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Moment Properties Of Generalized Order Statistics From Weibull-Geometric Distribution*

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Abstract

The concept of generalized order statistics was introduced by Kamps [21]. Generalized order statistics is a unified approach of other ordered random schemes, like order statistics, record values, sequential order statistics, progressively type II censored order statistics and Pfeifers records. Therefore, the study of moments and recurrence relations between moments of generalized order statistics are of special interest. In this paper, an attempt has been made to derive some recurrence relations for single and product moments of generalized order statistics from Weibull-geometric distribution, which was proposed by Barreto-Souza *et al*. [15]. Further, order statistics and record values are studied as special cases. At the end, some characterization results are also presented.

1 Introduction

The Weibull-geometric distribution was introduced by Barreto-Souza *et al*. [15] as a generalization of some of the commonly used distributions for modeling life time data, such as the extended exponential-geometric distribution, the exponential-geometric distribution and the Weibull distribution.

A random variable X is said to have the Weibull-geometric distribution if its probability density function (*pdf*) is of the form

$$f(x) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}[1-pe^{-(\beta x)^{\alpha}}]^{-2}, \quad x>0,\ \alpha>0,\ \beta>0,\ p\in(0,1) \tag{1}$$

and the corresponding survival function is

$$\bar{F}(x) = \frac{(1-p)e^{-(\beta x)^{\alpha}}}{1-pe^{-(\beta x)^{\alpha}}}, \quad x>0,\ \alpha>0,\ \beta>0,\ p\in(0,1), \tag{2}$$

where, $\bar{F}(x) = 1-F(x)$.

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Now, in view of (1) and (2), we have

$$\bar{F}(x) = \frac{[1 - pe^{-(\beta x)^{\alpha}}]}{\alpha\beta^{\alpha}x^{\alpha-1}} f(x). \tag{3}$$

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters, such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0, \quad \text{for} \quad 1 \leq i \leq n-1.$$

The random variables $X(1, n, \tilde{m}, k)$, $X(2, n, \tilde{m}, k)$, ..., $X(n, n, \tilde{m}, k)$ are said to be generalized order statistics (*gos*) from an absolutely continuous distribution function $F()$ with the probability density function (*pdf*) $f()$, if their joint *pdf* is of the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}\big[1 - F(x_i)\big]^{m_i} f(x_i)\Big)\big[1 - F(x_n)\big]^{k-1} f(x_n) \tag{4}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

If $m_i = m = 0;\ i = 1 \ldots n-1$, $k = 1$, we obtain the joint *pdf* of the order statistics and for $m \to -1$, $k \in N$, we get joint *pdf* of k^{th} record values.

Recurrence relations for the moments of *gos* for some specific as well as for general class of distribution are investigated by several authors in literature. For example see Kamps and Gather [23], Keseling [25], Cramer and Kamps [16], Ahsanullah [3], Kamps and Cramer [24], Pawlas and Szynal [32], Ahmad and Fawzy [2], Athar and Islam [8], Al-Hussaini *et al.* [5], Anwar *et al.* [6], Faizan and Athar [17], Ahmad [1], Khan *et al.* [26], Athar *et al.* [11, 12, 13, 14], Khwaja *et al.* [29], Athar and Nayabuddin [9, 10], Nayabuddin and Athar [31] and references therein.

The problem of characterization of distributions is another area that has attracted the interest of numerous researchers. Different approaches of characterization are available in the literature. Kamps [22] investigated the importance of recurrence relations between moments of order statistics in characterization. For more detailed survey one may refer to Khan and Zia [28], Athar and Nayabuddin [10], Khan and Khan [27] among others. Ahsanullah *et al.* [4] characterized certain continuous distributions by truncated moments. More information on characterization through truncated moments can be found in the works of Galambos and Kotz [18], Kotz and Shanbhag [30], Glänzel [19] and the references cited there.

2 Single Moments

Here we may consider two cases:

CASE I. $\gamma_i \neq \gamma_j, i, j = 1, 2, ..., n-1, i \neq j$.

In view of (4), the *pdf* of r^{th} *gos* $X(r,n,\tilde{m},k)$ is given as (Kamps and Cramer [24])

$$f_{X(r,n,\tilde{m},k)}(x) \;=\; C_{r-1} f(x) \sum_{i=1}^{r} [\bar{F}(x)]^{\gamma_i - 1}, \tag{5}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i \;,\; \gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \qquad 1 \le i \le r \le n.$$

CASE II. $m_i = m$, $i = 1, 2, ...n-1$.

