Applied Mathematics E-Notes, 19(2019), 250-256 © ISSN 1607-2510
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R Type Functions And Coincidence Points*

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Received 12 May 2018

Abstract

In this paper, we introduce a new class of contraction mappings using the family of $\mathfrak{R}$ functions introduced by A. F. R. L. de Hierro and N. Shahzad [Fixed Point Theory Appl. (2015) 2015:98] and proved an interesting result on the existence of coincidence points using such class of mappings. Also, to illustrate the usability of the result obtained, we provide an example which guarantees the existence of a solution for a nonlinear equation.

1 Introduction and Preliminaries

To begin with, we have the following definitions, notations and results which will be used in the sequel.

DEFINITION 1.1 ([3]). A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following conditions:

(1) $G(s, t) \leq s$;

(2) $G(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, +\infty)$.

For C-class functions see also [4, 6, 12].

In [12], the authors generalized the simulation function introduced by Khojasteh et al. ([11]) using the function of C-class as follows:

DEFINITION 1.2. A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ has the property C_G, if there exists an $C_G \geq 0$ such that

(3) $G(s, t) > C_G$ implies $s > t$;

(4) $G(t, t) \leq C_G$, for all $t \in [0, +\infty)$.

Some examples of C-class functions that have property C_G are as follows:

*Mathematics Subject Classifications: 47H10, 54H25, 46Nxx.

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a) $G(s,t) = s - t$, $C_G = r$, $r \in [0, +\infty)$;

b) $G(s,t) = s - \frac{(2+t)t}{1+t}$, $C_G = 0$;

c) $G(s,t) = \frac{s}{1+kt}$, $k \geq 1$, $C_G = \frac{r}{1+k}$, $r \geq 2$.

For more examples of C-class functions that have property C_G see [5, 6, 12].

Recently, Khojasteh et al. ([11]) (also see [2, 7, 13]) introduced a new approach in the fixed point theory by using the following:

DEFINITION 1.3**.** A simulation function is a mapping $\zeta : [0,\infty)^2 \to \mathbb{R}$ satisfying the following:

(5) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(6) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup\limits_{n\to\infty} \zeta(t_n, s_n) < 0$.

DEFINITION 1.4**.** A C_G-simulation function is a mapping $\zeta : [0,+\infty)^2 \to \mathbb{R}$ satisfying the following:

(7) $\zeta(t,s) < G(s,t)$ for all $t, s > 0$, where $G : [0,+\infty)^2 \to \mathbb{R}$ is a C-class function;

(8) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup\limits_{n\to\infty} \zeta(t_n, s_n) < C_G$.

Some examples of simulation functions:

d) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t, s \geq 0$.

e) $\zeta(t,s) = s - \varphi(s) - t$ for all $t, s \geq 0$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\varphi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and C_G-simulation functions see [5, 7, 11, 12, 13].

Let $\mathcal{Z}_G$ be the family of all C_G-simulation functions $\zeta : [0,+\infty)^2 \to \mathbb{R}$. Each simulation function as in Definition 1.3 is also a C_G-simulation function as in Definition 1.4, but the converse is not true. For this claim, see Example 3.3 of [7] using the C-class function $G(s,t) = s - t$.

Let f and g be self maps of a set X. Recall that if $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. A pair of self maps (f, g) is called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequencs in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. The pair (f, g) is weakly compatible if f and g commute at their coincidence points.

A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq X$ is a Picard-Jungck sequence of the pair (f, g) (based on x_0) if $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$ (see also [7, Definition 4.4]).

Now, we recall the following result of Abbas and Jungck [1] used in the sequel.

PROPOSITION 1.1. Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a unique common fixed point of f and g.

The following result will be used in the sequel.

LEMMA 1.1 (see [16, 17]). Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.1}$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to +\infty$:

$$\begin{gathered} d\left(x_{m(k)}, x_{n(k)}\right),\ d\left(x_{m(k)}, x_{n(k)+1}\right),\ d\left(x_{m(k)-1}, x_{n(k)}\right), \\ d\left(x_{m(k)-1}, x_{n(k)+1}\right),\ d\left(x_{m(k)+1}, x_{n(k)+1}\right). \end{gathered} \tag{1.2}$$

2 Main Results

In this section, we establish some results on the existence and uniqueness of coincidence point by using simulation functions in the framework of metric spaces. We begin with the following definition.

We consider the family $\mathcal{R}$ of R-functions introduced by Roldán López de Hierro and Shahzad in [8]. A function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ is called R-C_G function if the following conditions hold (where $G : [0,\infty)^2 \to \mathbb{R}$ has property C_G):

(η_1) for each sequence $\{t_n\} \subset]0, +\infty[$ such that $\eta(t_n, t_{n+1}) > C_G$ for all $n \in \mathbb{N}$, we have $\lim_{n\to+\infty} t_n = 0$;

(η_2) for every two sequences $\{t_n\}, \{s_n\} \subset]0, +\infty[$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = L \geq 0$, then $L = 0$ whenever $L < t_n$ and $\eta(t_n, s_n) > C_G$ for all $n \in \mathbb{N}$;

(η_3) $\eta(t, s) < G(s, t)$ for all $t, s > 0$; here function $G : [0,\infty)^2 \to \mathbb{R}$ is element of $C-$class function which has property C_G.

