Applied Mathematics E-Notes, 19(2019), 256-276 © ISSN 1607-2510
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Common Fixed Points Of (α, η, β)-b-Branciari F-Rational Type Contractions In (α, η)-Complete Branciari b-Metric Spaces*

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Received 17 May 2018

Abstract

The aim of this paper is to present the notion of (α, η, β)-b-Branciari F-rational type contractions. We also establish some new common fixed point theorems for such mappings in an (α, η)-complete Branciari b-metric spaces. We then derive some common fixed point results in complete Branciari b-metric spaces endowed with a graph or a partial order. We give examples in support of the obtained results.

1 Introduction

Since the introduction of Banach contraction principle in 1922, the study of existence and uniqueness of fixed points and common fixed points has become a subject of great interest because of its wide applications. Many authors proved the Banach contraction principle in various generalized metric spaces. In [10], Bakhtin introduced the concept of b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces and extensively applied by Czerwik in [11, 12]. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces, (see [1, 2, 4, 5, 6, 7, 14, 16, 18] and the references therein).

In the sequel, the letters $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$, $Fix\,(T)$ and $CFix\,(S,T)$ will denote the set of natural numbers, the set of all positive real numbers, the set of all real numbers, the set of all fixed points of T and the set of all common fixed points of S and T, respectively.

DEFINITION 1.1 ([11]). Let X be a nonempty set and $s \geq 1$ be a real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$;

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*Mathematics Subject Classifications: 47H10, 54H25

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(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s\left[d(x, z) + d(z, y)\right]$.

In this case, the pair (X, d) is called a b-metric space (with constant s).

In [13], Branciari introduced the following definition.

DEFINITION 1.2 ([13]). Let X be a non-empty set and $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each of them is different from x and y, one has

(i) $d(x, y) = 0 \Longleftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then (X, d) is called a Branciari metric space (for short, BMS). Roshan et al. [27] announced the following notion by combining conditions used for definitions of b-metric and Branciari metric spaces.

DEFINITION 1.3 ([27]). Let X be a non-empty set and $s \geq 1$ be a real number. Given $B_b : X \times X \longrightarrow [0, \infty)$. Suppose that for all $x, y \in X$ and for all distinct points $u, v \in X$ such that each of them is different from x and y, one has the following conditions:

(i) $B_b(x, y) = 0 \Longleftrightarrow x = y$;

(ii) $B_b(x, y) = B_b(y, x)$;

(iii) $B_b(x, y) \leq s\left[B_b(x, u) + B_b(u, v) + B_b(v, y)\right]$.

Then (X, B_b) is called a Branciari b-metric space (for short, BbMS).

EXAMPLE 1.1. Let $X = A \cup B$ where $A = \left\{\frac{1}{n} : n \in \{2, 3, 4, 5\}\right\}$ and $B = [1, 2]$. Define $B_b : X \times X \longrightarrow [0, \infty)$ such that $B_b(x, y) = B_b(y, x)$ for all $x, y \in X$, and

$$B_b\left(\frac{1}{2}, \frac{1}{3}\right) = B_b\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{100},$$
$$B_b\left(\frac{1}{2}, \frac{1}{5}\right) = B_b\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{2}{100},$$
$$B_b\left(\frac{1}{2}, \frac{1}{4}\right) = B_b\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{6}{100},$$
$$B_b(x, y) = |x - y|^2 \text{ otherwise.}$$

Then (X, B_b) is a Branciari b-metric space with coefficient $s = 4$. But, (X, B_b) is neither a metric space, nor a Branciari metric space.

LEMMA 1.1 ([27]). Let (X, B_b) be a Branciari b-metric space.

(i) Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$, with $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then

$$\frac{1}{s}B_b(x,y) \leq \lim_{n\longrightarrow\infty} \inf B_b(x_n,y_n) \leq \lim_{n\longrightarrow\infty} \sup B_b(x_n,y_n) \leq sB_b(x,y).$$

(ii) If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for infinitely many $m \neq n \in \mathbb{N}$, converging to $x \neq y$, then

$$\frac{1}{s}B_b(x,y) \leq \lim_{n\longrightarrow\infty} \inf B_b(x_n,y) \leq \lim_{n\longrightarrow\infty} \sup B_b(x_n,y) \leq sB_b(x,y),$$

for all $n \in \mathbb{N}$.

Hussain et al.[23] (see also [21]) extended the notions of α-ψ-contractive and α-admissible mappings. They stated some interesting results. Also, Hussain et al. [23] introduced a weaker notion than the concept of completeness and called it α-completeness for a metric space.

DEFINITION 1.4 ([23]). Let $T : X \to X$ be a self-mapping and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is (α, η)-admissible if

$$x, y \in X,\ \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1,$$

and

$$x, y \in X,\ \eta(x,y) \leq 1 \Longrightarrow \eta(Tx,Ty) \leq 1.$$

DEFINITION 1.5 ([23]). Given $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$. T is said triangular (α, η)-admissible if

(T_1) $\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1$, $x, y \in X$;

(T_2) $\eta(x,y) \leq 1 \Longrightarrow \eta(Tx,Ty) \leq 1$, $x, y \in X$;

(T_3) $\begin{cases} \alpha(x,u) \geq 1 \\ \alpha(u,y) \geq 1 \end{cases} \Longrightarrow \alpha(x,y) \geq 1$, for all $x, u, y \in X$;

(T_4) $\begin{cases} \eta(x,u) \leq 1 \\ \eta(u,y) \leq 1 \end{cases} \Longrightarrow \eta(x,y) \leq 1$, for all $x, u, y \in X$.

DEFINITION 1.6 ([23]). Let (X, d) be a metric space or a Branciari b-metric space and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions. Then X is said to be (α, η)-complete if every Cauchy sequence $\{x_n\}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, is convergent in X.

DEFINITION 1.7 ([23]). Let (X, d) be a metric space or a Branciari b-metric space. Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, +\infty)$ be two given functions.

T is (α, η)-continuous on (X, d) if for given $x \in X$ and a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to +\infty$, then $Tx_n \to Tx$ as $n \to +\infty$.

