left[(1 + |\zeta|^2)(1 - \alpha) - \operatorname{Re}(\zeta p_1)\right]$,
2. $|\zeta p_{n+1} - (1 + |\zeta|^2)p_n + \bar{\zeta} p_{n-1}| \leq 2\left[(1 + |\zeta|^2)(1 - \alpha) - \operatorname{Re}(\zeta p_1)\right]$,
3. $|(||\zeta| p_{n+1} - p_n|) - |\zeta|(||\zeta| p_n - p_{n-1}|)| \leq 2\left[(1 + |\zeta|^2)(1 - \alpha) - |\zeta| \operatorname{Re}(p_1)\right]$.

The results are sharp and the function defined by

$$p(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$$

acts as an extremal function for all the cases above.

Setting $\alpha = 0$ in the above theorem, we have the following corollary:

THEOREM 39 ([33, Corollary 2.2, p. 63]). For $\zeta \in \bar{\mathbb{D}}$, let $p \in \mathcal{P}$. Then, for $n \geq 2$,

1. $|\zeta p_2 - (1 + |\zeta|^2)p_1 + 2\bar{\zeta}| \leq 2\left[(1 + |\zeta|^2) - \operatorname{Re}(\zeta p_1)\right]$,

2. $|\zeta p_{n+1} - (1+|\zeta|^2)p_n + \bar{\zeta} p_{n-1}| \le 2\left[1+|\zeta|^2 - \mathrm{Re}(\zeta p_1)\right]$,
3. $\Big||\zeta|p_{n+1} - p_n| - |\zeta|\big||\zeta|p_n - p_{n-1}\big|\Big| \le 2\left[(1+|\zeta|^2) - |\zeta|\,\mathrm{Re}(p_1)\right]$.

The results are sharp and the extremal function is $\hat{p}_0$.

It should be noted that many useful results can be derived from Theorem 39 by assigning suitable values to ζ. The inequality (1) of Theorem 39 gives the following specific results:

1. if $\zeta = 1$, then $|p_2 - 2p_1 + 2| \le 2(2 - \mathrm{Re}(p_1))$,
2. if $\zeta = -1$, then $|p_2 + 2p_1 + 2| \le 2(2 + \mathrm{Re}(p_1))$,
3. if $\zeta = i$, then $|p_2 + 2ip_1 - 2| \le 2(2 + \mathrm{Im}(p_1))$,
4. if $\zeta = -i$, then $|p_2 - 2ip_1 - 2| \le 2(2 - \mathrm{Im}(p_1))$.

The inequality (2) of Theorem 39 gives the following results:

1. if $\zeta = 1$, then $|p_{n+1} - 2p_n + p_{n-1}| \le 2(2 - \mathrm{Re}(p_1))$,
2. if $\zeta = -1$, then $|p_{n+1} + 2p_n + p_{n-1}| \le 2(2 + \mathrm{Re}(p_1))$,
3. if $\zeta = i$, then $|p_{n+1} + 2ip_n - p_{n-1}| \le 2(2 + \mathrm{Im}(p_1))$,
4. if $\zeta = -i$, then $|p_{n+1} - 2ip_n - p_{n-1}| \le 2(2 - \mathrm{Im}(p_1))$,
5. if $\zeta = 1/n$ $(n = 2, 3, 4, \ldots)$, then
$$\left|p_{n+1} - \left(n + \frac{1}{n}\right)p_n + p_{n-1}\right| \le 2\left(n + \frac{1}{n} - \mathrm{Re}(p_1)\right),$$
6. if $\zeta = 1 - 1/n$ $(n = 2, 3, 4, \ldots)$, then
$$\left|p_{n+1} - \left(\frac{n}{n-1} + \frac{n-1}{n}\right)p_n + p_{n-1}\right| \le 2\left(\frac{n}{n-1} + \frac{n-1}{n} - \mathrm{Re}(p_1)\right).$$

Setting $|\zeta| = 1$ in the third inequality of the above theorem, we have
$$\big||p_{n+1} - p_n| - |p_n - p_{n-1}|\big| \le 2(2 - \mathrm{Re}(p_1)).$$

Lecko [33], further proved the following:

THEOREM 40 ([33, Theorem 2.7, p. 64]). For fixed $\alpha \in [0,1)$ and $\zeta \in \bar{\mathbb{D}}$, let $p \in \mathcal{P}(\alpha)$. Then, for $n \ge 2$,

$$|\zeta p_{n+1} - p_n| \le \begin{cases} 2\frac{1-|\zeta|^n}{1-|\zeta|}[(1-\alpha)(1+|\zeta|^2) - \mathrm{Re}(\zeta p_1)] + |2(1-\alpha) - \zeta p_1||\zeta|^n, & |\zeta| < 1; \\ (2n+1)|2(1-\alpha) - \zeta p_1|, & |\zeta| = 1. \end{cases}$$

and

$$\big||\zeta p_{n+1}|-|p_n|\big| \leq \begin{cases} 2\frac{1-|\zeta|^n}{1-|\zeta|}[(1-\alpha)(1+|\zeta|^2)-\operatorname{Re}(\zeta)|p_1|]+\big|2(1-\alpha)-\zeta|p_1|\big||\zeta|^n, & |\zeta|<1; \\ (2n+1)\big|2(1-\alpha)-\zeta|p_1|\big|, & |\zeta|=1. \end{cases}$$

The estimates are sharp for each $\zeta \in [0,1]$. When $\zeta \in [0,1)$, the function

$$p(z)=\frac{1+(1-2\alpha)z}{1-z}$$

acts as an extremal function,whereas if $\zeta=1$, then equality holds in case of the function

$$p(z)=\frac{1-2(\alpha\cos\theta)z-(1-2\alpha)z^2}{1-2(\cos\theta)z+z^2}=1+2(1-\alpha)\sum_{n=2}^{\infty}(\cos n\theta)z^n,$$

for sufficiently small $\theta \in [0,2\pi)$.

3.2 Tools for Bounds on Fourth and Fifth Coefficients

In this subsection, some results which could be useful in finding the sharp bounds of the fourth and fifth coefficients of normalized analytic functions belonging to certain subclasses of univalent functions are discussed.

Let f and g be analytic functions in the unit disk $\mathbb{D}$. Leverenz [36] gave a new derivation of the positive semi-definite Hermitian form equivalent to $|g(z)| \leq |f(z)|,\ z \in \mathbb{D}$ and used it to investigate the Hermitian forms for some subclasses of univalent functions. As a corollary to this main result, he established the following:

THEOREM 41 ([36, Theorem 4(b), p. 678]). A function $p \in \mathcal{P}$ if and only if

$$\sum_{j=0}^{\infty}\left\{\left|2z_j+\sum_{k=1}^{\infty}p_k z_{k+j}\right|^2-\left|\sum_{k=1}^{\infty}p_{k+1}z_{k+j}\right|^2\right\} \geq 0$$

for every sequence $\{z_k\}_{k=1}^{\infty}$ of complex numbers that satisfies $\lim_{k\to\infty}|z_k|^{1/k}<1$.

Further, he derived the sharp coefficient estimates for these subclasses. He also gave the specific functions required to make the Hermitian forms equal to zero.

By using Theorem 41, Ali and Singh [5], in 1996, proved the following result:

THEOREM 42 ([5, Lemma 3, p. 199]). Let $p \in \mathcal{P}$. Then

$$|p_4-(p_1p_3+ap_2^2)+ap_1^2p_2| \leq \begin{cases} 2, & 0<a\leq 1; \\ 2(2a-1), & a \geq 1. \end{cases}$$

Equality attained in the first case when $p(z)=(1+\epsilon z^4)/(1-\epsilon z^4), |\epsilon|=1$. In the second case, equality is attained when p is any one of the following $p(z)=(1+\epsilon z)/(1-\epsilon z)$ or $p(z)=(1+\epsilon z^2)/(1-\epsilon z^2), |\epsilon|=1$.

By using Theorems 32, 4 and 42, Ali and Singh [5] obtained the sharp upper bound for the fourth and fifth coefficients for strongly starlike functions. However, minor corrections were reported and the correct version of the proof of these results were given by Lecko and Sim [34]. Earlier, in 1993, Ma and Minda [43] prove the following results related to the coefficients of the functions in the class $\mathcal{P}$.

THEOREM 43 ([43, Lemma 1, p. 278]). Let $p \in \mathcal{P}$. Then the following holds:

1. $\left|p_{2n} - \frac{1}{2}p_n^2\right| \leq 2 - \frac{1}{2}|p_n|^2$,
2. $|\mu p_{2n}p_n^2 - p_n^4| \leq 8(\mu - 2) \ \ (\mu \geq 4)$,
3. $|\mu p_{2n}p_n - p_n^3| \leq 4(\mu - 2) \ \ (\mu \geq 6)$.

