Applied Mathematics E-Notes, 19(2019), 55-64 © ISSN 1607-2510
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On Three-Dimensional Quasi-Sasakian Manifolds And Magnetic Curves*

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Received 21 February 2018

Abstract

The object of the present paper is to study normal magnetic flows in three-dimensional quasi-Sasakian manifolds. It is shown that the structure function of a three-dimensional quasi-Sasakian manifold is necessarily constant if the manifold admits magnetic curve. In addition if the magnetic curve is biharmonic the manifold is locally ϕ-symmetric. An example of magnetic curve on three-dimensional quasi-Sasakian manifold is given.

1 Introduction

Geodesics on a curve can be defined as extremal curves for the following action energy functional

$$E(\gamma) = \frac{1}{2}\int g(\dot\gamma(t), \dot\gamma(t))dt,$$

where g is a Riemannian or pseudo-Riemannian metric. The geodesic equations can be obtained from Euler-Lagrange equations of motion for this action. Geodesics describe the motions of particles that are not experiencing any forces. On the other hand, the magnetic curves are trajectories of charged particles in presence of a time independent magnetic field ([13]). Biharmonic curves $\gamma : I \subset R \to (M, g)$ on a Riemannian manifold are the solutions of the differential equation

$$\nabla^3_{\dot\gamma}\dot\gamma + R(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma = 0.$$

For more details about biharmonic maps and curves we refer [7, 8, 11] and references therein.

A magnetic field on a manifold is expressed by a closed 2-form. Since the fundamental 2-form of a three-dimensional quasi-Sasakian manifold is closed, one can talk about magnetic curves on three-dimensional quasi-Sasakian manifolds.

The notion of quasi-Sasakian manifold was introduced by D. E. Blair [1] to unify Sasakian and cosymplectic manifolds. Three dimensional quasi-Sasakian manifolds was studied by Z. Olszak ([9, 10]). The first author of the present paper has also studied

*Mathematics Subject Classifications: 53 C 15, 53 D 25.

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three-dimensional quasi-Sasakian manifolds [5, 12]. Recently quasi-Sasakian structures have become a topic of growing interest due to its significant applications in string theory [6]. On three-dimensional quasi-Sasakian manifold the structure function was defined by Olszak. With the help of this structure function he obtained necessary and sufficient condition for such manifolds to be conformally flat [10]. In [5], U. C. De and the first author have deduced necessary and sufficient condition for three-dimensional quasi-Sasakian manifolds to be locally ϕ-symmetric. In [6] the first author has studied some further properties of three-dimensional quasi-Sasakian manifolds.

In this paper we would like to study normal flow lines of the magnetic curves in three-dimensional quasi-Sasakian manifolds. We also study biharmonic magnetic curves in three-dimensional quasi-Sasakian manifolds. The present paper is organized as follows: In Section 2, we give the required preliminaries. In Section 3, we study normal flow lines of magnetic curves. In Section 4, we analyze the nature of structure functions of the quasi-Sasakian manifolds admitting magnetic curves. Section 5 is devoted to study biharmonic magnetic curves on a three-dimensional quasi-Sasakian manifolds and it is proved that if a three-dimensional quasi-Sasakian manifold admits a biharmonic magnetic curve, the manifold becomes locally ϕ-symmetric. The last section contains an example of magnetic curve on a three-dimensional quasi-Sasakian manifold.

2 Preliminaries

Let M be a $(2n+1)$ dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is the Riemannian metric on M such that [2]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then also

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad g(X, \xi) = \eta(X),$$

for all $X, Y \in \chi(M)$.

Let Φ be the fundamental 2-form of M defined by $\Phi(X,Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$. Then $\Phi(X, \xi) = 0$, $X \in \chi(M)$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [1]. The normality condition gives that the induced almost contact structure of $M \times R$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of the quasi-Sasakian structure is always odd [1], it is equal to 1 if the structure is cosymplectic and it is equal to $2n + 1$ if the structure is Sasakian.

