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The Number Of Collisions In Biham-Middleton-Levine Model On A Square Lattice With Limited Number Of Cars*

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Abstract

In this paper, we have studied the behavior of the deterministic Biham-Middleton-Levine (BML) model on a square lattice of $N \times N$ sites with periodic boundary conditions. To this end, we have utilized the time-corrected diagonal map, and we have discussed for how many cars the system must self-organize to reach speed one, independent of the initial configuration. Indeed, we have demonstrated that when the number of cars, m, is equal or less than $[\frac{N}{2}]$, the system definitely attains velocity one. Also, for $m \leq [\frac{N}{2}]$, we have investigated the maximum number of possible collisions before attaining speed one. For this purpose, first, we have shown that there is at least a car which never stops, then we have proved that when the number of cars, m, is bounded above by $[\frac{N}{2}]$, then the number of collisions is bounded above by $\sum_{i=1}^{m-1}(m-i)$. Also, we give an example in which the number of collisions in the system reaches its maximum before the system attaining speed one.

1 Introduction

The traffic problem is a hot topic in different fields of science, and it has been investigated many years to improve the transportation infrastructure and to minimize the public expenses. The Biham-Middleton-Levine (BML) model is a simple and very efficient model which has been introduced to study the traffic problem [4]. Various versions of this model have been studied in literature [1, 2, 3, 6, 10] and this model has been applied in different contexts [7, 8, 9].

The original BML model is defined on $\mathbb{Z}^2$ lattice, and a site is occupied by a car with probability $p \in (0, 1)$ [4]. It is already proved that the BML model on $\mathbb{Z}^2$ lattice gets stuck for all p sufficiently close to 1 [2]. Computer simulations suggested that the system has a phase transition. In other words, there is some critical value of p, say p_c, such that for all $p > p_c$ asymptotically almost surely the system get stuck, although for all $p < p_c$, asymptotically almost surely the system self-organizes (the

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system attains speed one). A decade ago, by computer simulation, an impressive result had been revealed for the system. In fact, intermediate phases of behavior have been seen between attaining speed one and getting stuck; therefore it is possible there is no $p_c \in (0,1)$ [6].

In this note, the BML model is defined on a square lattice of $N \times N$ sites with periodic boundary conditions. Each site contains either a red car pointing to the right, a blue car pointing upwards, or is empty. When the time comes to move, first red cars run and then the blue ones; i.e., in any given time step, first all red cars try to move, then all the blue cars. If the site immediately to the right of a red car is already occupied by a blue car, then the red car and all red cars in a line immediately to the left of it, cannot move. Also, if any vertical line of blue cars is blocked by a red car above them, they cannot move, and they remain on their sites.

Notation: The size of the system and the number of cars are denoted by N and m, respectively. Also, the time step is represented by t. The square brackets $[\cdot]$ stands for the integer part of a positive real number.

2 The Main Result

Let $\mathbb{Z}_N \triangleq \frac{\mathbb{Z}}{N\mathbb{Z}}$, and assume the $N \times N$ discrete torus $\mathbb{Z}_N \times \mathbb{Z}_N$. Define $D_1, D_2, ..., D_N$ by

$$D_k \triangleq \{(x,y) \in \mathbb{Z}_N^2 : x + y = k \mod N\}.$$

In fact, $D_1, ..., D_N$ are the locations of the cars on the N North-West to South-East diagonals of the torus. See Figure 1.

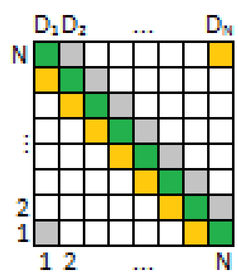

Figure 1: The positions of $D_1, ..., D_N$ are illustrated.

When a car moves in a time step, it moves up one diagonal otherwise it remains without movement. For each $t \geq 0$ let us denote by $\phi_t : \mathbb{Z}_N^2 \longrightarrow \mathbb{Z}_N$ the associated 'time-corrected diagonal map' [1] defined as $\phi_t(x,y) \triangleq x + y - t \mod N$.

