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Statistical Relative Equal Convergence Of Double Function Sequences And Korovkin-Type Approximation Theorem*

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Abstract

In this work, we introduce our new concept of the statistical equal convergence for double function sequences. Later, we introduce the concept of statistical relative equal convergence of double function sequences which is stronger than the notion of statistical relative convergence and statistical uniform convergence to demonstrate a Korovkin type approximation theorem and prove that our theorem is a non-trivial extension of some well-known Korovkin type approximation theorems which were proven by earlier authors. We present an example in support of our definition and result presented in this paper. Finally, we study Voronovskaya type theorem via statistical relative equal convergence.

1 Introduction

The well-known Korovkin type approximation theorem plays a very effective role in summability theory. It is principally related to limit $\lim \|L_n(f) - f\| = 0$, where (L_n) is a sequence of positive linear operators from $C[a, b]$ into itself and $\|f\|$ denotes the usual supremum norm of f in $C[a, b]$. Firstly, Korovkin [14] introduced the sufficient conditions for uniform convergence of $L_n(f)$ to a function f by using the functions x^i, $i = 0, 1, 2$. Quite a few researchers have generalized or extended this type approximation problems using different types of convergence, for a sequence of positive linear operators defined on various spaces([1, 2, 9, 10, 11, 21, 22]).

Now, let's remember some notations used in this article.

A double sequence $x = (x_{mn})$ is said to be convergent to ℓ in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{mn} - \ell| < \varepsilon$ whenever $m, n > N$, where ℓ is called the Pringsheim limit of x and denoted by $P - \lim x = \ell$ (see [18]). By a bounded double sequence we mean there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note

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that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Let $K \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Then the natural density *of* K, denoted by $\delta_2(K)$, is given by:

$$\delta_2(K) := P - \lim_{m,n} \frac{|\{j \le m, k \le n : (j,k) \in K\}|}{mn}$$

provided that the limit on the right-hand side exists in the Pringheim's sense. A real double sequence $x = (x_{mn})$ is said to be statistical convergent to ℓ if, for every $\varepsilon > 0$, the following set

$$K_\varepsilon := \left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - \ell| \ge \varepsilon\right\}$$

has natural density zero, i.e.,

$$\delta_2 \left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - \ell| \ge \varepsilon\right\}) = 0.$$

In this case, we write $st_2 - \lim x = \ell$ ([15]).

Let f and f_{mn} belong to $C(I)$, which is the space of all continuous real valued function on a compact subset I of the two dimensional real numbers. Also, $\|f\|$ denotes the usual supremum norm of f in $C(I)$.

DEFINITION 1 ([13]). (f_{mn}) is said to be statistical pointwise convergent to f on I if for every $\varepsilon > 0$ and for any $(x,y) \in I$, $\delta_2 \left\{(m,n) \in \mathbb{N}^2 : |f_{mn}(x,y) - f(x,y)| \ge \varepsilon\right\} = 0$. In this case, we write $f_{mn} \overset{st_2}{\to} f$ on I.

DEFINITION 2 ([13]). (f_{mn}) is said to be statistical uniform convergent to f on I if for every $\varepsilon > 0$,

$$\delta_2 \left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in I} |f_{mn}(x,y) - f(x,y)| \ge \varepsilon\right\} = 0.$$

In this case, we write $f_{mn} \overset{st_2}{\rightrightarrows} f$ on I.

2 Statistical Relative Equal Convergence for Double Function Sequences

E. H. Moore [16] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, E. W. Chittenden [4] gave the definition of relatively uniform convergence is equivalent to the definition given by Moore.

Similarly, a double sequence (f_{mn}) of functions, defined on any compact subset of the real two-dimensional space, converges *relatively uniform to a limit function* f if there exists a function $\sigma(x,y)$, called a scale function such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $m, n > n_\varepsilon$, the inequality

$$|f_{mn}(x,y) - f(x,y)| < \varepsilon\, |\sigma(x,y)|$$

holds uniformly in (x,y) . The double sequence (f_{mn}) is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniform.*

Okçu Şahin and Dirik gave the definition statistical relative uniform convergence which is stronger than statistical uniform convergence.

DEFINITION 3 ([17]). (f_{mn}) is said to be statistically *relatively uniform* convergent to f on I if there exists a function $\sigma(x,y)$, $|\sigma(x,y)|>0$ such that for every $\varepsilon>0$, the following set

$$M_{\varepsilon}:=\left\{(m,n)\ :\ \sup_{(x,y)\in I}\left|\frac{f_{mn}(x,y)-f(x,y)}{\sigma(x,y)}\right|\geq\ \varepsilon\right\}$$

has natural density zero, i.e.,

$$\delta_2\left\{(m,n)\ :\ \sup_{(x,y)\in I}\left|\frac{f_{mn}(x,y)-f(x,y)}{\sigma(x,y)}\right|\geq\ \varepsilon\right\}=0.$$

This limit is denoted by $st_2-f_{mn}\rightrightarrows f\ (I;\sigma)$.

It will be observed that uniform convergence is the special case of relatively uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [3, 4, 5, 11, 20]).

The definition of equal convergence for real functions was introduced by Császár and Laczkovich and they improved their investigations on this convergence([6, 7]). Also, Das, Dutta and Pal introduced the ideas of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence with the help of ideals by extending the equal convergence([8]).