For a three-dimensional quasi-Sasakian manifold, we have [10]

$$(\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in \chi(M). \tag{1}$$

$$\nabla_X \xi = -\beta\phi X, \quad X \in \chi(M), \tag{2}$$

for a function β defined on the manifold, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of the manifold.

The Ricci tensor of three-dimensional quasi-Sasakian manifolds was given in the paper [10]. The Riemann curvature of three-dimensional quasi-Sasakian manifold has been given in the paper [5] as follows:

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)[(\frac{r}{2}-\beta^2)X+(3\beta^2-\frac{r}{2})\eta(X)\xi \\ & +\eta(X)(\phi \mathrm{grad}\beta)-d\beta(\phi X)\xi] \\ & -g(X,Z)[(\frac{r}{2}-\beta^2)Y+(3\beta^2-\frac{r}{2})\eta(Y)\xi \\ & +\eta(Y)(\phi \mathrm{grad}\beta)-d\beta(\phi Y)\xi] \\ & +[(\frac{r}{2}-\beta^2)g(Y,Z)+(3\beta^2-\frac{r}{2})\eta(Y)\eta(Z) \\ & -\eta(Y)d\beta(\phi Z)-\eta(Z)d\beta(\phi Y)]X \\ & -[(\frac{r}{2}-\beta^2)g(X,Z)+(3\beta^2-\frac{r}{2})\eta(X)\eta(Z) \\ & -\eta(X)d\beta(\phi Z)-\eta(Z)d\beta(\phi X)]Y \\ & -\frac{r}{2}[g(Y,Z)X-g(X,Z)Y]. \end{aligned} \tag{3}$$

The notion of contact magnetic fields and normal flow lines were introduced by Cabrerizo and collaborators [3]. In a contact manifold the fundamental 2-form is closed. A magnetic field on a manifold M is a closed 2-form. So we can take the fundamental 2-form of a contact manifold as a magnetic field. The fundamental 2-form of a three-dimensional quasi-Sasakian manifold is also closed. So, we can consider the fundamental 2-form of the three-dimensional quasi-Sasakian manifold as a magnetic field which satisfies

$$\Phi(X,Y)=g(X,\phi Y).$$

Here we take the Lorentz equation as

$$\nabla_{\dot\gamma}\dot\gamma=\phi\dot\gamma, \tag{4}$$

where γ is a curve from $I \subseteq R$ to M, known as magnetic curve. A magnetic curve is called normal if $g(\dot\gamma,\dot\gamma)=1$.

A curve γ on a three-dimensional manifold is called a Frenet curve if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma=\kappa N,$$
$$\nabla_{\dot\gamma}N=-\kappa\dot\gamma+\tau B,$$
$$\nabla_{\dot\gamma}B=-\tau N,$$

where κ and τ are the curvature and the torsion of $\gamma(t)$, respectively and $\{\dot\gamma,N,B\}$ is Frenet frame. From Proposition 5.1 and Corollary 5.2 of [3], we have the following for a magnetic curve γ:

$$\nabla_{\dot\gamma}\dot\gamma=\xi\wedge\dot\gamma.$$

From (4) and the first Frenet formula, we have

$$\kappa N=\phi\dot\gamma. \tag{5}$$

3 Normal Flow Lines of Magnetic Field on Three-Dimensional Quasi-Sasakian Manifolds

The normal flow lines of magnetic fields in Sasakian manifolds were studied by J. L. Cabrerizo et al. [3]. In this section we study normal flow lines of magnetic fields corresponding to three-dimensional quasi-Sasakian manifolds. We prove the following:

THEOREM 3.1. Let (M, ϕ, ξ, g) be a three-dimensional quasi-Sasakian manifold. Then the curve $\gamma(t)$ is the normal flow line of the contact magnetic field if and only if the curve has constant curvature $\kappa = \sin\theta > 0$ and torsion $\tau = \beta + \cos\theta$, where θ is the constant angle between $\dot{\gamma}(t)$ and ξ along $\gamma(t)$.