The *pdf* of r^{th} *gos* $X(r,n,m,k)$ is given as (Kamps [21])

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\,[\bar{F}(x)]^{\gamma_r - 1}\, f(x)\, g_m^{r-1}(F(x)), \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}\,(1-x)^{m+1}, & m \neq -1 \\ \log\Big(\dfrac{1}{1-x}\Big), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0,1).$$

THEOREM 2.1. Let Case I be satisfied. For the Weibull-geometric distribution as given in (1) and $n \in N$, $\tilde{m} \in \mathbb{R}$, $k > 0$, $1 \le r \le n$, $j = 1, 2, \ldots,$

$$\begin{aligned} & E[X^j(r,n,\tilde{m},k)] \\ = \;& E[X^j(r-1,n,\tilde{m},k)] \\ & + \frac{j}{\gamma_r \alpha \beta^\alpha}\left[E[X^{j-\alpha}(r,n,\tilde{m},k)] - p\sum_{u=0}^{\infty}(-1)^u \frac{\beta^{u\alpha}}{u!} E[X^{j-\alpha(1-u)}(r,n,\tilde{m},k)]\right]. \end{aligned} \tag{7}$$

PROOF. We have by Athar and Islam [8],

$$E[\xi\,\{X(r,n,\tilde{m},k)\}] - E[\xi\,\{X(r-1,n,\tilde{m},k)\}] = C_{r-2}\int_{-\infty}^{\infty} \xi'(x) \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i} dx.$$

Let $\xi(x) = x^j$. Then

$$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = jC_{r-2}\int_{-\infty}^{\infty} x^{j-1}\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}dx.$$

Now in view of (3) , we have

$$\begin{aligned} & E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] \\ = \quad & \frac{jC_{r-1}}{\gamma_r \alpha\beta^\alpha}\int_0^{\infty} \frac{[1-pe^{-(\beta x)^\alpha}]}{x^{\alpha-1}} x^{j-1}\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1}\ f(x)dx, \end{aligned}$$

which after simplification yields (7).

Similarly, result for case II can be proved on the lines of Theorem 2.1 or by replacing $\tilde{m}$ by m.

REMARK 2.1. Let $m_i = m = 0$, $i = 1,2,...,n-1$ and $k = 1$. Then the recurrence relation for single moments of order statistics is given as

$$E(X^j_{r:n}) = E(X^j_{r-1:n}) + \frac{j}{(n-r+1)\alpha\beta^\alpha}\left[E(X^{j-\alpha}_{r:n}) - p\sum_{u=0}^{\infty}(-1)^u\frac{\beta^{u\alpha}}{u!}E(X_{r:n})^{j-\alpha(1-u)}\right].$$

REMARK 2.2. For $m_i = -1$, $i = 1,2,...,n-1$, the recurrence relation for single moments of k^{th} record values will be

$$E(X^{(k)}_{U(r)})^j = E(X^{(k)}_{U(r-1)})^j + \frac{j}{k\alpha\beta^\alpha}\left[E(X^{(k)}_{U(r)})^{j-\alpha} - p\sum_{u=0}^{\infty}(-1)^u\frac{\beta^{u\alpha}}{u!}E(X^{(k)}_{U(r)})^{j-\alpha(1-u)}\right].$$

3 Product Moments

CASE I. $\gamma_i \neq \gamma_j,\ i,j = 1,2,\ldots,n-1, i \neq j$.
The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$, is given as (Kamps and Cramer [24])

$$\begin{aligned} f_{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)}(x,y) \quad = \quad & C_{s-1}\sum_{i=r+1}^{s} a_i{}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right] \\ & \times\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},\quad x < y, \end{aligned} \tag{8}$$

where

$$a_i{}^{(r)}(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{(\gamma_j - \gamma_i)},\quad r+1 \le i \le s \le n.$$

CASE II. $m_i = m$, $i = 1,2,\ldots,n-1$.

The joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given as (Pawlas and Syznal [32])

$$\begin{aligned} & f_{X(r,n,m,k),X(s,n,m,k)}(x,y) \\ = \quad & \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\,[\bar{F}(x)]^m\, g_m^{r-1}(F(x)) \\ & \times[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar{F}(y)]^{\gamma_s-1}\,f(x)\,f(y), \quad -\infty \le x < y \le \infty. \end{aligned} \tag{9}$$

THEOREM 3.1. Let Case I be satisfied. For the Weibull-geometric distribution as given in (1) and $n \in N$, $\tilde{m} \in \mathbb{R}$, $k > 0$, $1 \le r < s \le n$, $i, j = 1, 2, \ldots$

$$\begin{aligned} & E[X^i(r,n,\tilde{m},k).X^j(s,n,\tilde{m},k)] \\ = \quad & E[X^i(r,n,\tilde{m},k).X^j(s-1,n,\tilde{m},k)] \\ & +\frac{j}{\gamma_s\alpha\beta^\alpha}\Big[E[X^i(r,n,\tilde{m},k).X^{j-\alpha}(s,n,\tilde{m},k)] \\ & -p\sum_{v=0}^{\infty}(-1)^v\frac{\beta^{v\alpha}}{v!}E[X^i(r,n,\tilde{m},k).X^{j-\alpha(1-v)}(s,n,\tilde{m},k)]\Big]. \end{aligned} \tag{10}$$

PROOF**.** We have by Athar and Islam [8],

$$\begin{aligned} & E[\xi\,\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}]-E[\xi\,\{X(r,n,\tilde{m},k).X(s-1,n,\tilde{m},k)\}] \\ = \quad & C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{d}{dy}\xi(x,y)\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}dydx. \end{aligned} \tag{11}$$

Now consider $\xi(x,y)=\xi_1(x)\xi_2(y)=x^iy^j$ in (11), then in view of (3), we get

$$\begin{aligned} & E[X^i(r,n,\tilde{m},k).X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)] \\ = \quad & \frac{jC_{r-1}}{\gamma_s\alpha\beta^\alpha}\int_0^\infty\int_x^\infty\frac{[1-pe^{-(\beta y)^\alpha}]}{y^{\alpha-1}}\,x^iy^{j-1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(x)}{\bar{F}(x)}\right]^{\gamma_i} \\ & \times\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dy\;dx, \end{aligned}$$

which leads to (10).

The expression for case II may be obtained on the lines of Theorem 3.1 or by replacing $\tilde{m}$ with m.

REMARK 3.1. Let $m_i = m = 0$, $i = 1, 2, \ldots, n-1$ and $k = 1$. Then the recurrence relation for product moments of order statistics is given as

$$\begin{aligned} & E(X_{r:n}^iX_{s:n}^j) \\ = \quad & E(X_{r:n}^iX_{s-1:n}^j) \\ & +\frac{j}{(n-s+1)\alpha\beta^\alpha}\times\left[E(X_{r:n}^iX_{s:n}^{j-\alpha})-p\sum_{v=0}^{\infty}(-1)^v\frac{\beta^{v\alpha}}{v!}E(X_{r:n}^iX_{s:n}^{j-\alpha(1-v)})\right]. \end{aligned}$$

REMARK 3.2. For $m_i = -1$, $i = 1, 2, \ldots, n-1$, the recurrence relation for product moments of k^{th} record values is

$$\begin{aligned}
& E[(X_{U(r)}^{(k)})^i (X_{U(s)}^{(k)})^j] \\
= \; & E[(X_{U(r)}^{(k)})^i (X_{U(s-1)}^{(k)})^j] \\
& + \frac{j}{k\alpha\beta^{\alpha}} \left[E[(X_{U(r)}^{(k)})^i (X_{U(s)}^{(k)})^{j-\alpha}] - p \sum_{u=0}^{\infty} (-1)^u \frac{\beta^{u\alpha}}{u!} E[(X_{U(r)}^{(k)})^i (X_{U(s)}^{(k)})^{j-\alpha(1-u)}] \right].
\end{aligned}$$

4 Characterizations

This section contains characterization results for the distribution under consideration through recurrence relations and conditional moment.