Firstly, we introduce the concept of statistical equal convergence for double function sequences belong to $C(I)$.

DEFINITION 4. (f_{mn}) is said to be statistical equal convergent to f on I if there is a double sequence of positive numbers (ε_{mn}) with $st_2-\lim\varepsilon_{mn}=0$ such that for any $(x,y)\in I$,

$$\delta_2\left(\left\{(m,n)\in\mathbb{N}^2:|f_{mn}(x,y)-f(x,y)|\geq\varepsilon_{mn}\right\}\right)=0.$$

In this case we write $f_{mn}\overset{eq-st_2}{\rightarrow}f$ on I.

Then, we give the result as follows:

LEMMA 1. $f_{mn}\rightrightarrows f$ on I implies $f_{mn}\overset{st_2}{\rightrightarrows}f$ on I, which also implies $f_{mn}\overset{eq-st_2}{\rightarrow}f$ on I. Furthermore, $f_{mn}\overset{eq-st_2}{\rightarrow}f$ on I implies $f_{mn}\overset{st_2}{\rightarrow}f$ on I.

PROOF. Let $f_{mn}\overset{st_2}{\rightrightarrows}f$ on I i.e., $st_2-\lim\|f_{mn}-f\|=0$. Let $\varepsilon>0$ be given. Define the set $D:=\left\{(m,n)\in\mathbb{N}^2:\|f_{mn}-f\|\geq\varepsilon\right\}$. Now we take the double sequence as follows:

$$\varepsilon_{mn}:=\begin{cases}\frac{1}{mn}, & (m,n)\in D,\\ \|f_{mn}-f\|+\frac{1}{mn}, & (m,n)\in D^c.\end{cases}$$

It is easy to see that, $st_2 - \lim \varepsilon_{mn} = 0$. Also, for all $(m,n) \in D^c$, we obtain $|f_{mn}(x,y) - f(x,y)| < \|f_{mn} - f\| < \varepsilon_{mn}$, which implies $f_{mn} \overset{eq-st_2}{\rightarrow} f$ on I.

Now we give an example which establishes the fact that statistical equal convergence is stronger than statistical uniform convergence for double function sequences.

EXAMPLE 1. Let $I = [0,1] \times [0,1]$, g is a function by $g(x,y) = 0$ for $(x,y) \in I$ and $A \subset \mathbb{N}^2$ with $\delta_2(A) = 0$. For each $(m,n) \in \mathbb{N}^2$, define $g_{mn} \in C(I)$ by

$$g_{mn}(x,y) = \begin{cases} \frac{x+y}{2} & , \quad (m,n) \in A, \\ 0 & , \quad (m,n) \in A^c. \end{cases}$$

Take a double sequence (ε_{mn}) by defined

$$\varepsilon_{mn} := \begin{cases} m+n, & (m,n) \in A, \\ \frac{1}{mn}, & (m,n) \in A^c. \end{cases}$$

Then we see that $st_2 - \lim \varepsilon_{mn} = 0$. Also, we obtain, for every $(x,y) \in I$,

$$\left\{(m,n) \in \mathbb{N}^2 : |g_{mn}(x,y) - g(x,y)| \geq \varepsilon_{mn}\right\} = \emptyset.$$

It is shown that $g_{mn} \overset{eq-st_2}{\rightarrow} g$ on I. But (g_{mn}) is not statistical uniform convergent and uniform convergent to the function $g = 0$ on I.

Now, we give the new concept of convergence with help of the concepts in Definition 3 and Definition 4 as follows:

DEFINITION 5. (f_{mn}) is said to be statistically relative equal convergent to f on I if there is a double sequence of positive numbers (ε_{mn}) with $st_2 - \lim \varepsilon_{mn} = 0$ and there exists a function $\sigma(x,y)$, $|\sigma(x,y)| > 0$, such that for any $(x,y) \in I$,

$$\delta_2 \left\{(m,n) \in \mathbb{N}^2 : \left|\frac{f_{mn}(x,y) - f(x,y)}{\sigma(x,y)}\right| \geq \varepsilon_{mn}\right\} = 0.$$

In this case we write $f_{mn} \overset{eq-st_2}{\rightarrow} f \ (I;\sigma)$.

REMARK 1. If the scale function is a non-zero constant, statistically relative equal convergence reduces to statistical equal convergence.

EXAMPLE 2. Take (g_{mn}) on I is given by

$$g_{mn}(x,y) := \begin{cases} \frac{xy}{2}, & m,n \text{ are squares}, \\ 0, & \text{otherwise}. \end{cases}$$

Also, take a double sequence (ε_{mn}) defined by

$$\varepsilon_{mn} := \begin{cases} 4(m^2+n^2), & m,n \text{ are squares}, \\ \frac{1}{m+n}, & \text{otherwise}. \end{cases}$$

Then we see that $st_2 - \lim \varepsilon_{mn} = 0$. Let $\sigma(x,y) = \frac{3}{x+y+1}$ for $(x,y) \in I$. Note that (g_{mn}) is statistically relatively equal convergent to $g = 0$ but it is not uniform convergent, statistical uniform convergent and statistically relatively uniform convergent to $g = 0$.