DEFINITION 1.8. Let (X, d) be a metric space or a Branciari b-metric space and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two given functions. We say that (X, d) is (α, η)-regular if $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ where $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$, for all $n \in \mathbb{N} \cup \{0\}$, imply that $\alpha(x_n, x^*) \geq 1$ or $\eta(x_n, x^*) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

In 2012, Wardowski [20] introduced the notion of F-contractions and proved variant fixed point theorems concerning F-contractions. For particular cases for functions F, one can obtain several known contractions from the literature, including the Banach contraction (see [3, 9, 22, 28]).

DEFINITION 1.9 ([20]). Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

where $\mathcal{F}$ is the set of functions $F : (0, \infty) \to (-\infty, \infty)$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;

$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

THEOREM 1.1 ([20]). Let (X, d) be a complete metric space and let $T : X \to X$ be a F- contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

Later, Piri and Kumam [17] modified the notion of F-contractions by changing $(F3)$ by $(F'3)$: F is continuous.

Denote Δ_F the set of functions $F : (0, \infty) \to (-\infty, \infty)$ satisfying $(F1), (F2)$ and $(F'3)$.

EXAMPLE 1.2. The following are some examples of functions belonging to Δ_F:

(1) $F_1(t) = \ln t$, (3) $F_3(t) = t - \frac{1}{t}$, (5) $F_5(t) = \frac{1}{1-e^t}$.

(2) $F_2(t) = \frac{1}{t^r},\ r > 0$, (4) $F_4(t) = \frac{e^t}{1-e^{2t}}$,

DEFINITION 1.10 ([28]). Let (X, d) be a Branciari metric space. Then $T : X \longrightarrow X$ is said to be a Branciari F-rational contraction, if there exist $F \in \digamma$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\right\}.$$

THEOREM 1.2 ([28]). Let (X, d) be a complete Branciari metric space and $T : X \longrightarrow X$ be a Branciari F-rational contraction. If T or F is continuous, then T has a unique fixed point in X.

As in [29], let Δ_β be the set of functions $\beta : (0, \infty) \longrightarrow (0, \infty)$ satisfying the following conditions:

($\beta 1$) $\liminf_{i \longrightarrow \infty} \beta(t_i) > 0$ for all real sequences $\{t_i\}$ with $t_i > 0$;

($\beta 2$) $\sum\limits_{i=0}^{\infty} \beta(t_i) = +\infty$ for each positive sequence $\{t_i\}$.

Hussain et al. [29] established some fixed point results for generalized F-contractive mappings in the setup of Branciari b-metric spaces as follows.

THEOREM 1.3 ([29]). Let (X, B_b) be a complete Branciari b-metric space with parameter $s \geq 1$. Given $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ and T: $X \longrightarrow X$. Assume that

(i) T is triangular (α, η)-admissible;

(ii) for all $x, y \in X$ (with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$) and $B_b(Tx, Ty) > 0$, we have

$$\beta\left(B_b(x, y)\right) + F\left(s^2 B_b(Tx, Ty)\right) \leq F\left(\begin{array}{c} \alpha_1 B_b(x, y) + \alpha_2 B_b(x, Tx) + \\ \alpha_3 B_b(y, Ty) + \alpha_4 B_b(y, Tx) \end{array}\right),$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $\alpha_i \geq 0$ for $i \in \{1, 2, 3, 4\}$ such that $\sum_{i=1}^{4} \alpha_i = 1$ and $\alpha_3 < \frac{1}{s}$;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(iv) T is (α, η)-continuous.

Then T has a fixed point. If in addition, $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in Fix(T)$, then such fixed point is unique.

2 Main Results

We begin with the following concepts.

DEFINITION 2.1. Let $S, T : X \to X$ be self-mappings and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that the pair (S, T) is (α, η)-admissible if

$$x, y \in X,\ \alpha(x, y) \geq 1 \Longrightarrow \alpha(Sx, Ty) \geq 1 \text{ and } \alpha(Sx, Ty) \geq 1,$$

and

$$x, y \in X,\ \eta(x, y) \leq 1 \Longrightarrow \eta(Sx, Ty) \leq 1 \text{ and } \eta(Sx, Ty) \leq 1.$$

DEFINITION 2.2. Let $S, T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$. We say that the pair (S, T) is triangular (α, η)-admissible if

(T_1) $\alpha(x, y) \geq 1 \Longrightarrow \alpha(Sx, Ty) \geq 1$ and $\alpha(Sx, Ty) \geq 1$ for all $x, y \in X$;

(T_2) $\eta(x, y) \leq 1 \Longrightarrow \eta(Sx, Ty) \leq 1$ and $\eta(Sx, Ty) \leq 1$ for all $x, y \in X$;

(T_3) $\left\{ \begin{array}{l} \alpha(x, u) \geq 1 \\ \alpha(u, y) \geq 1 \end{array} \right. \Longrightarrow \alpha(x, y) \geq 1$ for all $x, u, y \in X$;

(T_4) $\left\{ \begin{array}{l} \eta(x, u) \leq 1 \\ \eta(u, y) \leq 1 \end{array} \right. \Longrightarrow \eta(x, y) \leq 1$ for all $x, u, y \in X$.

Note that the concepts given in Definition 2.1 and Definition 2.2 are not concerned by the note of Berzig and Karapinar [30]. Now, we state and prove our main results.

DEFINITION 2.3. Let (X, B_b) be a Branciari b-metric space with parameter $s \geq 1$ and let S, T be self-mappings on X. Suppose that $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ are two functions. We say that the pair (S, T) is an (α, η, β)-b-Branciari F-rational contraction, if for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $B_b(Sx, Ty) > 0$, we have

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \beta\left(B_b(x, y)\right) + F\left(s^2 B_b(Sx, Ty)\right) \leq F\left(W_b(x, y)\right), \quad (1)$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and

$$W_b(x, y) = \max \left\{ \begin{array}{c} B_b(x, y), B_b(x, Sx), B_b(y, Ty), B_b(y, Sx), \\ \frac{B_b(x, Sx) B_b(y, Ty)}{s + B_b(x, y)}, \frac{B_b(x, Sx) B_b(y, Ty)}{s + B_b(Sx, Ty)} \end{array} \right\}. \quad (2)$$

THEOREM 2.1. Let (X, B_b) be a complete Branciari b-metric space with parameter s and let S, T: $X \longrightarrow X$ be self-mappings satisfying the following conditions:

(i) the pair (S, T) is triangular (α, η)-admissible;

(ii) (S,T) is an (α,η,β)-b-Branciari F-rational contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ or $\eta(x_0, Sx_0) \leq 1$;

(iv) S and T are (α,η)-continuous.

