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On The Hyers-Ulam Stability Of Operator Equations In Quasi-Banach Algebras*

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Abstract

This paper mainly discusses and proves the Hyers-Ulam stability of three types of polynomial operator equations in quasi-Banach algebras, under certain conditions about coefficient operators, by constructing contraction mappings and using fixed-point methods. It is also shown that the stability of operator equations depends heavily on the specified spaces, and the results are proposed with conditions as weak as possible.

1 Introduction and Preliminaries

In 1941, D. H. Hyers [1] proved the stability of additive mappings in Banach spaces associated with the Cauchy equations. In 1978, Th. M. Rassias [2] proved the stability of $\mathbb{R}$-linear mappings associated with the Cauchy equations, and in 2002 C. Park [3] proved the stability of $\mathbb{C}$-linear mappings in Banach modules. The Banach fixed point theorem [4] is an important tool in the theory of metric spaces because it assures the existence and uniqueness of fixed points of certain self mappings of metric spaces and provides a constructive method to find those fixed points. In consequence, the fixed point method for studying the stability of functional equations was used for the first time by Baker in 1991 (see [5]). Since then, the stability of some important functional equations and their applications have been extensively studied by many mathematicians [6–13].

In recent years, fixed point methods are demonstrated to be powerful in some problems of equations. Ali, et al. [21] investigated the properties of solutions to toppled systems of differential equations of noninteger order with fractional integral boundary conditions in 2017, by converting the system of differential equations to a system of fixed point problems for condensing mapping, and developed some conditions for the Hyers-Ulam stability. Khan, et al. [24] studied the Hyers-Ulam stability of solutions for coupled nonlinear fractional order differential equations (FODEs) with boundary conditions in 2017, by using Perovs fixed point theorem and Leray-Schauder-type fixed point theorem. In 2018, Ahmad, et al. [22] investigated the existence, uniqueness, and

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stability of the solutions to a nonlinear implicit type dynamical problem of impulsive fractional differential equation with nonlocal boundary conditions involving Caputo derivative by using the Krasnoselskii fixed point theorem. This paper will also broaden the use of fixed point methods.

Operator equations in Banach spaces are of great significance in pure and applied mathematics. With functional composition as multiplication and the operator norm as norm, the algebra of all continuous linear operators on a Banach space forms a Banach algebra. Polynomial operators are a natural generalization of linear operators. Such equations encompass a broad spectrum of applied problems including all linear equations. There have been some results about the Hyers-Ulam stability of equations in number fields. In 2009, Li and Hua [14] discussed and proved the Hyers-Ulam stability of the polynomial equation $x^n + \alpha x + \beta = 0$ over $\mathbb{R}$. In 2010 and 2011, M. Bidkham, H. A. Soleiman Mezerji, and M. Eshaghi Gordji [15, 16] generalized the results of Li and Hua, by discussing and proving the Hyers-Ulam stability of general real polynomial equations, and they established the Hyers-Ulam-Rassias stability of power series equations, and investigated the generalized Hyers-Ulam stability of them. Considering that quasi-Banach algebras are the generalization of Banach algebras, it is of great interest to investigate the operator equations in quasi-Banach algebras. Motivated by the results mentioned above and [17], this paper will generalize the results about Hyers-Ulam stability, from equations in number fields to operator equations in quasi-Banach algebras. In this paper, we will investigate the Hyers-Ulam stability of three typical operator equations in quasi-Banach algebras.

DEFINITION 1.1. Let X be a linear space over $\mathbb{C}$. A quasi-norm $\|\cdot\|$ is a real-valued function on X satisfying the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and all $x \in X$;

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is defined as a complete quasi-normed space. Specially, a quasi-Banach space is a Banach space when $K = 1$.

DEFINITION 1.2. Let $(X, \|\cdot\|)$ be a quasi-normed space. $(X, \|\cdot\|)$ is called a quasi-normed algebra if X is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in X$. A complete quasi-normed algebra is called a quasi-Banach algebra. Specially, a quasi-Banach algebra is a Banach algebra when $K = C = 1$.