PROOF. Suppose $\gamma(t)$ is a normal magnetic curve. Then

$$g(\nabla_{\dot{\gamma}}\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}) = g(\phi\dot{\gamma}, \phi\dot{\gamma}).$$

The above equation gives

$$\kappa^2 = \sin^2\theta,$$

where $\eta(\dot{\gamma}) = \cos\theta, 0 < \theta < \pi$, along $\gamma(t)$. Thus $\kappa(t) = \sin\theta$ is a constant. Now,

$$B = \dot{\gamma} \wedge N = \frac{1}{\kappa}(\xi - \cos\theta\dot{\gamma}).$$

Now from third Frenet formula $\nabla_{\dot{\gamma}}B = -\tau N$ and using (5) we have

$$\nabla_{\dot{\gamma}}\xi - \cos\theta\nabla_{\dot{\gamma}}\dot{\gamma} = -\tau\phi\dot{\gamma}.$$

Using (2) in the above equation, we get

$$\tau = \beta + \cos\theta. \tag{6}$$

Conversely, let the constant curvature of the magnetic curve is given by $\kappa = \sin\theta > 0$ and torsion $\tau = \beta + \cos\theta$, $0 < \theta < \pi$, where θ is the constant angle between $\dot{\gamma}(t)$ and ξ. Now taking the covariant derivative of $\cos\theta = g(\dot{\gamma}, \xi)$ along $\dot{\gamma}(t)$, we get

$$N = \mu\xi \wedge \dot{\gamma},$$

where $\mu(t)$ is a non vanishing function. Hence, we have $g(N, N) = g(\mu\xi \wedge \dot{\gamma}, \mu\xi \wedge \dot{\gamma})$. Thus we get $1 = |\mu(t)|\sin\theta$. So, we see that $\mu(t)$ is a non zero constant, say, μ_0. Then we have

$$B = \dot{\gamma} \wedge N = \mu_0(\xi - \cos\theta\dot{\gamma}). \tag{7}$$

Using third Frenet formula in (7), we have

$$\mu_0(\nabla_{\dot{\gamma}}\xi - \cos\theta\nabla_{\dot{\gamma}}\dot{\gamma}) = -\tau\mu_0\phi(\dot{\gamma}).$$

Now by (2) and (7), we get

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{(\tau - \beta)\phi(\dot{\gamma})}{\cos\theta}.$$

Now by (6), we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}.$$

This shows that $\gamma(t)$ is a normal flow line of the contact magnetic field.

4 Nature of the Structure Functions of Three-Dimensional Quasi-Sasakian Manifolds Admitting Magnetic Curves

In this section we study three-dimensional quasi-Sasakian manifolds admitting magnetic curves and prove the following:

THEOREM 4.1. The structure function of a three-dimensional quasi-Sasakian manifold admitting magnetic curve is constant.

PROOF. Since g is parallel, it follows that for a magnetic curve $\eta(\dot{\gamma})$ is constant. By (5) it follows that

$$\kappa^2 = 1 - \eta(\dot{\gamma})^2. \tag{8}$$

Again from (5)

$$N = \frac{1}{\kappa}\phi\dot{\gamma}.$$

Now taking covariant derivative of the above equation with respect to $\dot{\gamma}$, we get

$$\nabla_{\dot{\gamma}} N = -\frac{1}{\kappa^2}\kappa'\phi\dot{\gamma} + \frac{1}{\kappa}\nabla_{\dot{\gamma}}(\phi\dot{\gamma}). \tag{9}$$

We know $(\nabla_{\dot{\gamma}}\phi)\dot{\gamma} = \nabla_{\dot{\gamma}}(\phi\dot{\gamma}) - \phi(\nabla_{\dot{\gamma}}\dot{\gamma})$. Using (1) in the above equation, we get

$$\nabla_{\dot{\gamma}}(\phi\dot{\gamma}) = ((\beta - \eta(\dot{\gamma}))\xi + (1 - \beta\eta(\dot{\gamma}))\dot{\gamma}). \tag{10}$$