3 A Korovkin Type Approximation Theorem

In this section we implement the notion of statistical relative equal convergence of a double sequence of functions to demonstrate a Korovkin type approximation theorem in $C(I)$. Let T be a linear operator from $C(I)$ into $C(I)$. Then, as usual, we say that T is positive linear operator provided that $f \geq 0$ implies $T(f) \geq 0$. Also, we denote the value of $T(f)$ at a point $(x,y) \in I$ by $T(f(u,t);x,y)$ or, briefly, $T(f;x,y)$.

THEOREM 1. Let (T_{mn}) be a double sequence of positive linear operators acting from $C(I)$ into itself. Then, for all $f \in C(I)$,

$$T_{mn}(f) \overset{eq-st_2}{\to} f\left(I;\sigma\right), \tag{1}$$

if and only if

$$\begin{aligned} &T_{mn}(e_{00}) \overset{eq-st_2}{\to} e_{00}\left(I;\sigma_1\right), \\ &T_{mn}(e_{10}) \overset{eq-st_2}{\to} e_{10}\left(I;\sigma_2\right), \\ &T_{mn}(e_{01}) \overset{eq-st_2}{\to} e_{01}\ \left(I;\sigma_3\right), \\ &T_{mn}(e_{20}+e_{02}) \overset{eq-st_2}{\to} e_{20}+e_{02}\ \left(I;\sigma_4\right), \end{aligned} \tag{2}$$

where $e_{ij}\left(x,y\right) = x^i y^j$ for $i,j = 0,1,2,\ \sigma(x,y) = \max\{|\sigma_k(x,y)|\},\ |\sigma_k(x,y)| > 0$ for $k = 1,\ 2,\ 3,\ 4$.

PROOF. Since each $e_{ij}\left(x,y\right) = x^i y^j$, $i,j = 0,1,2$, belongs to $C\left(I\right)$, the implication (1)⇒(2) is obvious. Suppose now that (2) holds. Also, since f is continuous on I, we can write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\left|f\left(u,t\right) - f\left(x,y\right)\right| < \varepsilon$ for all $(u,t) \in I$ satisfying $|u-x| < \delta$ and $|t-y| < \delta$. Then, we get

$$\left|f\left(u,t\right) - f\left(x,y\right)\right| < \varepsilon + \frac{2\left\|f\right\|}{\delta^2}\psi\left(u,t\right), \tag{3}$$

where $\psi\left(u,t\right) = \left(u-x\right)^2 + \left(t-y\right)^2$. Since T_{mn} is positive linear operator, using the inequality (3), we obtain

$$\begin{aligned} & \left|T_{mn}\left(f;x,y\right) - f\left(x,y\right)\right| \\ = \ & \left|T_{mn}\left(f\left(u,t\right) - f\left(x,y\right);x,y\right) + f\left(x,y\right)\left(T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right)\right| \\ \leq \ & T_{mn}\left(\left|f\left(u,t\right) - f\left(x,y\right)\right|;x,y\right) + \left\|f\right\|\left|T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right| \\ \leq \ & T_{mn}\left(\varepsilon + \frac{2\left\|f\right\|}{\delta^2}\psi\left(u,t\right);x,y\right) + \left\|f\right\|\left|T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right| \\ = \ & \varepsilon T_{mn}\left(e_{00};x,y\right) + \frac{2\left\|f\right\|}{\delta^2}T_{mn}\left(\psi\left(u,t\right);x,y\right) \\ & + \left\|f\right\|\left|T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right|. \end{aligned} \tag{4}$$

Now, we calculate the term of "$T_{mn}(\psi(u,t);x,y)$";

$$\begin{aligned}
& T_{mn}(\psi(u,t);x,y) \\
= \; & T_{mn}\left((u-x)^2+(t-y)^2;x,y\right) \\
= \; & T_{mn}(e_{20}+e_{02};x,y)-2xT_{mn}(e_{10};x,y) \\
& -2yT_{mn}(e_{01};x,y)+\left(x^2+y^2\right)T_{mn}(e_{00};x,y) \\
\leq \; & \left|T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)\right|+ \\
& +2\|x\|\left|T_{mn}(e_{10};x,y)-e_{10}(x,y)\right|+2\|y\|\left|T_{mn}(e_{01};x,y)-e_{01}(x,y)\right| \\
& +\left\|x^2+y^2\right\|\left|T_{mn}(e_{00};x,y)-e_{00}(x,y)\right|. \qquad (5)
\end{aligned}$$