DEFINITION 1.3. A Banach algebra X is called a bounded Banach algebra if there exists a constant $0 < M < \infty$ such that $\|x\|\|y\| \leq M\|xy\|$ for all $x, y \in X$.

In this paper, the term algebra refers to an associative algebra over $\mathbb{C}$. For a Banach algebra or quasi-Banach algebra X, $0 \in X$ denotes the additive identity that satisfies $x+0=0+x=x$ for every $x \in X$, and e, if exists, denotes the multiplicative identity that satisfies $xe = ex = x$ for every $x \in X$. Notice that an algebra does not necessarily have a multiplicative identity. A Banach algebra is called unital if it has an identity element for the multiplication whose norm is 1. If an algebra has a multiplicative identity e, its norm $\|e\| \neq 0$ is assigned as 1 without loss of generality, because otherwise the norm $\|\cdot\|$ can be redefined by $\|\cdot\|_1 = \|e\|^{-1}\|\cdot\|$, which has the property that $\|e\|_1 = 1$. A Banach algebra is called commutative if its multiplication is commutative. If x is an element of algebra X with multiplicative identity e, we define $x^0 = e$. $x \in X$ is called regularly invertible if x is invertible and $\|x^{-1}\| = \|x\|^{-1}$.

DEFINITION 1.4. Let X be a quasi-Banach algebra and $F(x) = 0$ be an equation in X. If X_1 is a subset of X and there exists a constant $H > 0$ with the following property: for every $\epsilon > 0$ and $y \in X_1$, if $\|F(y)\| \leq \epsilon$, then there exists some $z \in X_1$ satisfying $F(z) = 0$, such that $\|y - z\| < H\epsilon$. Such H is called a Hyers-Ulam stability constant for equation $F(x) = 0$. Then we say that the equation $F(x) = 0$ has the Hyers-Ulam stability in X_1.

DEFINITION 1.5. Let $p \in \mathbb{R}$ be a real number. Let X be a quasi-Banach algebra with $K = C = 1$ and $\sum_{i=0}^{\infty} \alpha_i x^i = 0$ be a power series equation in X. If X_1 is a subset of X and there exists a constant $H > 0$ with the following property: for every $\epsilon > 0$ and $y \in X_1$, if

$$\left\|\sum_{i=0}^{\infty} \alpha_i y^i\right\| \leq \epsilon \sum_{i=0}^{\infty} \frac{\|\alpha_i\|^p}{2^i},$$

then there exists some $z \in X_1$ satisfying $\sum\limits_{i=0}^{\infty} \alpha_i z^i = 0$, such that $\|y - z\| < H\epsilon$. Such H is called a generalized Hyers-Ulam stability constant for the power series equation $\sum_{i=0}^{\infty} \alpha_i x^i = 0$. Then we say that the equation $\sum\limits_{i=0}^{\infty} \alpha_i x^i = 0$ has the generalized Hyers-Ulam stability in X_1.

Let X be a quasi-Banach algebra. This paper will mainly investigate the following three operator equations in X:

$$x^n + \alpha x + \beta = 0, \tag{1}$$

$$\sum_{i=0}^{n} \alpha_i x^i = 0, \tag{2}$$

$$\sum_{i=0}^{\infty} \alpha_i x^i = 0, \tag{3}$$

where α, β, α_i, $\beta_i \in X$ $(i = 0, 1, \cdots, n, \cdots)$ are called the coefficient operators of the corresponding equations. $B = \{x \in X | \|x\| \leq 1\}$ is a subset of X, and it is called the closed unit ball of X. This paper mainly proves that the above equations have Hyers-Ulam stability on the closed unit ball of X, under some conditions of their coefficient operators.

2 Main Results in Quasi-Banach Algebras

In this section, $(X, \|\cdot\|)$ is a unital and commutative quasi-Banach algebra over $\mathbb{C}$, with multiplicative identity e, over which the Hyers-Ulam stability of equation (1)–(3) will be investigated. The closed unit ball of X is defined as $B = \{x \in X | \|x\| \leq 1\}$. Let K be the modulus of concavity of $\|\cdot\|$. Let $C > 0$ be a constant that satisfies $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in X$. For $x \in \mathbb{R}$, $\lceil x \rceil$ is denoted as the minimum integer that is not less than x.