Putting (10) in (9) and using second Frenet formula, we get

$$-\kappa\dot{\gamma} + \tau B = -\frac{1}{\kappa^2}\kappa'\phi\dot{\gamma} + \frac{1}{\kappa}\big((\beta - \eta(\dot{\gamma}))\xi + (1 - \beta\eta(\dot{\gamma}))\dot{\gamma}\big). \tag{11}$$

Taking inner product in both sides of (11) with respect to ξ we obtain

$$-\kappa\eta(\dot{\gamma}) + \tau\eta(B) = \frac{\beta}{\kappa}(1 - \eta(\dot{\gamma})^2).$$

Using (8) in the above equation we get

$$-\kappa\eta(\dot{\gamma}) + \tau\eta(B) = \beta\kappa.$$

The above equation is true for arbitrary choice of the moving frame $\{\dot{\gamma}, N, B\}$. Let us consider $\dot{\gamma}$, N and ξ are coplaner at a point and angle between $\dot{\gamma}$ and ξ is θ and that between $\dot{\gamma}$ and N is $\frac{\pi}{2}$. Let us take B perpendicular to the plane determined by $\dot{\gamma}$, N and ξ. Then B is perpendicular to ξ. So, we have $\beta = -\eta(\dot{\gamma})$. But $\eta(\dot{\gamma})$ is constant. So β is also constant.

5 Biharmonic Magnetic Curve on Three-Dimensional Quasi-Sasakian Manifolds

In this section we study biharmonic magnetic curves on three-dimensional quasi-Sasakian manifolds. Let us recall the following:

DEFINITION 5.1. A magnetic curve $\gamma(t)$ on a three-dimensional quasi-Sasakian manifold is called biharmonic if it satisfies [4]

$$\nabla^3_{\dot\gamma}\dot\gamma + R(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma = 0.$$

Here we shall prove the following:

THEOREM 5.1. The scalar curvature of a three-dimensional quasi-Sasakian manifold admitting biharmonic magnetic curve is constant.

PROOF. By (1), (2) and (4), we have

$$\begin{aligned}\nabla^3_{\dot\gamma}\dot\gamma &= d\beta(\dot\gamma)g(\dot\gamma,\dot\gamma)\xi - \beta^2\phi\dot\gamma \\ &\quad +(\nabla_{\dot\gamma}\eta(\dot\gamma))(\xi - \beta\dot\gamma) \\ &\quad +\eta(\dot\gamma)(-\beta\phi\dot\gamma - d\beta(\dot\gamma)\dot\gamma - \beta\phi\dot\gamma) - \phi\dot\gamma. \end{aligned} \tag{12}$$

Putting $X = \phi X$ in (3) and then setting $X = Y = Z = \dot\gamma$, we get

$$\begin{aligned}R(\phi\dot\gamma,\dot\gamma)\dot\gamma &= g(\dot\gamma,\dot\gamma)[(\frac{r}{2} - \beta^2)\phi\dot\gamma - d\beta(\phi^2\dot\gamma)] \\ &\quad +[(\frac{r}{2} - \beta^2)g(\dot\gamma,\dot\gamma) + (3\beta^2 - \frac{r}{2})\eta(\dot\gamma)\eta(\dot\gamma) - 2\eta(\dot\gamma)d\beta(\phi\dot\gamma)]\phi\dot\gamma \\ &\quad +\eta(\dot\gamma)d\beta(\phi^2\dot\gamma) - \frac{r}{2}g(\dot\gamma,\dot\gamma)\phi\dot\gamma. \end{aligned} \tag{13}$$

Taking β as constant, from (12) and (13) we get

$$\nabla^3_{\dot\gamma}\dot\gamma = -\beta^2\phi\dot\gamma + (\nabla_{\dot\gamma}\eta(\dot\gamma))(\xi - \beta\dot\gamma) - 2\beta\eta(\dot\gamma)\phi\dot\gamma - \phi\dot\gamma, \tag{14}$$

$$R(\phi\dot\gamma,\dot\gamma)\dot\gamma = 2(\frac{r}{2} - \beta^2)\phi\dot\gamma + (3\beta^2 - \frac{r}{2})(\eta(\dot\gamma))^2\phi\dot\gamma - \frac{r}{2}\phi\dot\gamma. \tag{15}$$