Then S and T have a common fixed point. Moreover, S and T have a unique common fixed point when $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for all $x, y \in CFix(S,T)$.

PROOF. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Define a sequence $\{x_n\}$ by $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$ for $i = 0, 1, 2, ...$ Since the pair (S,T) is triangular (α,η)-admissible, we get $\alpha(x_1, x_2) = \alpha(Sx_0, Tx_1) \geq 1$ or $\eta(x_1, x_2) = \eta(Sx_0, Tx_1) \leq 1$. Continuing in this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ or } \eta(x_n, x_{n+1}) \leq 1,$$

for all $n \in \mathbb{N} \cup \{0\}$. If for some n, $x_n = x_{n+1}$, then x_n is a common fixed point of T and S. From now on, without loss of generality, we can assume that

$$x_n \neq x_{n+1}, \forall\ n \in \mathbb{N} \cup \{0\}.$$

Since (S,T) is an (α,η,β)-b-Branciari F-rational contraction, we derive

$$\begin{aligned} F\left(B_b(x_{2i+1}, x_{2i+2})\right) &= F\left(B_b(Sx_{2i}, Tx_{2i+1})\right) \\ &< \beta\left(B_b(x_{2i}, x_{2i+1})\right) + F\left(B_b(Sx_{2i}, Tx_{2i+1})\right) \\ &\leq F\left(W_b(x_{2i}, x_{2i+1})\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} W_b(x_{2i}, x_{2i+1}) &= \max\left\{ \begin{array}{c} B_b(x_{2i}, x_{2i+1}), B_b(x_{2i}, Sx_{2i}), \\ B_b(x_{2i+1}, Tx_{2i+1}), B_b(x_{2i+1}, Sx_{2i}), \\ \frac{B_b(x_{2i}, Sx_{2i}) B_b(x_{2i+1}, Tx_{2i+1})}{s + B_b(x_{2i}, x_{2i+1})}, \\ \frac{B_b(x_{2i}, Sx_{2i}) B_b(x_{2i+1}, Tx_{2i+1})}{s + B_b(Sx_{2i}, Tx_{2i+1})} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} B_b(x_{2i}, x_{2i+1}), B_b(x_{2i}, x_{2i+1}), \\ B_b(x_{2i+1}, x_{2i+2}), B_b(x_{2i+1}, x_{2i+1}), \\ \frac{B_b(x_{2i}, x_{2i+1}) B_b(x_{2i+1}, x_{2i+2})}{s + B_b(x_{2i}, x_{2i+1})}, \\ \frac{B_b(x_{2i}, x_{2i+1}) B_b(x_{2i+1}, x_{2i+2})}{s + B_b(x_{2i+1}, x_{2i+2})} \end{array} \right\} \\ &= \max\left\{B_b(x_{2i}, x_{2i+1}), B_b(x_{2i+1}, x_{2i+2})\right\}. \end{aligned}$$

If $W_b(x_{2i}, x_{2i+1}) = B_b(x_{2i+1}, x_{2i+2})$ for some i, then from (3), we have

$$F\left(B_b(x_{2i+1}, x_{2i+2})\right) < F\left(B_b(x_{2i+1}, x_{2i+2})\right),$$

which is a contradiction. We conclude that $W_b(x_{2i}, x_{2i+1}) = B_b(x_{2i}, x_{2i+1})$ for all i. By (3), we get that

$$F\left(B_b(x_{2i+1}, x_{2i+2})\right) < F\left(B_b(x_{2i}, x_{2i+1})\right).$$

Since F is strictly increasing, we deduce that

$$B_b(x_{2i+1}, x_{2i+2}) < B_b(x_{2i}, x_{2i+1}) \text{ for all } i \in \mathbb{N} \cup \{0\}.$$

This implies that

$$B_b(x_{n+1}, x_{n+2}) < B_b(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Again, (1) implies that

$$F(B_b(x_{n+1}, x_{n+2})) < F(B_b(x_n, x_{n+1})) - \beta(B_b(x_n, x_{n+1})).$$

Therefore,

$$\begin{aligned} F(B_b(x_{n+1}, x_{n+2})) &< F(B_b(x_n, x_{n+1})) - \beta(B_b(x_n, x_{n+1})) \\ &< F(B_b(x_n, x_{n+1})) - \beta(B_b(x_n, x_{n+1})) - \beta(B_b(x_{n-1}, x_n)) \\ &\quad ... \\ &< F(B_b(x_0, x_1)) - \sum_{z=0}^{n} \beta(B_b(x_z, x_{z+1})). \end{aligned}$$

Letting $n \longrightarrow \infty$ in above inequality and using ($\beta 2$), we have

$$\lim_{n\to\infty} F(B_b(x_{n+1}, x_{n+2})) = -\infty,$$

and from ($F2$), we obtain

$$\lim_{n\to\infty} B_b(x_{n+1}, x_{n+2}) = 0. \tag{4}$$

On the other hand,

$$\begin{aligned} F(B_b(x_{2i+1}, x_{2i+3})) &< F\left(s^2 B_b(x_{2i+1}, x_{2i+3})\right) \\ &< \beta(B_b(x_{2i}, x_{2i+2})) + F\left(s^2 B_b(Sx_{2i}, Tx_{2i+2})\right) \\ &\le F(W_b(x_{2i}, x_{2i+2})), \end{aligned} \tag{5}$$

where

$$\begin{aligned} W_b(x_{2i}, x_{2i+2}) &= \max\left\{ \begin{array}{c} B_b(x_{2i}, x_{2i+2}), B_b(x_{2i}, Sx_{2i}), \\ B_b(x_{2i+2}, Tx_{2i+2}), B_b(x_{2i+2}, Sx_{2i}), \\ \frac{B_b(x_{2i}, Sx_{2i}) B_b(x_{2i+2}, Tx_{2i+2})}{s + B_b(x_{2i}, x_{2i+2})}, \\ \frac{B_b(x_{2i}, Sx_{2i}) B_b(x_{2i+2}, Tx_{2i+2})}{s + B_b(Sx_{2i}, Tx_{2i+2})} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} B_b(x_{2i}, x_{2i+2}), B_b(x_{2i}, x_{2i+1}), \\ B_b(x_{2i+1}, x_{2i+3}), B_b(x_{2i+2}, x_{2i+1}), \\ \frac{B_b(x_{2i}, x_{2i+1}) B_b(x_{2i+2}, x_{2i+3})}{s + B_b(x_{2i}, x_{2i+2})}, \\ \frac{B_b(x_{2i}, x_{2i+1}) B_b(x_{2i+2}, x_{2i+3})}{s + B_b(x_{2i+1}, x_{2i+3})} \end{array} \right\} \\ &= \max\{B_b(x_{2i}, x_{2i+2}), B_b(x_{2i+1}, x_{2i+3})\}. \end{aligned}$$