LEMMA 2.1. If $x_1, x_2, \cdots, x_n \in X$ $(n \geq 2)$ and $r = \lceil \log_2 n \rceil$, then

$$\|x_1 + x_2 + \cdots + x_n\| \leq K^r \left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right).$$

PROOF. Firstly it is evident that $r \geq 1$. For $n = 2$, we have $r = \lceil \log_2 2 \rceil = 1$, and $\|x_1 + x_2\| \leq K^r \left(\|x_1\| + \|x_2\|\right)$. Consider a special case $n = 2^q$ $(q \in \mathbb{Z}_+)$, then $q = \log_2 n$, and we have

$$\begin{aligned}
\|x_1 + x_2 + \cdots + x_n\| &\leq K\left(\|x_1 + \cdots + x_{2^q-1}\| + \|x_{2^{q-1}+1} + \cdots + x_n\|\right) \\
&\leq K^2(\|x_1 + \cdots + x_{2^{q-2}}\| + \|x_{2^{q-2}+1} + \cdots + x_{2^{q-1}}\| \\
&\quad + \|x_{2^{q-1}+1} + \cdots + x_{3\cdot 2^{q-2}}\| + \|x_{3\cdot 2^{q-2}+1} + \cdots + x_{2^q}\|) \\
&\quad \vdots \\
&\leq K^q\left(\|x_1\| + \|x_2\| + \cdots + \|x_{2^q}\|\right) \\
&= K^{\log_2 n}\left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right).
\end{aligned}$$

More generally, when n can not be represented as 2^q $(q \in \mathbb{Z}_+)$, let q be the minimum integer that satisfies $n \leq 2^q$ and let the power s be the minimum real number such that

$$\|x_1 + x_2 + \cdots + x_n\| \leq K^s \left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right).$$

Then it is easy to see that s is not greater than q. Consequently, the power s is not greater than the minimum integer that is not less than $\log_2 n$, so we have

$$\begin{aligned}
\|x_1 + x_2 + \cdots + x_n\| &\leq K^s\left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right) \\
&\leq K^{\log_2 n}\left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right) \\
&= K^r\left(\|x_1\| + \|x_2\| + \cdots + \|x_n\|\right),
\end{aligned}$$

which completes the proof.

In the following context, for equation $\sum_{i=0}^{n} \alpha_i x^i = 0$ in a quasi-Banach algebra, r is used to denote $\lceil \log_2 n \rceil$ for convenience.

2.1 Hyers-Ulam Stability of Equation (2) in Quasi-Banach Algebras

THEOREM 2.2. If $\alpha_1 \in X$ is invertible and satisfies

$$\left\|\alpha_1^{-1}\right\|^{-1} > CK^r \max\left\{K \sum_{i \neq 1} \|\alpha_i\|, C \sum_{2 \leq i \leq n} i \|\alpha_i\|\right\},$$

and $y \in B$ satisfies the inequality

$$\left\| \sum_{i=0}^{n} \alpha_i y^i \right\| \leq \epsilon,$$

then there exists a solution $v \in B$ of equation (2) such that $\|y - v\| \leq H\epsilon$ where $H > 0$ is a constant.

PROOF. Let $\epsilon > 0$ and $y \in B$ such that $\left\|\sum_{i=0}^{n} \alpha_i y^i\right\| \leq \epsilon$. We will prove that there exists a positive constant H which is independent of ϵ and there exists a continuous linear operator v such that $\|y - v\| \leq H\epsilon$ for some $v \in B$ satisfying equation (2). Let us introduce a new operator on B:

$$g(x) = -\alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i (x \in B).$$

As a function of x, $g(x)$ is a continuous linear operator and it maps B to X, so we have the following inequality:

$$\|g(x)\| = \left\| \alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i \right\| \leq C \left\|\alpha_1^{-1}\right\| \left\| \sum_{i \neq 1} \alpha_i x^i \right\| \leq CK^r \left\|\alpha_1^{-1}\right\| \sum_{i \neq 1} \|\alpha_i\| \leq 1,$$

i.e. $g(x) \in B$. It comes that $g(x) \in B$ for all $x \in B$.