Hence, combining (4) and (5), we write

$$\begin{aligned}
& \left|T_{mn}(f;x,y)-f(x,y)\right| \\
\leq \; & \left(\varepsilon+\|f\|+\frac{2\|f\|}{\delta^2}\left\|x^2+y^2\right\|\right)\left|T_{mn}(e_{00};x,y)-e_{00}(x,y)\right| \\
& +\frac{4\|f\|}{\delta^2}\|x\|\left|T_{mn}(e_{10};x,y)-e_{10}(x,y)\right| \\
& +\frac{4\|f\|}{\delta^2}\|y\|\left|T_{mn}(e_{01};x,y)-e_{01}(x,y)\right| \\
& +\frac{2\|f\|}{\delta^2}\left|T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)\right|+\varepsilon.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\left|T_{mn}(f;x,y)-f(x,y)\right| \leq \; & \varepsilon+\theta\{\left|T_{mn}(e_{00};x,y)-e_{00}(x,y)\right| \\
& +\left|T_{mn}(e_{10};x,y)-e_{10}(x,y)\right| \\
& +\left|T_{mn}(e_{01};x,y)-e_{01}(x,y)\right| \\
& +\left|T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)\right|\}
\end{aligned}$$

where

$$\theta=\max\left\{\varepsilon+\|f\|+\frac{2\|f\|}{\delta^2}\left\|x^2+y^2\right\|,\frac{4\|f\|}{\delta^2}\|x\|,\frac{4\|f\|}{\delta^2}\|y\|,\frac{2\|f\|}{\delta^2}\right\}.$$

Since ε is arbitrary, we can write

$$\begin{aligned}
\left|T_{mn}(f;x,y)-f(x,y)\right| \leq \; & \theta\{\left|T_{mn}(e_{00};x,y)-e_{00}(x,y)\right| \\
& +\left|T_{mn}(e_{10};x,y)-e_{10}(x,y)\right|+\left|T_{mn}(e_{01};x,y)-e_{01}(x,y)\right| \\
& +\left|T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)\right|\}. \qquad (6)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\left|\frac{T_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right| \\
\leq\ & \theta\left\{\left|\frac{T_{mn}(e_{00};x,y)-e_{00}(x,y)}{\sigma_1(x,y)}\right|\right. \\
&+\left|\frac{T_{mn}(e_{10};x,y)-e_{10}(x,y)}{\sigma_2(x,y)}\right|+\left|\frac{T_{mn}(e_{01};x,y)-e_{01}(x,y)}{\sigma_3(x,y)}\right| \\
&\left.+\left|\frac{T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)}{\sigma_4(x,y)}\right|\right\}
\end{aligned}$$

where $\sigma(x,y)=\max\{|\sigma_k(x,y)|\}$, $k=1,2,3,4.$

Since $T_{mn}(e_{00}) \overset{eq-st_2}{\to} e_{00}(I;\sigma_1)$, there is a double sequence of positive numbers $\left(\varepsilon'_{mn}\right)$ with $st_2-\lim\varepsilon'_{mn}=0$ and a function $\sigma_1(x,y)$, $|\sigma_1(x,y)|>0$, such that for any $(x,y)\in I$,

$$\delta_2\left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}(e_{00};x,y)-e_{00}(x,y)}{\sigma_1(x,y)}\right|\geq\varepsilon'_{mn}\right\}=0.$$

Since $T_{mn}(e_{10}) \overset{eq-st_2}{\to} e_{10}(I;\sigma_2)$, there is a double sequence of positive numbers $\left(\varepsilon''_{mn}\right)$ with $st_2-\lim\varepsilon''_{mn}=0$ and a function $\sigma_2(x,y)$, $|\sigma_2(x,y)|>0$, such that for any $(x,y)\in I$,

$$\delta_2\left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}(e_{10};x,y)-e_{10}(x,y)}{\sigma_2(x,y)}\right|\geq\varepsilon''_{mn}\right\}=0.$$

Since $T_{mn}(e_{01}) \overset{eq-st_2}{\to} e_{01}(I;\sigma_3)$, there is a double sequence of positive numbers $\left(\varepsilon'''_{mn}\right)$ with $st_2-\lim\varepsilon'''_{mn}=0$ and a function $\sigma_3(x,y)$, $|\sigma_3(x,y)|>0$, such that for any $(x,y)\in I$,

$$\delta_2\left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}(e_{01};x,y)-e_{01}(x,y)}{\sigma_3(x,y)}\right|\geq\varepsilon'''_{mn}\right\}=0.$$

Since $T_{mn}(e_{20}+e_{02}) \overset{eq-st_2}{\to} e_{20}+e_{02}\ (I;\sigma_4)$, there is a double sequence of positive numbers $\left(\varepsilon''''_{mn}\right)$ with $st_2-\lim\varepsilon''''_{mn}=0$ and a function $\sigma_4(x,y)$, $|\sigma_4(x,y)|>0$, such that for any $(x,y)\in I$,

$$\delta_2\left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)}{\sigma_4(x,y)}\right|\geq\varepsilon''''_{mn}\right\}=0.$$

For every $(x,y)\in I$, setting

$$\begin{aligned} U &= \left\{(m,n)\in\mathbb{N}^2 : \left|\frac{T_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\geq\varepsilon_{mn}\right\}, \\ U_0 &= \left\{(m,n)\in\mathbb{N}^2 : \left|\frac{T_{mn}(e_{00};x,y)-e_{00}(x,y)}{\sigma_1(x,y)}\right|\geq\varepsilon'_{mn}\right\}, \\ U_1 &= \left\{(m,n)\in\mathbb{N}^2 : \left|\frac{T_{mn}(e_{10};x,y)-e_{10}(x,y)}{\sigma_2(x,y)}\right|\geq\varepsilon''_{mn}\right\}, \\ U_2 &= \left\{(m,n)\in\mathbb{N}^2 : \left|\frac{T_{mn}(e_{01};x,y)-e_{01}(x,y)}{\sigma_3(x,y)}\right|\geq\varepsilon'''_{mn}\right\}, \\ U_3 &= \left\{(m,n)\in\mathbb{N}^2 : \left|\frac{T_{mn}(e_{20}+e_{02};x,y)-(e_{20}+e_{02})(x,y)}{\sigma_4(x,y)}\right|\geq\varepsilon''''_{mn}\right\}, \end{aligned}$$