If $W_b(x_{2i}, x_{2i+2}) = B_b(x_{2i+1}, x_{2i+3})$ for some i, then from (5), we have

$$F\left(B_b(x_{2i+1}, x_{2i+3})\right) < F\left(B_b(x_{2i+1}, x_{2i+3})\right),$$

which is a contradiction. We conclude that $W_b(x_{2i}, x_{2i+1}) = B_b(x_{2i}, x_{2i+2})$ for all i. By (5), we get that

$$F\left(B_b(x_{2i+1}, x_{2i+2})\right) < F\left(B_b(x_{2i}, x_{2i+1})\right).$$

Since F is strictly increasing, we deduce that

$$B_b(x_{2i+1}, x_{2i+3}) < B_b(x_{2i}, x_{2i+2}) \text{ for all } i \in \mathbb{N} \cup \{0\}.$$

This implies that

$$B_b(x_{n+1}, x_{n+3}) < B_b(x_n, x_{n+2}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Taking the limit as $n \longrightarrow \infty$ in the above and using (4), we have

$$\lim_{n\to\infty} B_b(x_{n+1}, x_{n+3}) = 0. \tag{6}$$

Next, we show that $\{x_n\}$ is a B_b-Cauchy sequence in X. Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exist $m_j > n_j > j$ such that $B_b\left(x_{m_j}, x_{n_j}\right) \geq \varepsilon$. Let n_j be the smallest number satisfying the condition above. We have

$$B_b\left(x_{m_j}, x_{n_j-1}\right) < \varepsilon. \tag{7}$$

Therefore,

$$\begin{aligned} \varepsilon &\leq B_b\left(x_{m_j}, x_{n_j}\right) \\ &\leq s\left[B_b\left(x_{m_j}, x_{m_j+1}\right) + B_b\left(x_{m_j+1}, x_{n_j+1}\right) + B_b\left(x_{n_j}, x_{n_{j+1}}\right)\right]. \end{aligned} \tag{8}$$

By taking the upper limit as $j \to \infty$ in (8) and using (4), we get

$$\frac{\varepsilon}{s} \leq \lim_{j\to\infty} \sup B_b\left(x_{m_j+1}, x_{n_{j+1}}\right). \tag{9}$$

From rectangular inequality, we have

$$B_b\left(x_{m_j}, x_{n_j}\right) \leq s[B_b\left(x_{m_j}, x_{n_j-1}\right) + B_b\left(x_{n_j-1}, x_{n_j+1}\right) + B_b\left(x_{n_j-1}, x_{n_j}\right)]. \tag{10}$$

By (4), (6) and (7), we have

$$\lim_{j\to\infty} \sup B_b\left(x_{m_j}, x_{n_j}\right) \leq s\varepsilon. \tag{11}$$

Also,

$$B_b\left(x_{n_j}, x_{m_j+1}\right) \leq s[B_b\left(x_{n_j}, x_{n_j-1}\right) + B_b\left(x_{n_j-1}, x_{m_j}\right) + B_b\left(x_{m_j}, x_{m_j+1}\right)].$$

Again, from (4) and (7),

$$\lim_{j\to\infty}\sup B_b\left(x_{n_j},x_{m_j+1}\right)\le s\varepsilon. \tag{12}$$

Applying (1) to conclude that

$$\begin{aligned} F\left(s^2B_b(x_{m_j+1},x_{nj+1})\right) &= F\left(s^2B_b(Sx_{m_j},Tx_{nj})\right)\\ &\le F\left(W_b(x_{m_j},x_{n_j})\right)-\beta\left(B_b(x_{m_j},x_{n_j})\right), \end{aligned}$$

where

$$W_b(x_{m_j},x_{n_j})=\max\left\{\begin{array}{c} B_b(x_{m_j},x_{n_j}),B_b(x_{m_j},x_{m_j+1}),\\ B_b(x_{n_j},x_{n_j+1}),B_b(x_{n_j},x_{m_j+1}),\\ \frac{B_b(x_{m_j},x_{m_j+1})B_b(x_{n_j},x_{n_j+1})}{s+B_b(x_{m_j},x_{n_j})},\frac{B_b(x_{m_j},x_{m_j+1})B_b(x_{n_j},x_{n_j+1})}{s+B_b(x_{m_j+1},x_{n_j+1})} \end{array}\right\}.$$

Taking the upper limit as $j\to\infty$ and using $(F1)$, (9), (11) and (12), we have

$$\begin{aligned} F\left(s^2\frac{\varepsilon}{s}\right) &\le F\left(s^2\lim_{j\to\infty}\sup B_b(x_{m_j+1},x_{nj+1})\right)\\ &\le F\left(\max\left\{\begin{array}{c}\lim_{j\to\infty}\sup B_b(x_{m_j},x_{n_j}),\\ \lim_{j\to\infty}\sup B_b(x_{n_j},x_{m_j+1})\end{array}\right\}\right)-\lim_{j\to\infty}\inf\beta\left(B_b(x_{m_j},x_{n_j})\right)\\ &\le F\left(\max\left\{s\varepsilon,s\varepsilon\right\}\right)-\lim_{j\to\infty}\inf\beta\left(B_b(x_{m_j},x_{n_j})\right), \end{aligned}$$