Now, we use R-functions to define a new class of contractions. Let (X, d) be a metric space. Denote by Λ the family of lower semi-continuous functions $\lambda : X \to [0, +\infty[$. In the sequel, we will use the following notation

$$D(u, v; \lambda) := d(u, v) + \lambda(u) + \lambda(v) \quad \text{for all } u, v \in X \text{ and } \lambda \in \Lambda.$$

Now, we define the new family of contractions.

DEFINITION 2.1. Let (X, d) be a metric space and let $h, g : X \to X$ be self mappings. A mapping h is a $(RC_G, g)-$contraction if there exist a R-C_G- function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\eta(D(hu, hv; \lambda), D(gu, gv; \lambda)) \geq C_G \tag{1}$$

for all $u, v \in X$ with $D(gu, gv; \lambda) > 0$.

In the case, $g = i_X$ (identity mapping on X) and $C_G = 0$ we get a contraction mapping of Nastasi et al. [14].

Now, we state our result for the notion of (RC_G, g)-contraction. It generalizes the corresponding results of [5, 7, 11, 15] in several directions.

THEOREM 2.1. Let (X, d) be a metric space, $f, g : X \to X$ be self-mappings and f be a (RC_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that at least one of the following conditions hold:

(i) $(f(X), d)$ or $(g(X), d)$ is complete;

(ii) (X, d) is complete, g is continuous and (f, g) is compatible.

Then f and g have a unique point of coincidence.

PROOF. First of all we shall prove that the point of coincidence of f and g is unique (if it exists). Suppose that z_1 and z_2 are distinct points of coincidence of f and g. From this it follows that there exist two points v_1 and v_2 ($v_1 \neq v_2$) such that $fv_1 = gv_1 = z_1$ and $fv_2 = gv_2 = z_2$. Then (1) implies that

$$\begin{aligned} C_G &\leq \eta(D(fv_1, fv_2; \lambda), D(gv_1, gv_2; \lambda)) \\ &= \eta(D(z_1, z_2; \lambda), D(z_1, z_2; \lambda)) \\ &< G(D(z_1, z_2; \lambda), D(z_1, z_2; \lambda)) \leq C_G, \end{aligned}$$

which is a contradiction.

In order to prove that f and g have a point of coincidence, suppose that there is a Picard-Jungck sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N}\cup\{0\}$. If $y_k = y_{k+1}$ for some $k \in \mathbb{N}\cup\{0\}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. Substituting $u = x_{n+1}$, $v = x_{n+2}$ in (1) we obtain that

$$\begin{aligned} C_G &\leq \eta(D(fx_{n+1}, fx_{n+2}; \lambda), D(gx_{n+1}, gx_{n+2}; \lambda)) = \eta(D(y_{n+1}, y_{n+2}; \lambda), D(y_n, y_{n+1}; \lambda)) \\ &< G(D(y_n, y_{n+1}; \lambda), D(y_{n+1}, y_{n+2}; \lambda)). \end{aligned}$$

Using (3) of Definition 1.2, we have $D(y_n, y_{n+1}; \lambda) > D(y_{n+1}, y_{n+2}; \lambda)$. Hence, for all $n \in \mathbb{N}\cup\{0\}$ we get that $D(y_{n+1}, y_{n+2}; \lambda) < D(y_n, y_{n+1}; \lambda)$. Also, by using property

(η_1), we have $\lim_{n\to\infty} D(y_n, y_{n+1}; \lambda) = 0$. Consequently, $d(y_n, y_{n+1}) \to 0$ and $\lambda(y_n) \to 0$.

Further we have to prove that $y_n \neq y_m$ for $n \neq m$. Indeed, suppose that $y_n = y_m$ for some $n > m$. Then we choose $x_{n+1} = x_{m+1}$ (which is obviously possible by the definition of Picard-Jungck sequence $\{y_n\}$) and hence also $y_{n+1} = y_{m+1}$. Then following the previous arguments, we have

$$D(y_n, y_{n+1}; \lambda) < D(y_{n-1}, y_n; \lambda) < \cdots < D(y_m, y_{m+1}; \lambda) = D(y_n, y_{n+1}; \lambda),$$

which is a contradiction.

Now, we have to show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that it is not true. Putting $u = y_{m(k)+1}$, $v = y_{n(k)+1}$ in (1), we obtain

$$\begin{aligned} C_G &\leq \eta\left(D\left(y_{m(k)+1}, y_{n(k)+1}; \lambda\right), D\left(y_{m(k)}, y_{n(k)}; \lambda\right)\right) \\ &< G\left(D\left(y_{m(k)}, y_{n(k)}; \lambda\right), D\left(y_{m(k)+1}, y_{n(k)+1}; \lambda\right)\right). \end{aligned} \tag{2}$$

Using (3) of Definition 1.2, it follows that $D\left(y_{m(k)}, y_{n(k)}; \lambda\right) > D\left(y_{m(k)+1}, y_{n(k)+1}; \lambda\right)$.