Let us define a metric $d(x, y) = \|x - y\|$ on B, and then it is obvious that (B, d) is a complete metric space. Next, we will prove that g is a contraction operator from B to B. For any $x, y \in B$ and $x \neq y$, we have

$$\begin{aligned} d(g(x), g(y)) &= \left\| \alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i - \alpha_1^{-1} \sum_{i \neq 1} \alpha_i y^i \right\| \\ &\leq C^2 K^r \left\|\alpha_1^{-1}\right\| \|x - y\| \sum_{2 \leq i \leq n} \|\alpha_i\| \left\| \sum_{0 \leq j \leq i-1} x^j y^{i-1-j} \right\| \\ &\leq C^2 K^r \left\|\alpha_1^{-1}\right\| \|x - y\| \sum_{2 \leq i \leq n} i \|\alpha_i\| = \lambda d(x, y), \end{aligned}$$

i.e. $d(g(x), g(y)) \leq \lambda d(x, y)$, where $\lambda = C^2 K^r \left\|\alpha_1^{-1}\right\| \sum_{2 \leq i \leq n} i \|\alpha_i\| \in (0, 1)$. Thus $g(x)$ is a contraction operator that maps B to B. According to Banach fixed-point theorem [18], there exists a unique continuous linear operator $v \in B$ that satisfies $g(v) = v$. Hence the equation (2) has a solution in B.

Finally, we will prove that the equation (2) has the Hyers-Ulam stability. Since $\left\|\alpha_1^{-1}\right\|^{-1} > CK^{r+1} \sum_{i \neq 1} \|\alpha_i\|$ and then $1 - \lambda K > 0$, let $H = CK(1 - \lambda K)^{-1} \left\|\alpha_1^{-1}\right\| > 0$, then

$$\begin{aligned}\|y-v\| &= \|y-g(y)+g(y)-g(v)\| \\ &\le K\|y-g(y)\| + K\|g(y)-g(v)\| \\ &\le K\left\|y+\alpha_1^{-1}\sum_{i\neq 1}\alpha_i y^i\right\| + \lambda K\|y-v\| \\ &\le CK\left\|\alpha_1^{-1}\right\|\left\|\sum_{0\le i\le n}\alpha_i y^i\right\| + \lambda K\|y-v\|.\end{aligned}$$

Therefore, we have the inequality

$$\|y-v\| \le CK(1-\lambda K)^{-1}\left\|\alpha_1^{-1}\right\|\left\|\sum_{0\le i\le n}\alpha_i y^i\right\| \le H\epsilon,$$

so the equation (2) has the Hyers-Ulam stability.

Furthermore, we can derive the following corollary.

COROLLARY 2.3. If $\alpha_1 \in X$ is regularly invertible and

$$\|\alpha_1\| > CK^r \max\left\{K\sum_{i\neq 1}\|\alpha_i\|, C\sum_{2\le i\le n} i\,\|\alpha_i\|\right\},$$

then the equation (2) has the Hyers-Ulam stability in B, and $H = CK(1-\lambda K)^{-1}\|\alpha_1\|^{-1} > 0$ is a Hyers-Ulam stability constant for the equation (2) where

$$\lambda = C^2K^r\left\|\alpha_1\right\|^{-1}\sum_{i=2}^{n} i\|\alpha_i\| \in (0,1).$$

2.2 Hyers-Ulam Stability of Equation (1) in Quasi-Banach Algebras

Using the similar methods, we get some results about the Hyers-Ulam stability of equation (1) in quasi-Banach algebras.

THEOREM 2.4. If $\alpha \in X$ is invertible and satisfies $\left\|\alpha^{-1}\right\|^{-1} > nC^2K^{r+1}$, $\|\beta\| < C^{-1}K^{-1}\left\|\alpha^{-1}\right\|^{-1} - 1$, and $y \in B$ satisfies the inequality $\|y^n + \alpha y + \beta\| \le \epsilon$, then there exists a solution $v \in B$ of equation (1) such that $\|y-v\| \le H\epsilon$ where $H > 0$ is a constant.