Let us denote by $X^1, X^2, \cdots, X^m$ the initial positions of the cars, where m is the number of cars, and denote the position of the car i at time $t \geq 0$ by X_t^i ($X_0^i = X^i$) and $Y_t^i \triangleq \phi_t(X_t^i)$. Therefore, at a given time t, Y_t^i gives us some information about the configuration $(X_t^i)_{i \leq m}$, but it does not specify it uniquely. For any t, the points

Y_t^i are distributed within $\mathbb{Z}_N$, such that some points of $\mathbb{Z}_N$ can be occupied by more than one Y_t^i and others will be empty; note that many cars can hold different sites on the same diagonal.

By [1, Proposition 1], we know that if $m < \frac{1}{2}N$, the system must attain speed one, where m is the number of cars. In fact, we can deduce the following proposition:

PROPOSITION 1. If $m \leq [\frac{N}{2}]$, the system must reach speed one.

PROOF. Assume that N is an odd number. Since $m \in \mathbb{Z}$, we have $m \leq [\frac{N}{2}]$ if and only if $m < \frac{N}{2}$. Therefore by [1, Proposition 1], the system must attain speed one.

Now suppose that N is an even number and $m \leq [\frac{N}{2}] = \frac{N}{2}$. For $m < \frac{N}{2}$ we have the same result by [1, Proposition 1]. Assume that $m = \frac{N}{2}$. If for all times there exists an empty arc in $\mathbb{Z}_N$ of length at least two then, by [1, Proposition 1], the system must attain speed one in limited time. Suppose that at time t^* there is not an empty arc in $\mathbb{Z}_N$ of length greater than 1. Hence, $Y_{t^*}^1, Y_{t^*}^2, \cdots, Y_{t^*}^m$ are distributed in $\mathbb{Z}_{N=2m}$ so that none of them are adjacent. In fact, there is an empty arc of length one between each $Y_{t^*}^i$ and $Y_{t^*}^{i+1}$, for $1 \leq i \leq m-1$. Therefore, for each diagonal which contains a car we have two facts:

(1) That diagonal contains exactly one car.

(2) Its above diagonal is empty.

Clearly, in the next move, we do not have any collision, and all cars move freely. Thus, from t^* to t^*+1, the configuration of $Y_{t^*}^1, Y_{t^*}^2, \cdots, Y_{t^*}^m$ does not change in $\mathbb{Z}_N$. By the same argument, for all $t \geq t^*$, we do not have any change in the configuration of $Y_t^1, \cdots, Y_t^m$ in $\mathbb{Z}_N$ and no collision will happen in the system. It means that the speed is one.

Answer the following question is the primary purpose of this note:

QUESTION ([1, Question 5]). *Suppose we place a configuration of $m \leq [\frac{N}{2}]$ cars on $\mathbb{Z}_N \times \mathbb{Z}_N$ uniformly at random. By Proposition 1 the system will self-organize to attain speed one, but how many collisions will occur before it does so?*

To answer this question, first we state two facts:

(a) Suppose there exists a diagonal having some cars which do not collide with other cars (the cars out of the diagonal). Assume that the image of these cars is $Y \in \mathbb{Z}_N$. If some collisions happen between these cars, under the time-corrected diagonal map, the image of cars where an accident affected them and could not move will take one step to the left; i.e., their new image is $Y-1 \in \mathbb{Z}_N$. Now cars whose images are pushed to $Y-1$ never will collide with the cars whose images are Y.

(b) Suppose we have some cars on two consecutive diagonals with images $Y-1, Y \in \mathbb{Z}_N$. If some cars whose images are $Y-1$ collide with some cars whose images are Y, then the image of them will take one step to the left (to $Y-2 \in \mathbb{Z}_N$), and they do not have collision anymore with the cars whose images are always Y.

LEMMA 1. Consider a system of size $N \times N$ with $m \leq [\frac{N}{2}]$ cars. There exists at least a car which never stops.