THEOREM 4.1. For any non-negative random variable ($r.v.$) X having absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all x. Fix a positive integer j. A necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$\begin{aligned}
& E[X^j(r,n,m,k)] \\
= \; & E[X^j(r-1,n,m,k)] + \frac{j}{\gamma_r \alpha\beta^{\alpha}} \Bigg[E[X^{j-\alpha}(r,n,m,k)] \\
& - p \sum_{u=0}^{\infty} (-1)^u \frac{\beta^{\alpha u}}{u!} E[X^{j-\alpha(1-u)}(r,n,m,k)] \Bigg]. \qquad (12)
\end{aligned}$$

PROOF. The necessary part follows from (7) with $\tilde{m} = m$. On the other hand, if the relation (12) is satisfied, that is

$$\begin{aligned}
& E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] \\
= \; & \frac{j}{\gamma_r \alpha\beta^{\alpha}} \left[E[X^{j-\alpha}(r,n,m,k)] - p \sum_{u=0}^{\infty} (-1)^u \frac{\beta^{\alpha u}}{u!} E[X^{j-\alpha(1-u)}(r,n,m,k)] \right].
\end{aligned}$$

Now in view of Athar and Islam [8] for $\xi(x) = x^j$, we have

$$\begin{aligned}
& \frac{j}{\gamma_r} \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j-1} [\bar{F}(x)]^{\gamma_r} \, g_m^{r-1}(F(x)) dx \\
= \; & \frac{C_{r-1}}{(r-1)!} \frac{j}{\gamma_r \alpha\beta^{\alpha}} \int_0^{\infty} x^{j-1} [\bar{F}(x)]^{\gamma_{r-1}} f(x) \, g_m^{r-1}(F(x)) \\
& \times \left\{ x^{1-\alpha} f(x) - p \sum_{u=0}^{\infty} (-1)^u \frac{\beta^{\alpha u}}{u!} x^{1-\alpha+\alpha u} f(x) \right\} dx
\end{aligned}$$

or

$$\frac{C_{r-1}}{(r-1)!}\frac{j}{\gamma_r\alpha\beta^\alpha}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_{r-1}}f(x)\,g_m^{r-1}(F(x))$$
$$\times\left\{\alpha\beta^\alpha\bar{F}(x)-x^{1-\alpha}f(x)+x^{1-\alpha}pe^{-(\beta x)^\alpha}f(x)\right\}dx=0. \quad (13)$$

Applying the extension of *Müntz-Sźasz* theorem (Hwang and Lin [20]) to (13), we get

$$f(x)=\frac{\alpha\beta^\alpha x^{\alpha-1}}{[1-pe^{-(\beta x)^\alpha}]}\bar{F}(x).$$

This proves the theorem.

THEOREM 4.2. For the condition as stated in Theorem 4.1. Fix positive integers i and j. A necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$\begin{aligned}&E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]\\=\;&E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]\\&+\frac{j}{\gamma_s\alpha\beta^\alpha}\Big[E[X^i(r,n,\tilde{m},k)X^{j-\alpha}(s,n,\tilde{m},k)]\\&-p\sum_{v=0}^{\infty}(-1)^v\frac{\beta^{v\alpha}}{v!}E[X^i(r,n,\tilde{m},k)X^{j-\alpha(1-v)}(s,n,\tilde{m},k)]\Big].\end{aligned} \quad (14)$$

PROOF. The necessary part follows from (10) with $\tilde{m}=m$. On the other hand, if the relation (14) is satisfied, that is

$$\begin{aligned}&E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]\\=\;&\frac{j}{\gamma_s\alpha\beta^\alpha}\Big[E[X^i(r,n,\tilde{m},k)X^{j-\alpha}(s,n,\tilde{m},k)]\\&-p\sum_{v=0}^{\infty}(-1)^v\frac{\beta^{v\alpha}}{v!}E[X^i(r,n,\tilde{m},k)X^{j-\alpha(1-v)}(s,n,\tilde{m},k)]\Big].\end{aligned}$$