Furthermore, we can derive the following corollary.

COROLLARY 2.5. If $\alpha \in X$ is regularly invertible and $\|\alpha\| > nC^2K^{r+1}$, $\|\beta\| < \frac{\|\alpha\|}{CK} - 1$, then the equation (1) has the Hyers-Ulam stability in B, and $H = CK(1-\gamma K)^{-1}\|\alpha\|^{-1} > 0$ is a Hyers-Ulam stability constant for the equation (1), where $\gamma = nC^2K^r\left\|\alpha\right\|^{-1} \in (0,1)$.

3 Special Cases in Banach Algebras

In this section, $(X, \|\cdot\|)$ is a unital and commutative Banach algebra over $\mathbb{C}$, with multiplicative identity e, over which the Hyers-Ulam stability of equation (1)–(3) will be investigated. The closed unit ball of X is defined as $B = \{x \in X \mid \|x\| \leq 1\}$.

THEOREM 3.1 (Gelfand-Mazur, [19]). If Banach algebra X is also a division algebra, then X is isometrically isomorphic to the complex field $\mathbb{C}$.

THEOREM 3.2. Let X be a unital Banach algebra. If X is a division algebra, then $\|x^{-1}\| = \|x\|^{-1}$ for every nonzero operator $x \in X$.

PROOF. Firstly, X is isometrically isomorphic to the complex field $\mathbb{C}$, according to the Gelfand-Mazur theorem. Let $d(x, y) = \|x - y\|$ be the induced metric from the norm $\|\cdot\|$, hence (X, d) is a metric space. So there exists an isometrical isomorphism $\phi : X \to \mathbb{C}$ satisfying $\|\phi(x)\|_{\mathbb{C}} = \|x\|$ for all $x \in X$, where $\|\cdot\|_{\mathbb{C}}$ denotes the natural norm of $\mathbb{C}$.

Let x be a nonzero operator in X, then $\|x\| \neq 0$ and its multiplicative inverse x^{-1} exists. It is evident that $\phi(x) \in \mathbb{C}$, so we have $\|\phi(x)\|_{\mathbb{C}} = |\phi(x)| = \|x\|$. Since ϕ is an isometrical isomorphism, we have

$$\begin{aligned} 1 = \|e\| = |\phi(e)| &= |\phi(xx^{-1})| \\ &= |\phi(x)\phi(x^{-1})| \\ &= |\phi(x)||\phi(x^{-1})| \\ &= \|x\| \left\|x^{-1}\right\|. \end{aligned}$$

Therefore, $\|x^{-1}\| = \|x\|^{-1}$ holds for every nonzero operator $x \in X$.

THEOREM 3.3 ([19]). If X is a bounded Banach algebra, then X is isometrically isomorphic to the complex field $\mathbb{C}$.

Similar to the proof of Theorem 3.2, we can derive the following corollary.

COROLLARY 3.4. If X is a unital bounded Banach algebra, then $\|x^{-1}\| = \|x\|^{-1}$ for every nonzero operator $x \in X$.

3.1 Hyers-Ulam Stability of Equation (2) in Banach Algebras

Using the methods in the proof of Theorem 2.2, we have the following results.

THEOREM 3.5. If $\alpha_1 \in X$ is invertible and satisfies

$$\left\|\alpha_1^{-1}\right\|^{-1} > \max\left\{\sum_{i\neq 1} \|\alpha_i\|, \sum_{2\leq i\leq n} i\,\|\alpha_i\|\right\},$$

and $y \in B$ satisfies the inequality

$$\left\| \sum_{i=0}^{n} \alpha_i y^i \right\| \leq \epsilon,$$

then there exists a solution $v \in B$ of equation (2) such that $\|y - v\| \leq H\epsilon$ where $H > 0$ is a constant.

Furthermore, we can derive the following corollaries.