which implies that

$$\lim_{j\to\infty}\inf\beta\left(B_b(x_{m_j},x_{n_j})\right)=0. \tag{13}$$

It is a contradiction with respect to the fact that $B_b\left(x_{m_j},x_{n_j}\right)\ge\varepsilon$, because of the property $(\beta1)$. Therefore, $\{x_n\}$ is a B_b-Cauchy sequence. Since (X,B_b) is (α,η)-complete and $\alpha\left(x_n,x_{n+1}\right)\ge 1$ or $\eta\left(x_n,x_{n+1}\right)\le 1$ for all $n\in\mathbb{N}\cup\{0\}$, the sequence $\{x_n\}$ B_b-converges to some point $x^*\in X$, that is, $\lim_{n\longrightarrow\infty}B_b\left(x_n,x^*\right)=0$. This implies that $\lim_{i\longrightarrow\infty}B_b\left(x_{2i+1},x^*\right)=0$ and $\lim_{i\longrightarrow\infty}B_b\left(x_{2i+2},x^*\right)=0$. Since T is (α,η)-continuous, by Lemma 1.1, one writes

$$\begin{aligned} \frac{1}{s}B_b\left(x^*,Tx^*\right) &= \liminf_{i\longrightarrow\infty}B_b\left(x_{2i+1},Tx_{2i+1}\right)\\ &\le \limsup_{i\longrightarrow\infty}B_b\left(x_{2i+1},Tx_{2i+1}\right)=\limsup_{i\longrightarrow\infty}B_b\left(x_{2i+1},x_{2i+2}\right)=0. \end{aligned}$$

Hence $B_b\left(x^*,Tx^*\right)=0$, and so $x^*=Tx^*$. Similarly, $x^*=Sx^*$. Therefore, x^* is a common fixed point of S and T. Let $y^*\in CFix(S,T)$ such that $y^*\neq x^*$, and $\alpha\left(x^*,y^*\right)\ge 1$ or $\eta\left(x^*,y^*\right)\le 1$. Then

$$\begin{aligned} &\beta\left(B_b(x^*,y^*)\right)+F\left(B_b(Sx^*,Ty^*)\right)\\ \le\;&\beta\left(B_b(x^*,y^*)\right)+F\left(s^2B_b(Sx^*,Ty^*)\right)\\ \le\;&F\left(\max\left\{\begin{array}{c} B_b(x^*,y^*),B_b(x^*,Sx^*),B_b(y^*,Ty^*),\\ B_b(y^*,Sx^*),\\ \frac{B_b(x^*,Sx^*)B_b(y^*,Ty^*)}{s+B_b(x^*,y^*)},\\ \frac{B_b(x^*,Sx^*)B_b(y^*,Ty^*)}{s+B_b(Sx^*,Ty^*)} \end{array}\right\}\right). \end{aligned}$$

We get

$$\beta\left(B_b(x^*,y^*)\right)+F\left(B_b(x^*,y^*)\right)\leq F\left(B_b(x^*,y^*)\right)$$

which is a contradiction. Hence $x^*=y^*$. Therefore, S and T have a unique common fixed point.

Theorem 2.1 is illustrated by the following example.

EXAMPLE 2.1. Let $X=\{1,2,3,4,5\}$. It is easy to check that the mapping $B_b:X\times X\to[0,+\infty)$ given by

$$\begin{aligned}
B_b\left(x,x\right) &= 0, \quad \text{for all } x\in X,\\
B_b\left(1,3\right) &= B_b\left(1,5\right)=B_b\left(2,3\right)=B_b\left(3,5\right)=1,\\
B_b\left(2,4\right) &= B_b\left(2,5\right)=B_b\left(4,5\right)=4,\\
B_b\left(1,2\right) &= 9,\\
B_b\left(1,4\right) &= B_b\left(3,4\right)=10,\\
B_b\left(x,y\right) &= B_b\left(y,x\right), \quad \text{for all } x,y\in X,
\end{aligned}$$

is a Branciari b-metric on X with $s=3$. Define $\beta:(0,\infty)\longrightarrow(0,\infty)$ by $\beta\left(t\right)=t+\frac{1}{150}$. Then $\beta\in\Delta_\beta$. Also, define $F:(0,\infty)\longrightarrow(-\infty,\infty)$ by $F\left(t\right)=t+\ln t$, for all $t>0$. Then $F\in\Delta_F$. Define the mappings $S,T:X\longrightarrow X$ and $\alpha,\eta:X\times X\longrightarrow[0,\infty)$ by

$$Sx=3 \text{ for all x} \in X,$$

$$\begin{aligned}
&T\left(1\right)=3, \quad T\left(2\right)=5, \quad T\left(3\right)=3,\\
&T\left(4\right)=1, \quad T\left(5\right)=2,
\end{aligned}$$

and

$$\alpha\left(x,y\right)=\left\{\begin{array}{ll} 1+\cosh\left(x+y\right), & (x,y)\in\left\{\begin{array}{c}(1,4),\\ (3,4),(3,1)\end{array}\right\},\\ \frac{1}{2+e^{(x+y)}}, & \text{otherwise,}\end{array}\right.$$

$$\eta\left(x,y\right)=\left\{\begin{array}{ll} \tanh\left(x+y\right), & (x,y)\in\left\{\begin{array}{c}(1,4),\\ (3,4),(3,1)\end{array}\right\},\\ 3+e^{-(x+y)}, & \text{otherwise.}\end{array}\right.$$

Then S and T are (α,η)-continuous and the pair (S,T) is triangular (α,η)-admissible. Let $x_0=1$. We have

$$\alpha\left(1,S\left(1\right)\right)=\alpha\left(1,1\right)\geq1 \text{ or } \eta\left(1,S\left(1\right)\right)=\eta\left(1,1\right)\leq1.$$

For $(x,y)\in\left\{(1,4),(3,4)\right\}$, $\alpha(x,y)\geq1$ or $\eta(x,y)\leq1$ and $B_b\left(Sx,Ty\right)>0$, we have

$$\beta\left(B_b(x,y)\right)+F\left(s^2B_b(Sx,Ty)\right)\leq F\left(W_b(x,y)\right).$$

Thus all conditions of Theorem 2.1 are satisfied and 3 is the unique common fixed point of S and T.