PROOF. Assume that $m = [\frac{N}{2}]$. By contradiction, suppose all cars in the system have some stops. Let car i be the first car which has a stop and whose image is $Y \in \mathbb{Z}_N$, therefore, the image of car i moves to $Y-1$ after its stop. For the cars whose images are in $Y-1$ and $Y-2$, after a stop, their image move to $Y-2$ and $Y-3$, respectively. Note that for cars which immediately right to their images exist an empty arc, it is necessary some image appear directly right to their image so that they can have their stop; otherwise there is a car which never stops. Now, by [1, Lemma 3], we know that the dynamics cannot create new empty arcs of length greater than one. Therefore, since all cars must have a stop, we conclude that there is no empty arc of length greater than one after passing enough time. For a given car i with the image Y, since it had a stop (now its image is in $Y-1$) then necessarily we had a car j with the image in Y or $Y+1$. Now:

(i) If car j had image Y, after a stop its image is $Y-1$, and since we do not have any empty arc of length greater than one we have m occupied places in $\mathbb{Z}_N$, which each of them contains at least one car, and two cars have image $Y-1$. Therefore we have at least $m+1$ cars, which is a contradiction.

(ii) If car j had image $Y+1$, after a stop its image is Y (immediately next to $Y-1$), and since we do not have any empty arc of length greater than one, we have at least m cars, and places Y and $Y-1$ next to each other are occupied. Therefore we have at least $m+1$ cars, which is a contradiction.

Hence, there is a car which never stops.

In Figure 2, we give some initial configurations such that for the images with red color never appear another image immediately to its right. In fact, by the argument in the proof of Lemma 1, we see there is an image which never appear another image directly to its right. In other words, if for all images enter another image immediately to their right (even temporarily) we reach to a contradiction with the number of cars in the system.

Figure 2: No image will appear immediately right to the red images.

THEOREM 1. Consider a system of size $N \times N$ with $m \leq [\frac{N}{2}]$ cars. Then we have at most $\sum_{i=1}^{m-1}(m-i)$ collisions in this system.

PROOF. Assume that $0 \leq Y^1 \leq \cdots \leq Y^l \leq N$, $l \leq m$, are all occupied places (images) in $\mathbb{Z}_N$. Let Y^k, $1 \leq k \leq l$, be an image which does not appear another image immediately to its right. Consider the new order $Y^{k+1}, Y^{k+2}, \cdots, Y^l, Y^1, \cdots, Y^k$, and denote them without changing the order $Y^{1,1}, Y^{2,1}, \cdots, Y^{r_1,1} = Y^k$ where $r_1 = l$. Suppose $Y^{i,1}$ is the image of $y^{i,1}$ cars for $1 \leq i \leq r_1$, and $\sum_{i=1}^{r_1} y^{i,1} = m$. At least there exists one car with image $Y^{r_1,1} = Y^k$ so that it moves without any stop. Thus, for cars with the image $Y^{r_1,1}$ we have at most $y^{r_1,1} - 1$ collisions and for cars with the image $Y^{i,1}$ for $i \neq r_1$, we have at most $y^{i,1}$ collisions. Therefore, we have $y^{1,1} + \cdots + y^{r_1-1,1} + y^{r_1,1} - 1 = m - 1$ collisions. Now for the cars which had a collision their images move one step to the left, and the new configuration of the images is $Y^{1,2}, Y^{2,2}, \cdots, Y^{r_2,2}, Y^{r_1,1}$. We had $y^{r_1,1} - 1$ collisions in $Y^{r_1,1}$, thus we know $Y^{r_1,1}$ is the image of just one car (the car which moves without any collision) and $y^{i,2}$ is the number of cars which their image is $Y^{i,2}$ for $i = 1, \cdots, r_2$. Now by facts (a) and (b), cars with the image $Y^{r_2,2}$ do not have any collisions anymore with the car whose image is $Y^{r_1,1}$, and at least there exists one car with image $Y^{r_2,2}$ so that it does not stop anymore. Thus, for cars with the image $Y^{r_2,2}$ we have at most $y^{r_2,2} - 1$ collisions and for cars with the image $Y^{i,2}$, $i \neq r_2$, we have at most $y^{i,2}$, $i \neq r_2$, collisions. Therefore, we have at most $y^{1,2} + \cdots + y^{r_2-1,2} + y^{r_2,2} - 1 = m - 2$ collisions. For the cars which had a collision their images move one step to the left and the new configuration of the images is $Y^{1,3}, Y^{2,3}, \cdots, Y^{r_3,3}, Y^{r_2,2}, Y^{r_1,1}$. By continuing this process, we reach to the configuration