where $\varepsilon_{mn}:=4\theta\varepsilon'''''_{mn}$, $\varepsilon'''''_{mn}:=\max\left\{\varepsilon'_{mn},\varepsilon''_{mn},\varepsilon'''_{mn},\varepsilon''''_{mn}\right\}$ and $\sigma(x,y)=\max\left\{|\sigma_k(x,y)|\right\}, k=1,2,3,4$. It is clear that $st_2-\lim\varepsilon_{mn}=0$. Now it is easy to see that

$$U\subseteq\bigcup_{i=0}^{3}U_i$$

which gives, using (6),

$$0\leq P-\lim_{m,n}\frac{|U|}{mn}\leq\sum_{i=0}^{3}P-\lim_{m,n}\frac{|U_i|}{mn}=0.$$

The proof is complete.

EXAMPLE 3. Let $I=[0,1]\times[0,1]$. Consider the double Bernstein type polynomials

$$B_{mn}(f;x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}f\left(\frac{j}{m},\frac{k}{n}\right)\binom{m}{j}\binom{n}{k}x^j(1-x)^{m-j}y^k(1-y)^{n-k} \qquad (7)$$

on $C(I)$. Using these polynomials, we introduce the following positive linear operators on $C(I)$:

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))B_{mn}(f;x,y),\ \ (x,y)\in I \text{ and } f\in C(I), \qquad (8)$$

where (g_{mn}) on I is given by

$$g_{mn}(x,y):=\begin{cases}\frac{xy}{2}, & m,n \text{ are squares,}\\ 0, & \text{otherwise.}\end{cases}$$

Take a double sequence (ε_{mn}) by defined

$$\varepsilon_{mn}:=\begin{cases}4\left(m^2+n^2\right), & m,n \text{ are squares,}\\ \frac{1}{m+n}, & \text{otherwise.}\end{cases}$$

Then we see that $st_2 - \lim \varepsilon_{mn} = 0$. Let $\sigma(x,y) = \frac{3}{x+y+1}$ for $(x,y) \in I$. Note that (g_{mn}) is statistically relatively equal convergent to $g = 0$. Also, observe that

$$\begin{aligned} P_{mn}(e_{00}; x,y) &= (1 + g_{mn}(x,y))e_{00}(x,y), \\ P_{mn}(e_{01}; x,y) &= (1 + g_{mn}(x,y))e_{01}(x,y), \\ P_{mn}(e_{10}; x,y) &= (1 + g_{mn}(x,y))e_{10}(x,y), \\ P_{mn}(e_{20}+e_{02}; x,y) &= (1 + g_{mn}(x,y)) \left[(e_{20}+e_{02})(x,y) + \frac{x(1-x)}{m} + \frac{y(1-y)}{n} \right]. \end{aligned}$$

Since $g_{mn} \overset{eq-st_2}{\to} g = 0\,(I;\sigma)$, we conclude that

$$\begin{aligned} &P_{mn}(e_{00}) - e_{00} \overset{eq-st_2}{\to} 0\,(I;\sigma)\,, \\ &P_{mn}(e_{10}) - e_{10} \overset{eq-st_2}{\to} 0\,(I;\sigma)\,, \\ &P_{mn}(e_{01}) - e_{01} \overset{eq-st_2}{\to} 0\,(I;\sigma)\,, \\ &P_{mn}(e_{20}+e_{02}) - (e_{20}+e_{02}) \overset{eq-st_2}{\to} 0\,(I;\sigma)\,. \end{aligned}$$

Notwithstanding, since (g_{mn}) is not uniform convergent, statistically uniform convergent and statistically relatively uniform convergent, we can say that the classical, statistical and statistical relative cases of the Korovkin results introduced respectively in [19], [12] , [11] do not work for our operators given by (8). Therefore, this example clearly shows that our Theorem 1 is a non-trivial generalization of the classical, statistical and statistical relative ones.

4 Rate of Statistical Relative Equal Convergence

In this section, we compute the corresponding rate of statistical relative equal convergence of positive linear operators on $C(I)$ via the modulus of continuity.

Now we recall the concept of modulus of continuity. For $f \in C(I)$, the modulus of continuity of f, denoted by $\omega_2(f;\delta)$, is defined by

$$\omega_2(f;\delta) := \sup \left\{ |f(u,t) - f(x,y)| : (u,t), (x,y) \in I, \sqrt{(u-x)^2 + (t-y)^2} \le \delta \right\}$$

where $f \in C(I)$ and $\delta > 0$. In order to obtain our result, we will make use of the following elementary inequality, for all $f \in C(I)$ and for λ, $\delta > 0$,

$$\omega_2(f; n\delta_1) \le (1 + [n])\,\omega_2(f;\delta) \tag{9}$$

where $[n]$ is defined to be the greatest integer less than or equal to n.