Now from (14) and (15) we get

$$\nabla^3_{\dot\gamma}\dot\gamma + R(\phi\dot\gamma,\dot\gamma)\dot\gamma = \left\{2r - 3\beta^2 - 2\beta\eta(\dot\gamma) - 1 + (3\beta^2 - \frac{r}{2})(\eta(\dot\gamma))^2 - \frac{r}{2}\right\}\phi\dot\gamma. \tag{16}$$

In Section 4, we have seen $\beta = \eta(\dot\gamma)$ is conatant. Hence

$$\nabla^3_{\dot\gamma}\dot\gamma + R(\phi\dot\gamma,\dot\gamma)\dot\gamma = \left\{2r - 5\beta^2 - 1 + (3\beta^2 - \frac{r}{2})\beta^2 - \frac{r}{2}\right\}\phi\dot\gamma.$$

If the curve is biharmonic we have

$$2r - 5\beta^2 - 1 + (3\beta^2 - \frac{r}{2})\beta^2 - \frac{r}{2} = 0.$$

The above equation gives

$$r = \frac{2(1+5\beta^2-3\beta^4)}{3-\beta^2}.$$

Since β is constant, r is constant.

From [5], we know that a three-dimensional quasi-Sasakian manifold with constant structure function is locally ϕ-symmetric if and only if the scalar curvature is constant. Now, obviously we can state the following:

THEOREM 5.2. A three-dimensional quasi-Sasakian manifold admitting biharmonic magnetic curve is locally ϕ-symmetric.

6 Example of a Quasi-Saskian Manifold and Magnetic Curve

In this section we shall give an example of a three-dimensional quasi-Sasakian manifold [5] and we shall also give an example of a magnetic curve. Let us consider the three-dimensional manifold $M = \{(x,y,z) \in R^3, (x,y,z) \neq (0,0,0)\}$, where (x,y,z) are the standard co-ordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad , e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \qquad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $M(\phi,\xi,\eta,g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of the manifold. Then we have

$$[e_1,e_2] = e_3, \qquad [e_1,e_3] = -e_2, \qquad [e_2,e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) \;\; = \;\; & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) \\ & -g([Y,Z],X) + g([Z,X],Y), \end{aligned}$$

which is known as Koszul's formula. Taking $e_3 = \xi$ and using the above formula for Riemannian metric g, it can be easily calculated that

$$\nabla_{e_1} e_3 = -\tfrac{1}{2} e_2, \qquad \nabla_{e_1} e_2 = \tfrac{1}{2} e_3, \qquad \nabla_{e_1} e_1 = 0,$$

$$\nabla_{e_2} e_3 = \tfrac{1}{2} e_1, \qquad \nabla_{e_2} e_2 = 0, \qquad \nabla_{e_2} e_1 = -\tfrac{1}{2} e_3,$$

$$\nabla_{e_3} e_3 = 0, \qquad \nabla_{e_3} e_2 = \tfrac{1}{2} e_1, \qquad \nabla_{e_3} e_1 = -\tfrac{1}{2} e_2.$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula $\nabla_X \xi = -\beta\phi(X)$. Hence $M(\phi, \xi, \eta, g)$ is a three-dimensional quasi-Sasakian manifold with the structure function with $\beta = -\frac{1}{2}$.