THEOREM 2.2. Let (X,B_b) be a complete Branciari b-metric space with parameter $s\geq1$ and let S,T: $X\longrightarrow X$ be self-mappings satisfying the following conditions:

(i) the pair (S,T) is triangular (α,η)-admissible;

(ii) (S,T) is an (α,η,β)-b-Branciari F-rational contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ or $\eta(x_0, Sx_0) \leq 1$;

(iv) (X, B_b) is an (α,η)-regular Branciari b-metric space.

Then S and T have a common fixed point. Moreover, S and T have a unique common fixed point when $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for all $x, y \in CFix(S,T)$.

PROOF. Let $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ or $\eta(x_0, Sx_0) \leq 1$. As in the proof as in Theorem 2.1, we construct a sequence $\{x_n\}$ in X defined by $x_{2i+1} \in Sx_{2i}$ and $x_{2i+2} \in Tx_{2i+1}$ $(i \geq 0)$ such that $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \longrightarrow x^* \in X$ as $n \longrightarrow \infty$. By condition (iv), we have $\alpha(x_n, x^*) \geq 1$ or $\eta(x_n, x^*) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (1), we have

$$\begin{aligned}
& \beta\left(B_b(x_{2n}, x^*)\right) + F\left(B_b(Sx_{2n}, Tx^*)\right) \\
\leq\ & \beta\left(B_b(x_{2n}, x^*)\right) + F\left(s^2 B_b(Sx_{2n}, Tx^*)\right) \\
\leq\ & F\left(\max\left\{\begin{array}{c} B_b(x_{2n}, x^*), B_b(x_{2n}, Sx_{2n}), B_b(x^*, Tx^*), \\ B_b(x^*, Sx_{2n}), \\ \frac{B_b(x_{2n}, Sx_{2n}) B_b(x^*, Tx^*)}{s + B_b(x_{2n}, x^*)}, \\ \frac{B_b(x_{2n}, Sx_{2n}) B_b(x^*, Tx^*)}{s + B_b(Sx_{2n}, Tx^*)} \end{array}\right\}\right),
\end{aligned}$$

which implies

$$F\left(B_b(x_{2n+1}, Tx^*)\right) \leq F\left(\max\left\{\begin{array}{c} B_b(x_{2n}, x^*), B_b(x_{2n}, x_{2n+1}), B_b(x^*, Tx^*), \\ B_b(x^*, x_{2n+1}), \\ \frac{B_b(x_{2n}, x_{2n+1}) B_b(x^*, Tx^*)}{s + B_b(x_{2n}, x^*)}, \\ \frac{B_b(x_n, x_{2n+1}) B_b(x^*, Tx^*)}{s + B_b(x_{2n+1}, Tx^*)} \end{array}\right\}\right).$$

From $(F1)$, we have

$$B_b(x_{2n+1}, Tx^*) \leq \max\left\{\begin{array}{c} B_b(x_{2n}, x^*), B_b(x_{2n}, x_{2n+1}), B_b(x^*, Tx^*), \\ B_b(x^*, x_{2n+1}), \\ \frac{B_b(x_{2n}, x_{2n+1}) B_b(x^*, Tx^*)}{s + B_b(x_{2n}, x^*)}, \\ \frac{B_b(x_{2n}, x_{2n+1}) B_b(x^*, Tx^*)}{s + B_b(x_{2n+1}, Tx^*)} \end{array}\right\}.$$

Suppose that $x^* \neq Tx^*$, then $B_b(x^*, Tx^*) > 0$. From Lemma 1.1, we get

$$\begin{aligned}
\frac{1}{s} B_b(x^*, Tx^*) &= \liminf_{n \longrightarrow \infty} B_b(x_{2n+1}, Tx^*) \\
&\leq \limsup_{n \longrightarrow \infty} B_b(x_{2n+1}, Tx^*) \leq B_b(x^*, Tx^*).
\end{aligned}$$

Hence $B_b(x^*, Tx^*) = 0$, which is a contradiction. Therefore, $x^* = Tx^*$. Similarly, $x^* = Sx^*$, so x^* is a common fixed point of S and T. The uniqueness follows similarly as in Theorem 2.1.

Now, we state the following corollaries. The first one is easy.

COROLLARY 2.1. Let $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions and (X, B_b) be an (α, η)-complete Branciari b-metric space. Consider S, T: $X \longrightarrow X$ two self-mappings satisfying the following conditions:

(i) for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $B_b(Sx, Ty) > 0$, we have

$$\begin{aligned} & \beta\left(B_b(x,y)\right) + F\left(s^2 B_b(Sx,Ty)\right) \\ \leq \quad & F\left(\alpha_1 B_b(x,y) + \alpha_2 B_b(x,Sx) + \alpha_3 B_b(y,Ty) + \alpha_4 B_b(y,Sx)\right), \end{aligned}$$

where $\beta \in \Delta_\beta$ and $\alpha_i \geq 0$ for $i \in \{1,2,3,4\}$ such that $\sum_{i=1}^{4} \alpha_i = 1$, $\alpha_3 < \frac{1}{s}$ and $F \in \Delta_F$;

(ii) the pair (S, T) is triangular (α, η)-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ or $\eta(x_0, Sx_0) \leq 1$;

(iv) either S and T are (α, η)-continuous, or (X, B_b) is an (α, η)-regular Branciari b-metric space.

Then S and T have a common fixed point.

Taking $S = T$ in Corollary 2.1, we state the following result.

COROLLARY 2.2 ([29]). Let $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions and (X, B_b) be an (α, η)-complete Branciari b-metric space. Let T: $X \longrightarrow X$ be a self-mapping satisfying the following conditions:

(i) for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $B_b(Tx, Ty) > 0$, we have

$$\begin{aligned} & \beta\left(B_b(x,y)\right) + F\left(s^2 B_b(Tx,Ty)\right) \\ \leq \quad & F\left(\alpha_1 B_b(x,y) + \alpha_2 B_b(x,Tx) + \alpha_3 B_b(y,Ty) + \alpha_4 B_b(y,Tx)\right), \end{aligned}$$

where $\beta \in \Delta_\beta$ and $\alpha_i \geq 0$ for $i \in \{1,2,3,4\}$ such that $\sum_{i=1}^{4} \alpha_i = 1$, $\alpha_3 < \frac{1}{s}$ and $F \in \Delta_F$;

(ii) T is triangular (α, η)-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(iv) either T is (α, η)-continuous, or (X, B_b) is an (α, η)-regular Branciari b-metric space.