Theorem 41 is very useful in deducing the sharp upper bound on initial coefficients. In particular, to obtain the sharp bound of the fourth and fifth coefficients of certain analytic functions, Ali [2] gave an alternate proof of Theorem 43 proved by Ma and Minda [44, 43] by choosing a particular sequence $\{z_k\}_{k=1}^{\infty}$ in Theorem 41, see [2, Lemma 2, p. 65]. In 2003, Ali [2] proved the following:

THEOREM 44 ([2, Lemma 3, p. 66]). Let $p \in \mathcal{P}$ and $0 \leq \beta \leq 1$ and $\beta(2\beta - 1) \leq \delta \leq \beta$, then

$$|p_3 - 2\beta p_1 p_2 + \delta p_1^3| \leq 2.$$

Further, if $\delta = \beta$, then by Theorem 44, we have

THEOREM 45 ([2, Corollary 1, p. 67]). Let $p \in \mathcal{P}$ and $0 \leq \beta \leq 1$. Then

$$|p_3 - 2\beta p_1 p_2 + \beta p_1^3| \leq 2.$$

When $\beta = 0$, equality holds if and only if

$$p(z) = p_3(z) = \sum_{k=1}^{3} \lambda_k \frac{1 + \epsilon e^{(-2\pi i k)z/3}}{1 - \epsilon e^{(-2\pi i k)z/3}} \ \ (|\epsilon| = 1),$$

where $\lambda_k > 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. If $\beta = 1$, equality holds if and only if p is the reciprocal of p_3. If $0 < \beta < 1$, equality holds if and only if $p(z) = (1 + \epsilon z)/(1 - \epsilon z)$ or $p(z) = (1 + \epsilon z^3)/(1 - \epsilon z^3)$ $(|\epsilon| = 1)$.

THEOREM 46 ([2, Corollary 1, p. 68]). Let $p \in \mathcal{P}$. Then

$$|p_3 - (\mu + 1)p_1 p_2 + \mu p_1^3| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|\mu - 1|, & \text{elsewere.} \end{cases}$$

Ali [2] used Theorems 44 and 45 to determine the sharp bounds on the first four coefficients and the estimate on the Fekete-Szegö functional of the inverse for strongly starlike functions. In 2015, Ravichandran and Verma [55] proved the following results by using Theorem 41.

THEOREM 47 ([55, Lemma 2.1, p. 506]). Let α, β, γ and a satisfy the inequalities $0 < \alpha < 1$, $0 < a < 1$ and

$$8a(1-a)[(\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2] + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a).$$

If $p \in \mathcal{P}$, then

$$|\gamma p_1^4 + a p_2^2 + 2\alpha p_1 p_3 - (3/2)\beta p_1^2 p_2 - p_4| \leq 2.$$

As an application of Theorem 47, Ravichandran and Verma [55] proved some conjectures related to the sharp bound on the fifth coefficient for certain subclasses of starlike functions. They also reproved the inequalities $(iii), (iv)$ and (v) of Theorem 27 by using Theorem 41, see [55, Lemma 2.2, p. 507].

THEOREM 48 ([55, Lemma 2.3, p. 507]). Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$,

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \text{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, then the inequality is sharp for the function $p(z) = (1+z^{m+n})/(1-z^{m+n})$. In the other cases, the inequality is sharp for the function $\hat{p}_0$.

THEOREM 49 ([55, Corollary 2.4, p. 508]). Let $p \in \mathcal{P}$. Then, for all $n \in \mathbb{N}$ and $\mu \leq 1$,

$$|\mu p_n p_{2n} - p_n^3| \leq 4(2-\mu), \quad \text{and} \quad |\mu p_n^2 p_{2n} - p_n^4| \leq 8(2-\mu)$$

The inequality is sharp for the function $\hat{p}_0$.

For normalized analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, Zaprawa [65], in 2017, mentioned a problem of finding the sharp estimate on the generalized form of the second Hankel determinant $J_n = a_{n+1}a_{n+2} - a_n a_{n+3}$. In his paper, it was shown that for starlike functions, $|J_2| \leq 2$. He further conjectured that for starlike functions, $|J_n| \leq 2$ for all natural numbers n. To prove the estimate on J_2, he proved that if $p \in \mathcal{P}$, then $|p_1^5 + 2p_1^3 p_2 - 4p_1^2 p_3 + 3p_1 p_2^2 - 6p_1 p_4 + 4p_2 p_3| \leq 48$, see [65, Theorem 2.2, p. 5].

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