Then we have the following result.

THEOREM 2. Let (T_{mn}) be a double sequence of positive linear operators acting from $C(I)$ into $C(I)$. Assume that the following conditions hold:

$$T_{mn}(e_{00}) \overset{eq-st_2}{\to} e_{00}\,(I;\sigma_1)\ , \tag{10}$$

and

$$\omega_2\left(f;\delta_{mn}\right) \overset{eq-st_2}{\rightarrow} 0\left(I;\sigma_2\right) \ , \tag{11}$$

where $\delta_{mn} := \sqrt{T_{mn}(\nabla;x,y)}$ with $\nabla(u,t) = (u-x)^2 + (t-y)^2$.Then we have, for all $f \in C\left(I\right)$,

$$T_{mn}(f) \overset{eq-st_2}{\rightarrow} f\left(I;\sigma\right) \ ,$$

where $e_{ij}\left(x,y\right) = x^i y^j$, $i,j = 0,1,2$, $\sigma(x,y) = \max\left|\sigma_k(x,y)\right|; \left|\sigma_k(x,y)\right| > 0$, $k = 1,2$.

PROOF. Let $f \in C\left(I\right)$ and $(x,y) \in I$ be fixed. Using the properties of ω_2 and the positivity and monotonicity of T_{mn}, we get, for any $\delta > 0$ and $m,n \in \mathbb{N}$,

$$\begin{aligned}
& \left|T_{mn}\left(f;x,y\right) - f\left(x,y\right)\right| \\
\leq\ & T_{mn}\left(\left|f\left(u,t\right) - f\left(x,y\right)\right|;x\right) + \left|f\left(x,y\right)\right|\left|T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right| \\
\leq\ & T_{mn}\left(\left(1 + \frac{\nabla(u,t)}{\delta^2}\right)\omega_2\left(f;\delta\right);x,y\right) + \left|f\left(x,y\right)\right|\left|L_m\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right| \\
\leq\ & \omega_2\left(f;\delta\right)T_{mn}\left(e_{00};x,y\right) + \omega_2\left(f;\delta\right)T_{mn}\left(\frac{\nabla(u,t)}{\delta^2};x,y\right) \\
& + \left|f\left(x,y\right)\right|\left|L_m\left(e_{00};x,y\right) - e_{00}\left(x,y\right)\right|.
\end{aligned}$$

So, we get

$$\begin{aligned}
\left|\frac{T_{mn}\left(f;x,y\right) - f\left(x,y\right)}{\sigma(x,y)}\right| \leq\ & \frac{\omega_2\left(f;\delta\right)}{\left|\sigma_2(x,y)\right|}\left|\frac{T_{mn}\left(e_{00};x,y\right) - e_{00}(x,y)}{\sigma_1(x,y)}\right| \\
& + \left|f\left(x,y\right)\right|\left|\frac{T_{mn}\left(e_{00};x,y\right) - e_{00}\left(x,y\right)}{\sigma_1(x,y)}\right| \\
& + \frac{\omega_2\left(f;\delta\right)}{\left|\sigma_2(x,y)\right|} + \frac{\omega_2\left(f;\delta\right)}{\left|\sigma_2(x,y)\right|\delta^2}T_{mn}\left(\nabla(u,t);x,y\right) \\
\leq\ & \frac{\omega_2\left(f;\delta\right)}{\left|\sigma_2(x,y)\right|}\left|\frac{T_{mn}\left(e_{00};x,y\right) - e_{00}(x,y)}{\sigma_1(x,y)}\right| \\
& + M\left|\frac{T_{mn}\left(e_{00};x,y\right) - e_{00}(x,y)}{\sigma_1(x,y)}\right| + 2\frac{\omega_2\left(f;\delta\right)}{\left|\sigma_2(x,y)\right|} \quad (12)
\end{aligned}$$

where $\delta_{mn} := \delta := \sqrt{T_{mn}(\nabla;x,y)}$ and $M := \|f\|$.

Since $T_{mn}\left(e_{00}\right) \overset{eq-st_2}{\rightarrow} e_{00}\left(I;\sigma_1\right)$, there is a double sequence of positive numbers $\left(\varepsilon'_{mn}\right)$ with $st_2 - \lim \varepsilon'_{mn} = 0$ and there exist a function $\sigma_1(x,y), |\sigma_1(x,y) > 0|$ such that for any $(x,y) \in I$,

$$\delta_2\left\{(m,n) \in \mathbb{N}^2 : \left|\frac{T_{mn}(e_{00};x,y) - e_{00}\left(x,y\right)}{\sigma_1(x,y)}\right| \geq \varepsilon'_{mn}\right\} = 0.$$

Also, since $\omega_2\left(f;\delta_{mn}\right) \overset{eq-st_2}{\rightarrow} 0\left(I;\sigma_2\right)$, then there is a double sequence of positive numbers $\left(\varepsilon''_{mn}\right)$ with $st_2 - \lim \varepsilon''_{mn} = 0$ and there exist a function $\sigma_2(x,y), |\sigma_2(x,y) > 0|$

such that for any $(x,y)\in I$,

$$\delta_2\left\{(m,n)\in\mathbb{N}^2:\left|\frac{\omega_2\left(f;\delta_{mn}\right)}{\sigma_2(x,y)}\right|\geq\varepsilon_{mn}^{''}\right\}=0.$$