Then T has a fixed point. Taking $\beta(t) = \tau (> 0)$ in Theorem 2.1 and Theorem 2.2, we state the following corollary (an extension of Wardowski result [20]).

COROLLARY 2.3. Let $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions. Let (X, B_b) be an (α, η)-complete Branciari b-metric space. Consider S, T: $X \longrightarrow X$ two self-mappings satisfying the following conditions:

(i) the pair (S,T) is triangular (α,η)-admissible;

(ii) for all $x,y\in X$ with $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ and $B_b(Sx,Ty)>0$, we have
$$\tau+F\left(s^2B_b(Sx,Ty)\right)\leq F\left(B_b(x,y)\right),$$
where $\tau>0$ and $F\in\Delta_F$;

(iii) there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\geq 1$ or $\eta(x_0,Sx_0)\leq 1$;

(iv) either S and T are (α,η)-continuous, or (X,B_b) is an (α,η)-regular Branciari b-metric space.

Then S and T have a common fixed point.

Taking $S=T$ in Corollary 2.3, we have

COROLLARY 2.4. Let $\alpha,\eta: X\times X\longrightarrow[0,\infty)$ be two functions, (X,B_b) be an (α,η)-complete Branciari b-metric space and let S: $X\longrightarrow X$ be a self-mapping satisfying the following conditions:

(i) S is a triangular (α,η)-admissible mapping;

(ii) for all $x,y\in X$ with $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ and $B_b(Sx,Sy)>0$, we have
$$\tau+F\left(s^2B_b(Sx,Sy)\right)\leq F\left(B_b(x,y)\right),$$
where $\tau>0$ and $F\in\Delta_F$;

(iii) there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\geq 1$ or $\eta(x_0,Sx_0)\leq 1$;

(iv) either S is (α,η)-continuous, or (X,B_b) is an (α,η)-regular Branciari b-metric space.

Then S has a fixed point.

EXAMPLE 2.2. Let $X=\{0,1,2,3\}$. Define $B_b: X\times X\longrightarrow[0,\infty)$ by
$$\begin{aligned}B_b(0,3)&=B_b(2,3)=B_b(0,2)=1,\\ B_b(1,3)&=3,\quad B_b(0,1)=6,\quad B_b(1,2)=5,\end{aligned}$$
$$B_b(x,x)=0\quad\text{and}\quad B_b(x,y)=B_b(y,x),\quad\text{for all } x,y\in X.$$
Obviously, (X,B_b) is a Branciari b-metric space with $s=\frac{6}{5}$, but (X,B_b) is not a b-metric space with the same coefficient s because the triangle inequality does not hold for all $x,y,z\in X$. Indeed,
$$6=B_b(0,1)>\frac{6}{5}\left[B_b(0,3)+B_b(3,1)\right]=\frac{6}{5}[1+3]=\frac{24}{5}.$$

Note that (X, B_b) is not a Branciari metric space because the rectangular inequality does not hold for all all $x, y, u, v \in X$. Indeed,

$$6 = B_b(0,1) > B_b(0,2) + B_b(2,3) + B_b(3,1) = 1 + 1 + 3 = 5.$$

Let $S, T : X \longrightarrow X$ be defined as

$$S(x) = \begin{cases} 0, & x \in \{0,1,2\}, \\ 2, & x = 3, \end{cases} \quad \text{and } T(x) = \begin{cases} 0, & x = 0, \\ 2, & x \in \{1,2\}, \\ 1, & x = 3. \end{cases}$$

Define $F : (0, \infty) \longrightarrow (-\infty, \infty)$ by $F(t) = \ln t$, for all $t > 0$. Also, we define $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 2, & x \in (x,y) \in \{0,1,2\}, \\ \frac{1}{5}, & \text{otherwise}, \end{cases} \quad \text{and } \eta(x,y) = \begin{cases} 1, & x \in \{0,1,2\}, \\ \frac{1}{3}, & \text{otherwise}. \end{cases}$$

For $(x,y) \in \{(0,1),(1,2)\}$ such that $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ and $B_b(Sx, Ty) > 0$, we have

$$\tau + F\left(s^2 B_b(Sx, Ty)\right) \leq F\left(B_b(x,y)\right).$$

Thus all the conditions of Corollary 2.3 are satisfied with $\tau \in (0,1]$. Thus S and T has a common fixed point, which is, $x = 0$.

Now, taking $F(t) = \ln t$ in Theorem 2.1 and Theorem 2.2, we state the following result.

COROLLARY 2.5. Let $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions. Let (X, B_b) be an (α, η)-complete Branciari b-metric space. Consider S, T: $X \longrightarrow X$ two self-mappings satisfying the following conditions:

(i) the pair (S, T) is triangular (α, η)-admissible;

(ii) for all $x, y \in X$ with $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ and $B_b(Sx, Ty) > 0$, we have

$$s^2 B_b(Sx, Ty) \leq e^{-\beta(B_b(x,y))} W_b(x,y),$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $W_b(x,y)$ is defined by (2)

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ or $\eta(x_0, Sx_0) \leq 1$;

(iv) either S and T are (α, η)-continuous, or (X, B_b) is an (α, η)-regular Branciari b-metric space.

Then S and T have a common fixed point.

3 G-β-b-Branciari F-Rational Contractions

Consistent with Jachymski [31], let (X, B_b) be a Branciari b-metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

DEFINITION 3.1 ([29]). Let (X, B_b) be a Branciari b-metric space endowed with a graph and let S: $X \longrightarrow X$ be a given mapping.

(i) (X, B_b) is said to be G-complete if every Cauchy sequence $\{x_n\}$ in X satisfying $(x_n, x_{n+1}) \in E(G)$ or $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$, is convergent in X.