Now by using Athar and Islam [8], for $\xi(x,y)=x^i.y^j$

$$\begin{aligned}&\frac{j}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^iy^{j-1}[\bar{F}(x)]^mf(x)g_m^{r-1}(F(x))\\&\qquad\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dydx\\=\;&\frac{j}{\gamma_s\alpha\beta^\alpha}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^iy^{j-1}[\bar{F}(x)]^mf(x)g_m^{r-1}(F(x))\\&\qquad\times[h_m(F(y))-h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_s-1}\end{aligned}$$

$$\times \left\{ y^{1-\alpha} f(y) - p \sum_{v=0}^{\infty} (-1)^v \frac{(\beta y)^{\alpha v}}{v!} y^{1-\alpha} f(y) \right\} dy dx,$$

which implies

$$\frac{j}{\gamma_s \alpha \beta^\alpha} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^{j-1} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1}$$
$$\times \left\{ \alpha \beta^\alpha \bar{F}(y) - y^{1-\alpha} f(y) + y^{1-\alpha} p e^{-(\beta y)^\alpha} f(y) \right\} dy dx = 0. \qquad (15)$$

Applying the extension of $M\ddot{u}ntz - S\acute{z}asz$ theorem (Hwang and Lin [20]) to (15), we get

$$f(y) = \frac{\alpha \beta^\alpha y^{\alpha-1}}{[1 - p e^{-(\beta y)^\alpha}]} \bar{F}(y).$$

Hence the Theorem.

THEOREM 4.3**.** Suppose that an absolutely continuous (with respect to Lebesgue measure) random variable X has the *df* $F(x)$ and *pdf* $f(x)$ for $0 < x < \infty$, such that $f'(x)$ and $E(X|X \leq x)$ exist for all $x, 0 < x < \infty$, then

$$E(X|X \leq x) = g(x)\eta(x), \qquad (16)$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{x^{1-\alpha} e^{(\beta x)^\alpha} \left(1 - p e^{-(\beta x)^\alpha}\right)^2}{p \alpha \beta^\alpha} \Big\{ - \frac{x}{\left(1 - p e^{-(\beta x)^\alpha}\right)} + \int_0^x \frac{1}{\left(1 - p e^{-(\beta u)^\alpha}\right)} du \Big\}.$$

if and only if

$$f(x) = \alpha \beta^\alpha (1-p) x^{\alpha-1} e^{-(\beta x)^\alpha} [1 - p e^{-(\beta x)^\alpha}]^{-2}, \ \ x > 0, \ \alpha > 0, \ \beta > 0, \ p \in (0,1), \qquad (17)$$

which is the *pdf* of the Weibull-geometric distribution.

PROOF. First we shall prove the necessary part. For the *pdf* given in (17), we have

$$E(X|X \leq x) = \frac{\alpha \beta^\alpha (1-p)}{F(x)} \int_0^x u.\, u^{\alpha-1} e^{-(\beta u)^\alpha} [1 - p e^{-(\beta u)^\alpha}]^{-2} du. \qquad (18)$$

Integrating (18) by parts, taking $u^{\alpha-1} e^{-(\beta u)^\alpha} [1 - p e^{-(\beta u)^\alpha}]^{-2}$ as the part to be integrated and the rest of the integrand for differentiation, we get

$$E(X|X \leq x)$$
$$= \frac{1}{F(x)} \left\{ - \frac{(1-p)}{p} \frac{x}{[1 - p e^{-(\beta x)^\alpha}]} + \frac{(1-p)}{p} \int_0^x \frac{1}{[1 - p e^{-(\beta u)^\alpha}]} du \right\}. \qquad (19)$$

Multiplying and dividing (19) by $f(x)$, we obtain the result given in (16).

To prove the sufficiency part, we have from Ahsanullah *et al.* [4],

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}$$

or

$$\frac{f'(x)}{f(x)} = -\frac{2p\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{[1 - pe^{-(\beta x)^{\alpha}}]} + \frac{(\alpha-1)}{x} - \alpha\beta^{\alpha}x^{\alpha-1}, \tag{20}$$

where

$$g'(x) = x + g(x)\left(\frac{2p\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{[1 - pe^{-(\beta x)^{\alpha}}]} - \frac{(\alpha-1)}{x} + \alpha\beta^{\alpha}x^{\alpha-1}\right).$$

Integrating both sides of (20) with respect to x, we have

$$f(x) = cx^{\alpha-1}e^{-(\beta x)^{\alpha}}[1 - pe^{-(\beta x)^{\alpha}}]^{-2}.$$

Now using the condition $\int_{-\infty}^{\infty} f(x)dx = 1$, we obtain

$$f(x) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}[1 - pe^{-(\beta x)^{\alpha}}]^{-2},$$

which completes the proof.

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