For every $(x,y)\in I$, setting

$$\begin{aligned} U &:= \left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}\left(f;x,y\right)-f\left(x,y\right)}{\sigma(x,y)}\right|\geq\varepsilon_{mn}^{'''}\right\}, \\ U_1 &:= \left\{(m,n)\in\mathbb{N}^2:\left|\frac{T_{mn}(e_{00};x,y)-e_{00}\left(x,y\right)}{\sigma_1(x,y)}\right|\geq\varepsilon_{mn}^{\prime}\right\}, \\ U_2 &:= \left\{(m,n)\in\mathbb{N}^2:\left|\frac{\omega_2\left(f;\delta_{mn}\right)}{\sigma_2(x,y)}\right|\geq\varepsilon_{mn}^{''}\right\}, \end{aligned}$$

where $\varepsilon_{mn}=\max\left\{\varepsilon_{mn}^{'},\varepsilon_{mn}^{''}\right\}$, $\varepsilon_{mn}^{'''}:=\varepsilon_{mn}^2+(M+2)\varepsilon_{mn}$, $\sigma(x,y)=\max\left\{|\sigma_k(x,y)|\right\}$ and $|\sigma_k(x,y)|>0$, $k=1,2$. It is clear that $st_2-\lim\varepsilon_{mn}^{'''}=0$. Now, it is clear to write that

$$U\subseteq U_1\cup U_2$$

which gives, using (12),

$$0\leq P-\lim_{m,n}\frac{|U|}{mn}\leq\left\{P-\lim_{m,n}\frac{|U_1|}{mn}\right\}+\left\{P-\lim_{m,n}\frac{|U_2|}{mn}\right\}=0.$$

for every $(x,y)\in I$. The proof is completed.

5 A Voronovskaya-Type Theorem

In this section, we give a Voronovskaya-type theorem in statistical relative equal case for the positive linear operators $\{P_{nn}\}$ given by (8) for $n=m$.

LEMMA 2. For the positive linear operators $\{B_{nn}\}$ given by (7), we obtain, for all $(x,y)\in I=[0,1]\times[0,1]$,

$$\begin{aligned} B_{nn}\left(e_{03};x,y\right) &= \frac{(n-1)(n-2)}{n^2}y^3+\frac{3(n-1)}{n^2}y^2+\frac{1}{n^2}y, \\ B_{nn}\left(e_{30};x,y\right) &= \frac{(n-1)(n-2)}{n^2}x^3+\frac{3(n-1)}{n^2}x^2+\frac{1}{n^2}x, \\ B_{nn}\left(e_{04};x,y\right) &= \frac{(n-1)(n-2)(n-3)}{n^3}y^4+\frac{6(n-1)(n-2)}{n^3}y^3+\frac{7(n-1)}{n^3}y^2+\frac{1}{n^3}y, \\ B_{nn}\left(e_{40};x,y\right) &= \frac{(n-1)(n-2)(n-3)}{n^3}x^4+\frac{6(n-1)(n-2)}{n^3}x^3+\frac{7(n-1)}{n^3}x^2+\frac{1}{n^3}x, \end{aligned}$$

where $e_{ij}\left(x,y\right)=x^iy^j$, $i,j=0,1,2,3,4.$

LEMMA 3. Let $(x,y)\in I=[0,1]\times[0,1]$. Then, we get

$$n^2P_{nn}\left((u-x)^4;x,y\right)\overset{eq-st_2}{\rightarrow}3x^2(x-1)^2\ \ (I;\sigma), \tag{13}$$

$$n^2 P_{nn}\left((t-y)^4 ; x,y\right) \overset{eq-st_2}{\rightarrow} 3y^2(y-1)^2 \ \ (I;\sigma), \tag{14}$$

where $e_{ij}(x,y) = x^i y^j$, $for\ i,j = 0,1,2.$

PROOF. We shall prove (13). It is similar to the proof of (14), we omit it. For Lemma 2, we can write

$$n^2 P_{nn}\left((u-x)^4 ; x,y\right) = (1+g_{nn}(x,y))\left[3x^4 - 6x^3 + 3x^2 + \frac{-6x^4+12x^3-7x^2+x}{n}\right].$$

Herefrom, we get

$$\begin{aligned}
&\left|n^2 P_{nn}\left((u-x)^4 ; x,y\right) - 3e_{20}(x,y)\left[e_{20}(x,y) - 2e_{10}(x,y) + e_{00}(x,y)\right]\right| \\
\leq\ & 12 g_{nn}(x,y) + (1+g_{nn}(x,y))\frac{26}{n}
\end{aligned} \tag{15}$$

for every $x \in [0,1]$. Since $g_{mn} \overset{eq-st_2}{\rightarrow} g = 0(I;\sigma)$, it is easy to observe that

$$12 g_{nn}(x,y) + (1+g_{nn}(x,y))\frac{26}{n} \overset{eq-st_2}{\rightarrow} 0\,(I;\sigma). \tag{16}$$

Combining (15) and (16), the proof is complete.