COROLLARY 3.6. If X is also a division algebra and

$$\|\alpha_1\| > \max \left\{ \sum_{i \neq 1} \|\alpha_i\|, \sum_{2 \leq i \leq n} i \|\alpha_i\| \right\},$$

then the equation (2) has the Hyers-Ulam stability in B, and $H = (1-\lambda)^{-1}\|\alpha_1\|^{-1} > 0$ is a Hyers-Ulam stability constant for the equation (2) where

$$\lambda = \left\|\alpha_1^{-1}\right\| \sum_{i=2}^{n} i\|\alpha_i\| \in (0, 1).$$

COROLLARY 3.7. If X is bounded and

$$\|\alpha_1\| > \max \left\{ \sum_{i \neq 1} \|\alpha_i\|, \sum_{2 \leq i \leq n} i \|\alpha_i\| \right\},$$

then the equation (2) has the Hyers-Ulam stability in B, and $H = (1-\lambda)^{-1}\|\alpha_1\|^{-1} > 0$ is a Hyers-Ulam stability constant for the equation (2) where

$$\lambda = \left\|\alpha_1^{-1}\right\| \sum_{i=2}^{n} i\|\alpha_i\| \in (0, 1).$$

COROLLARY 3.8 (Hyers-Ulam stability of equation (1) in Banach algebras). If $\alpha \in X$ is invertible and $\left\|\alpha^{-1}\right\|^{-1} > n$, $\|\beta\| < \left\|\alpha^{-1}\right\|^{-1} - 1$, then the equation (1) has the Hyers-Ulam stability in B, and $H = \frac{\|\alpha^{-1}\|}{1-n\|\alpha^{-1}\|} > 0$ is a Hyers-Ulam stability constant for the equation (1).

3.2 Generalized Hyers-Ulam Stability of Equation (3) in Banach Algebras

THEOREM 3.9. Let $p \in \mathbb{R}$ be a real number. If $\alpha_1 \in X$ is invertible and satisfies

$$\left\|\alpha_1^{-1}\right\|^{-1} > \max \left\{ \sum_{i \neq 1} \|\alpha_i\|, \sum_{i \geq 2} i \|\alpha_i\| \right\},$$

and $y \in B$ satisfies the inequality

$$\left\|\sum_{i=0}^{\infty} \alpha_i y^i\right\| \leq \epsilon \sum_{i=0}^{\infty} \frac{\|\alpha_i\|^p}{2^i},$$

then there exists a solution $v \in B$ of equation (3) such that $\|y-v\| \leq H\epsilon$ where $H > 0$ is a constant.

PROOF. Let $\epsilon > 0$ and $y \in B$ such that $\left\|\sum_{i=0}^{\infty} \alpha_i y^i\right\| \leq \epsilon \sum_{i=0}^{\infty} \frac{\|\alpha_i\|^p}{2^i}$. We shall show that there exists a constant $H > 0$ independent of ϵ and v such that $\|y-v\| \leq H\epsilon$ for some $v \in B$ satisfying equation (3). If we set $g(x) = -\alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i$ $(x \in B)$, then we have

$$\|g(x)\| = \left\|\alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i\right\| \leq \left\|\alpha_1^{-1}\right\| \left\|\sum_{i \neq 1} \alpha_i x^i\right\| \leq \left\|\alpha_1^{-1}\right\| \sum_{i \neq 1} \|\alpha_i\| \leq 1,$$

i.e. $g(x) \in B$. Define a metric $d(x,y) = \|x-y\|$ on B and then (B,d) is a complete metric space. The mapping g maps B to B.

Next, we shall show that g is a contraction mapping from B to B. For any $x, y \in B$ and $x \neq y$, we have

$$\begin{aligned} d(g(x), g(y)) &= \left\|\alpha_1^{-1} \sum_{i \neq 1} \alpha_i x^i - \alpha_1^{-1} \sum_{i \neq 1} \alpha_i y^i\right\| \\ &\leq \left\|\alpha_1^{-1}\right\| \|x-y\| \sum_{i=2}^{\infty} \|\alpha_i\| \left\|\sum_{j=0}^{i-1} x^j y^{i-1-j}\right\| \\ &\leq \left\|\alpha_1^{-1}\right\| \|x-y\| \sum_{i=2}^{\infty} i \|\alpha_i\| = \theta d(x,y), \end{aligned}$$

i.e. $d(g(x), g(y)) \leq \theta d(x,y)$, where $\theta = \left\|\alpha_1^{-1}\right\| \sum_{i=2}^{\infty} i \|\alpha_i\| \in (0,1)$.