(ii) (X, B_b) is said to be G-regular if for each sequence $\{x_n\}$ in X satisfying $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ (resp. $(x_{n+1}, x_n) \in E(G)$), we have $(x_n, x) \in E(G)$ (resp. $(x, x_n) \in E(G)$) for all $n \in \mathbb{N}$, we have

$$x_n \longrightarrow x \Longrightarrow Sx_n \longrightarrow Sx.$$

The main result of this section is

THEOREM 3.1. Let (X, B_b) be a G-complete Branciari b-metric space such that for all $(x, y) \in E(G)$ and $(y, z) \in E(G)$, we have $(x, z) \in E(G)$. Let $S, T : X \longrightarrow X$ be self-mappings satisfying the following assertions:

(i) for all $x, y \in X$ with $(x, y) \in E(G)$, we have $(Sx, Ty) \in E(G)$;

(ii) S and T are monotone and the following inequality holds

$$\beta(B_b(x, y)) + F\left(s^2 B_b(Sx, Ty)\right) \leq F(W_b(x, y)),$$

for all $x, y \in X$ with $((x, y) \in E(G)$ or $(y, x) \in E(G))$ and $B_b(Sx, Ty) > 0$, where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $W_b(x, y)$ is defined by (2);

(iii) there exists $x_0 \in X$ such that $(x_0, Sx_0) \in E(G)$ or $(Sx_0, x_0) \in E(G)$;

(iv) either S and T are G-continuous, or (X, B_b) is a G-regular Branciari b-metric space.

Then S and T have a common fixed point. Moreover, S and T have a unique common fixed point when $(x, y) \in E(G)$ or $(y, x) \in E(G)$ for all $x, y \in CFix(S, T)$.

PROOF. This result is obtained as a consequence of Theorem 2.1 and Theorem 2.2 by taking

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and } \eta(x, y) = \begin{cases} 1, & (y, x) \in E(G), \\ 2, & \text{otherwise}. \end{cases}$$

As a consequence of Theorem 3.1, we have

COROLLARY 3.1. Let (X, B_b) be a G-complete Branciari b-metric space such that for all $(x, y) \in E(G)$ and $(y, z) \in E(G)$, we have $(x, z) \in E(G)$. Let $S, T : X \longrightarrow X$ be self-mappings satisfying the following assertions:

(i) for all $x, y \in X$ with $(x, y) \in E(G)$, we have $(Sx, Ty) \in E(G)$;

(ii) S and T are monotone and the following inequality holds for all $x, y \in X$ with $((x, y) \in E(G)$ or $(y, x) \in E(G))$ such that $B_b(Sx, Ty) > 0$ and

$$s^2 B_b(Sx, Ty) \leq e^{-\beta(B_b(x,y))} W_b(x, y),$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $W_b(x, y)$ is defined by (2);

(iii) there exists $x_0 \in X$ such that $(x_0, Sx_0) \in E(G)$ or $(Sx_0, x_0) \in E(G)$;

(iv) either S and T are G-continuous, or (X, B_b) is a G-regular Branciari b-metric space.

Then S and T have a common fixed point.

4 Ordered β-b-Branciari F-Rational Contractions

Fixed point theorems for monotone operators in ordered metric spaces have been widely investigated and have had various applications in differential and integral equations and other branches, (see [23, 24, 25, 26] and the references therein).

Let $\preceq$ be a partial order on X. Recall that $T : X \longrightarrow X$ is nondecreasing if for all $x, y \in X$,

$$x \preceq y \Longrightarrow Tx \preceq Ty.$$

DEFINITION 4.1 ([29]). Let $(X, B_b, \preceq)$ be an ordered Branciari b-metric space and S: $X \longrightarrow X$ be a given mapping.

(i) (X, B_b) is said to be $\preceq$-complete if every Cauchy sequence $\{x_n\}$ in X satisfying $x_n \preceq x_{n+1}$ or $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$, is convergent in X.

(ii) (X, B_b) is said to be $\preceq$-regular if for each sequence $\{x_n\}$ in X satisfying $x_n \longrightarrow x$ and $x_n \preceq x_{n+1}$ (resp. $x_{n+1} \preceq x_n$), we have $x_n \preceq x$ (resp. $x \preceq x_n$) for all $n \in \mathbb{N}$, we have

$$x_n \longrightarrow x \Longrightarrow Sx_n \longrightarrow Sx.$$

Our result is

THEOREM 4.1. Let $(X, B_b, \preceq)$ be an $\preceq$-complete partially ordered Branciari b-metric space. Let S, T: $X \longrightarrow X$ be two self-mappings satisfying the following assertions:

(i) the pair (S, T) is triangular (α, η)-admissible;

(ii) S and T are monotone and the following inequality holds

$$\beta\left(B_b(x,y)\right) + F\left(s^2 B_b(Sx,Ty)\right) \leq F\left(W_b(x,y)\right),$$

for all $x, y \in X$ with $(x \preceq y$ or $y \preceq x)$ and $B_b\left(Sx, Ty\right) > 0$, where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $W_b(x,y)$ is defined by (2);

(iii) there exists $x_0 \in X$ such that $x_0 \preceq Sx_0$ or $Sx_0 \preceq x_0$;

(iv) either S and T are $\preceq$-continuous, or (X, B_b) is $\preceq$-regular.

Then S and T have a common fixed point.

PROOF. It suffices to take in Theorems 2.1 and 2.2,

$$\alpha\left(x,y\right) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and } \eta\left(x,y\right) = \begin{cases} 1, & y \preceq x, \\ 2, & \text{otherwise}. \end{cases}$$

Taking $F\left(t\right) = \ln t$ in Theorem 4.1, we state the following corollary.

COROLLARY 4.1. Let $(X, B_b, \preceq)$ be an $\preceq$-complete partially ordered Branciari b-metric space. Let $S, T \colon X \longrightarrow X$ be self-mappings satisfying the following assertions:

(i) the pair (S, T) is triangular (α, η)-admissible;

(ii) S and T are monotone and the following inequality holds

$$s^2 B_b(Sx,Ty) \leq e^{-\beta(B_b(x,y))} W_b(x,y),$$

for all $x, y \in X$ with $(x \preceq y$ or $y \preceq x)$ and $B_b\left(Sx, Ty\right) > 0$, where $\beta \in \Delta_\beta$, $F \in \Delta_F$ and $W_b(x,y)$ is defined by (2);

(iii) there exists $x_0 \in X$ such that $x_0 \preceq Sx_0$ or $Sx_0 \preceq x_0$;

(iv) either S and T are $\preceq$-continuous, or (X, B_b) is $\preceq$-regular.

Then S and T have a common fixed point.

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