$$Y^{1,m-1}, Y^{2,m-1}, \cdots, Y^{r_{m-1},m-1}, Y^{r_{m-2},m-2}, Y^{r_{m-3},m-3}, \cdots, Y^{r_2,2}, Y^{r_1,1}$$

where just one collision, $(m - (m-1))$, can happen and this collision is between cars with the image $Y^{1,m-1}$, and since after all these collisions $Y^{1,m-1}$ cannot reach directly right to $Y^{r_1,1}$, we cannot have more accident in the system. Now, if we sum the number of all possible collisions in the system, we have

$$(m-1) + (m-2) + (m-3) + \cdots + 1 = \sum_{i=1}^{m-1} (m-i).$$

The proof is complete.

For more clarification about the proof of Theorem 1, see Figure 3. In this figure, $N = 8$ and $m = 4$. We have at most $6 = 3 + 2 + 1$ collisions. Here we check the images of four initial configurations. The interested reader can check all possible initial configurations.

EXAMPLE 1. In Figure 4, we give an example of an initial configuration with $N = 8$ and $m = 4$ so that the number of collisions reaches to its maximum.

3 Conclusions

In this paper, by means of the time-corrected diagonal map, we have discussed the behavior of the deterministic Biham-Middleton-Levine (BML) model of finite sites

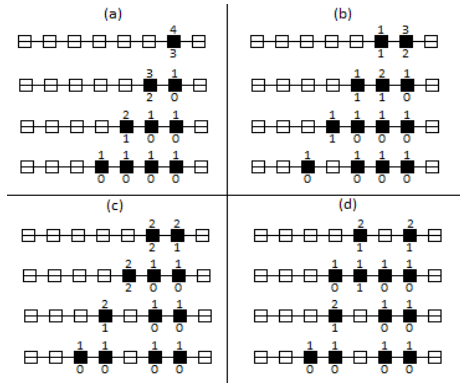

Figure 3: The numbers above and below each occupied place show the number of cars and the number of possible collisions, respectively.

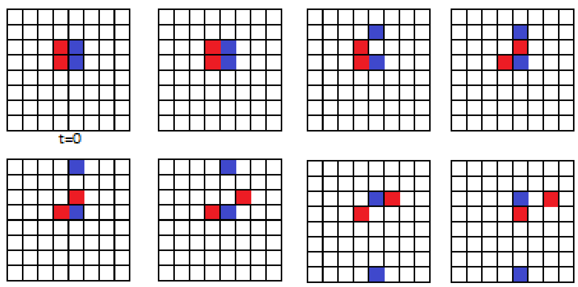

Figure 4: There are exactly six collisions in this system.

with periodic boundary conditions. First, we have shown that when the number of cars is less than or equal to $[\frac{N}{2}]$, the system definitely attains speed one. Then, we have proved that there is at least one car in the system so that it never stops provided that $m \leq [\frac{N}{2}]$. Finally, we have demonstrated that when $m \leq [\frac{N}{2}]$, the maximum number of collisions in the system is $\sum_{i=1}^{m-1}(m-i)$. Moreover, we have presented an example in which the system attains the maximum number of collisions. The present work has involved addressing the deterministic behavior of the BML model; however, future work will investigate random starting configurations with more cars.

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