Thus g is a contraction mapping from B to B. According to Banach fixed-point theorem [18], there exists a unique $v \in B$ such that $g(v) = v$. Hence the equation (3) has a solution in B.

Finally, we show that the equation (3) has the generalized Hyers-Ulam stability. Since $\left\|\alpha_1^{-1}\right\|^{-1} > \sum_{i \neq 1} \|\alpha_i\|$ and then $1 - \theta K > 0$, let us introduce the constant $H = 2(1-\theta)^{-1} \left\|\alpha_1^{-1}\right\| \|\alpha_1\|^p > 0$, then

$$\begin{aligned} \|y-v\| &= \|y - g(y) + g(y) - g(v)\| \\ &\leq \|y - g(y)\| + \|g(y) - g(v)\| \\ &\leq \left\|y + \alpha_1^{-1} \sum_{i \neq 1} \alpha_i y^i\right\| + \theta \|y-v\| \\ &\leq \left\|\alpha_1^{-1}\right\| \left\|\sum_{0 \leq i \leq n} \alpha_i y^i\right\| + \theta \|y-v\|. \end{aligned}$$

Therefore, we easily have the inequality

$$\begin{aligned}
\|y-v\| &\le (1-\theta)^{-1}\left\|\alpha_1^{-1}\right\|\left\|\sum_{i=0}^{\infty}\alpha_i y^i\right\| \\
&\le (1-\theta)^{-1}\left\|\alpha_1^{-1}\right\|\epsilon\sum_{i=0}^{\infty}\frac{\|\alpha_i\|^p}{2^i} \\
&\le (1-\theta)^{-1}\left\|\alpha_1^{-1}\right\|\epsilon\sum_{i=0}^{\infty}\frac{\|\alpha_1\|^p}{2^i} \\
&= 2(1-\theta)^{-1}\left\|\alpha_1^{-1}\right\|\|\alpha_1\|^p\epsilon \le H\epsilon,
\end{aligned}$$

which completes the proof.

Furthermore, we can derive the following corollaries.

COROLLARY 3.10. Let $p \in \mathbb{R}$ be a real number. If X is also a division algebra and

$$\|\alpha_1\| > \max\left\{\sum_{i\neq 1}\|\alpha_i\|, \sum_{i\geq 2} i\,\|\alpha_i\|\right\},$$

then the equation (3) has the generalized Hyers-Ulam stability in B, and $H = 2(1-\theta)^{-1}\|\alpha_1\|^{p-1} > 0$ is a generalized Hyers-Ulam stability constant for the equation (3) where

$$\theta = \|\alpha_1\|^{-1}\sum_{i=2}^{\infty} i\|\alpha_i\| \in (0,1).$$

COROLLARY 3.11. Let $p \in \mathbb{R}$ be a real number. If X is bounded and

$$\|\alpha_1\| > \max\left\{\sum_{i\neq 1}\|\alpha_i\|, \sum_{i\geq 2} i\,\|\alpha_i\|\right\},$$

then the equation (3) has the generalized Hyers-Ulam stability in B, and $H = 2(1-\theta)^{-1}\|\alpha_1\|^{p-1} > 0$ is a generalized Hyers-Ulam stability constant for the equation (3) where

$$\theta = \|\alpha_1\|^{-1}\sum_{i=2}^{\infty} i\|\alpha_i\| \in (0,1).$$

Unfortunately, we could not prove the generalized Hyers-Ulam stability of the power series operator equations in quasi-Banach algebras via the existing methods used above. It is an interesting open problem whether the power series operator equations have the generalized Hyers-Ulam stability for the case they have some solutions in quasi-Banach algebras.

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