Using the above relations we obtain the components of the curvature tensor as follows.

$$R(e_1, e_2)e_3 = 0, \qquad R(e_2, e_3)e_3 = -\tfrac{1}{4} e_2, \qquad R(e_1, e_3)e_3 = -\tfrac{1}{4} e_1,$$

$$R(e_1, e_2)e_2 = \tfrac{3}{4} e_1, \qquad R(e_3, e_2)e_2 = -\tfrac{1}{4} e_3, \qquad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = \tfrac{3}{4} e_2, \qquad R(e_2, e_3)e_1 = 0, \qquad R(e_3, e_1)e_1 = \tfrac{3}{4} e_3.$$

Now we see that

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = \frac{1}{2},$$

$$S(e_2, e_2) = g(R(e_2, e_1)e_1, e_2) + g(R(e_2, e_3)e_3, e_2) = \frac{1}{2},$$

$$S(e_3, e_3) = g(R(e_3, e_1)e_1, e_3) + g(R(e_3, e_2)e_2, e_3) = \frac{1}{2}$$

and

$$S(e_i, e_j) = 0, (i \neq j).$$

Therefore the scalar curvature $r = \frac{3}{2}$. We also see that

$$(\nabla_{e_1} R)(e_1, e_2)e_1 = (\nabla_{e_2} R)(e_1, e_2)e_2 = \frac{1}{2} e_3, \;\; (\nabla_{e_2} R)(e_1, e_2)e_1 = (\nabla_{e_1} R)(e_1, e_2)e_2 = 0.$$

Hence, M is locally ϕ-symmetric. Let us choose a curve $\gamma : I \to M$ by

$$\gamma(s) = (1, -s, s).$$

We see that

$$\dot{\gamma}(s) = (0, -1, 1) = -e_2 + e_3.$$

Now

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{(-e_2 + e_3)}(-e_2 + e_3) = -e_1.$$

Again

$$\phi\dot{\gamma} = \phi(-e_2 + e_3) = -\phi e_2 = -e_1.$$

So

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}.$$

Hence, the curve γ is a magnetic curve. It is also biharmonic.

From the above example we obtain the following:

• The structure function of the manifold is $\beta = -\frac{1}{2}$. Hence it is a constant. So it verifies Theorem 4.1.

• The scalar curvature of the manifold is $r = \frac{3}{2}$, a constant, which shows validity of Theorem 5.1.

• The manifold is locally ϕ-symmetric that validates Theorem 5.2.

Acknowledgment. The authors are thankful to the referee for his valuable suggestions for the improvement of the paper.

References

[1] D. E. Blair, The theory of quasi-Sasakian structure, J. Differential Geom., 1(1967), 331–345.

[2] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress Math., Vol. 203. Birkhauser, Boston-Basel-Berlin, 2002.

[3] J. L. Cabrerizo, M. Fernandez, and J. S. Gomez, The contact magnetic flow in 3D Sasakian manifolds, J. Phys. A: Math, Theor. 42(2009) 195201 (10pp).

[4] R. Caddeo, S. Montaldo and P. Piu, Biharmonic curves on a surface, Rend. Mat. Appl. 21 (2001), 143–157.

[5] U. C. De and A. Sarkar, On three-dimensional quasi-Sasakian manifolds, SUT Journal of Mathematics. 45(2009), 59–71.

[6] T. Friedrichand and S. Ivanov, Parallel spinors and connections with skew symmetric torsion in string theory, Asian J. Math., 6(2002), 303–336.

[7] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, Chinese Ann. Math. Ser. A, 7(1986), 389–402.

[8] S. Montaldo and C. Oniciuc, A short survey on biharmonic maps between Riemannian manifolds, Revista Un. Mat. Argentina, 47(2006), 1–22.

[9] Z. Olszak, Normal almost contact metric manifolds of dimension three, Ann. Polon. Math., 47(1986), 41–50.

[10] Z. Olszak, On three-dimensional conformally flat quasi-Sasakian manifolds, Period. Math. Hungar., 33(1996), 105–113.

[11] A. Sarkar and A. Mondal, Certain curves on some classes of of three-dimensional almost contact metric manifolds, Revista Un. Mat. Argentina, 58(2017), 107–125.

[12] A. Sarkar, A. Sil and D. Biswas, A study on three-dimensional quasi-Sasakian amnifolds, Indian J. Math., 59(2017), 209–225.

[13] B. Thide, Electromagnetic Field Theory, Upsilon Books, Communa AB, Uppasala, Sweden, 2004.