THEOREM 3. Let $(x,y) \in I = [0,1]\times[0,1]$ and $f,\ f_x,\ f_y,\ f_{xx},\ f_{xy},\ f_{yy} \in C(I)$. Then,

$$n\left\{P_{nn}(f;x,y) - f(x,y)\right\} \overset{eq-st_2}{\rightarrow} \frac{1}{2}\left\{(x-x^2)f_{xx} + (y-y^2)f_{yy}\right\}(I;\sigma).$$

PROOF. Let $(x,y) \in I$ and $f_x,\ f_y,\ f_{xx},\ f_{xy},\ f_{yy} \in C(I)$. Define the function Ψ by

$$\begin{aligned}
&\Psi(u,t) \\
=\ & \begin{cases} \frac{1}{\sqrt{(u-x)^4+(t-y)^4}}\Big\{f(u,t) - f(x,y) - f_x(u-x) - f_y(t-y) & \\ -\frac{1}{2}\left[f_{xx}(u-x)^2 + 2f_{xy}(u-x)(t-y) + f_{yy}(t-y)^2\right]\Big\}, & (u,t) \neq (x,y), \\ 0, & (u,t) = (x,y). \end{cases}
\end{aligned}$$

Hence, by assumption we obtain $\Psi(u,t) = 0$ and Ψ belongs to $C(I)$. By the Taylor formula for $f \in C(I)$, we have

$$\begin{aligned}
f(u,t) &= f(x,y) + (u-x)f_x + (t-y)f_y + \frac{1}{2}\Big\{(u-x)^2 f_{xx} \\
&\quad + 2(u-x)(t-y)f_{xy} + (t-y)^2 f_{yy}\Big\} \\
&\quad + \Psi(u,t)\sqrt{(u-x)^4 + (t-y)^4}.
\end{aligned}$$

Because of linearity of the operator P_{nn}, we get

$$\begin{aligned} P_{nn}(f;x,y) &= f(x,y)(1+g_{nn}(x,y)) + f_x P_{nn}(u-x;x,y) \\ &+ f_y P_{nn}(t-y;x,y) + \frac{1}{2}\{f_{xx}P_{nn}\left((u-x)^2;x,y\right) \\ &+ 2f_{xy}P_{nn}((u-x)(t-y);x,y) + f_{yy}P_{nn}\left((t-y)^2;x,y\right)\} \\ &+ P_{nn}\left(\Psi(u,t)\sqrt{(u-x)^4+(t-y)^4};x,y\right). \end{aligned}$$

Now if we use the properties of this operator, we obtain

$$\begin{aligned} P_{nn}(f;x,y) &= f(x,y)(1+g_{nn}(x,y)) + \frac{1+g_{nn}(x,y)}{2n}\left\{(x-x^2)f_{xx}+(y-y^2)f_{yy}\right\} \\ &+ P_{nn}\left(\Psi(u,t)\sqrt{(u-x)^4+(t-y)^4};x,y\right) \end{aligned}$$

which yields

$$\begin{aligned} n\{P_{nn}(f;x,y)-f(x,y)\} &= n g_{nn}(x,y) f(x,y) \\ &+ \frac{1+g_{nn}(x,y)}{2}\left\{(x-x^2)f_{xx}+(y-y^2)f_{yy}\right\} \\ &+ nP_{nn}\left(\Psi(u,t)\sqrt{(u-x)^4+(t-y)^4};x,y\right). \end{aligned} \tag{17}$$

Applying the Cauchy-Schwarz inequality for the third term in (17), we get

$$\begin{aligned} & n\left|P_{nn}\left(\Psi(u,t)\sqrt{(u-x)^4+(t-y)^4};x,y\right)\right| \\ \leq\; & \left(P_{nn}\left(\Psi^2(u,t);x,y\right)\right)^{1/2}\cdot\left(n^2P_{nn}\left((u-x)^4+(t-y)^4;x,y\right)\right)^{1/2} \\ =\; & \left(P_{nn}\left(\Psi^2(u,t);x,y\right)\right)^{1/2}\left\{n^2P_{nn}\left((u-x)^4;x,y\right)+n^2P_{nn}\left((t-y)^4;x,y\right)\right\}^{1/2}. \end{aligned}$$

Let $\beta(u,t)=\Psi^2(u,t)$. In this case, we show that $\beta(x,y)=0$ and β belongs to $C(I)$. From Theorem 1,

$$P_{nn}\left(\Psi^2(u,t);x,y\right) = P_{nn}\left(\beta_{(x,y)}(u,t);x,y\right) \overset{eq-st_2}{\to} \beta_{(x,y)}(x,y) = 0\,(I;\sigma). \tag{18}$$

Using (18) and Lemma 3, we get from (17)

$$nP_{nn}\left(\Psi_{(x,y)}(u,t)\sqrt{(u-x)^4+(t-y)^4};x,y\right) \overset{eq-st_2}{\to} 0\,(I;\sigma). \tag{19}$$

Considering (19) and (17) and also $g_{nn}(x,y) \overset{eq-st_2}{\to} g=0\,(I;\sigma)$, we have

$$n\{P_{nn}(f;x,y)-f(x,y)\} \overset{eq-st_2}{\to} \frac{1}{2}\left\{(x-x^2)f_{xx}+(y-y^2)f_{yy}\right\}(I;\sigma).$$

So the proof is completed.

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