Now, since the sequence $\{y_n\}$ is not a Cauchy sequence, then by Lemma 1.1, we have $d\left(y_{m(k)}, y_{n(k)}\right)$ and $d\left(y_{m(k)+1}, y_{n(k)+1}\right)$ tend to $\varepsilon > 0$, as $k \to \infty$. Also, we have

$$d\left(y_{m(k)}, y_{n(k)}\right) \leq D\left(y_{m(k)}, y_{n(k)}; \lambda\right).$$

Therefore, using inequality (2) and (η_2), we have $L = \varepsilon = 0$, which is a contradiction. Therefore, the Picard-Jungck sequence $\{y_n\}$ is a Cauchy sequence.

Suppose that (i) holds, i.e., $(g(X), d)$ is complete. Then there exists $v \in X$ such that $gx_n \to gv$ as $n \to \infty$. We shall prove that $fv = gv$. It is clear that we can suppose $y_n \neq fv, gv$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by (1), we have

$$C_G \leq \eta(D(fx_n, fv; \lambda), D(gx_n, gv; \lambda)) < G(D(gx_n, gv; \lambda), D(fx_n, fv; \lambda)).$$

Using (3) of Definition 1.2, we get $D(fx_n, fv; \lambda) < D(gx_n, gv; \lambda)$. It implies that $fx_n \to fv$ as $n \to \infty$. Hence, $fv = gv$ is a (unique) point of coincidence of f and g.

Similarly, we can prove that $fv = gv$ is a (unique) point of coincidence of f and g, when $(f(X), d)$ is complete.

Finally, suppose that (ii) holds. Since (X, d) is complete, then there exists $v \in X$ such that $fx_n \to v$, when $n \to \infty$. As g is continuous, $g(fx_n) \to gv$ when $n \to \infty$. Consider

$$C_G \leq \eta(D(f(gx_n), fv; \lambda), D(g(fx_n), gv; \lambda)) < G(D(g(fx_n), gv; \lambda), D(f(gx_n), fv; \lambda)).$$

Using (3) of Definition 1.2 and the continuity of g, we have

$$D(f(gx_n), fv; \lambda) < D(g(fx_n), gv; \lambda) \to 0, \quad as \quad n \to \infty.$$

It implies that $d(f(gx_n), fv) \to 0$, as $n \to \infty$. Further, as f and g are compatible, we have

$$d(fv, gv) \leq d(fv, f(gx_n)) + d(f(gx_n), g(fx_n)) + d(g(fx_n), gv) \to 0 + 0 + 0 = 0.$$

Hence, the result is proved in both cases, i.e., the mappings f and g have a unique point of coincidence. The proof is complete.

REMARK 2.1. (a) If (i) holds and the pair (f, g) is weakly compatible then by Proposition 1.1, f and g have a unique common fixed point.

(b) Also, if (ii) holds then f and g have a unique common fixed point again by Proposition 1.1, because each compatible pair (f, g) is a weakly compatible.

EXAMPLE 2.1. Let $X = [0, +\infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in [0, +\infty)$, and consider the mappings $f, g : [0, +\infty) \to [0, +\infty)$ given, for all $x \in [0, +\infty)$, by

$$fx = x + 2, \quad gx = 4x + e^{2x}.$$

In order to solve the nonlinear equation

$$x + 2 = 4x + e^{2x},$$

Theorem 2.1 can be applied using the function $\eta(t, s) = \frac{9}{10}\left(s - \frac{(2+t)t}{1+t}\right)$ for $s, t \in [0, +\infty)$ and $C_F = 0$, $F(s, t) = s - \frac{(2+t)t}{1+t}$ and the lower semi-continuous function $\lambda : X \to [0, \infty)$ defined by $\lambda(u) = u$ for all $u \in X$. Now, we have that

$$\begin{aligned}
&\eta\left(D\left(fx, fy; \lambda\right), D\left(gx, gy; \lambda\right)\right) \\
&= \frac{9}{10}\left(D\left(gx, gy; \lambda\right) - \frac{\left(2 + D\left(fx, fy; \lambda\right)\right) D\left(fx, fy; \lambda\right)}{1 + D\left(fx, fy; \lambda\right)}\right) \\
&= \frac{9}{10}\left(\left|4\left(x - y\right) + \left(e^{2x} - e^{2y}\right)\right| + x + y - \frac{\left(2 + |x - y| + x + y\right)\left[|x - y| + x + y\right]}{1 + |x - y| + x + y}\right) \\
&\geq 0.
\end{aligned}$$

Since $f(X) = [2, +\infty), g(X) = [1, +\infty)$, using Theorem 2.1 (i) the result follows.

CLAIM. Finally, we have an open question, Does the Theorem 2.1 hold, if we replace $D(u, v; \lambda)$ with $P(u, v; \lambda) := p(u, v) + \lambda(u) + \lambda(v)$ where p is a partial metric on X?

Acknowledgements. The authors are thankful to the learned referee and editor for their valuable remarks and suggestions to improve this work.

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