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On Generalized Absolute Cesàro Summability Of Factored Infinite Series*

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Abstract

In this paper, we generalize a known result dealing with the absolute Cesàro summability factors of infinite series. Some new and known results are also obtained.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n), that is (see [5])

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1,\quad \text{and}\quad A_{-n}^{\alpha+\beta}=0\quad \text{for}\quad n>0.$$

A series $\sum a_n$ is said to be summable $\mid C,\alpha,\beta,\sigma;\delta\mid_k$, $k\geq 1$, $\delta\geq 0$, $\alpha+\beta>-1$, and $\sigma\in\mathbb{R}$, if (see [2])

$$\sum_{n=1}^{\infty}n^{\sigma(\delta k+k-1)}\frac{\left|t_n^{\alpha,\beta}\right|^k}{n^k}<\infty.$$

If we take $\sigma=1$, then $\mid C,\alpha,\beta,\sigma;\delta\mid_k$ summability reduces to $\mid C,\alpha,\beta;\delta\mid_k$ summability (see [3]). If we set $\sigma=1$ and $\delta=0$, then we obtain the $\mid C,\alpha,\beta\mid_k$ summability (see [6]). Also, if we take $\beta=0$, then we have $\mid C,\alpha,\sigma;\delta\mid_k$ summability (see [10]). Furthermore, if we take $\sigma=1$, $\beta=0$, and $\delta=0$, then we get $\mid C,\alpha\mid_k$ summability (see [7]). Finally, if we set $\sigma=1$ and $\beta=0$, then we get $\mid C,\alpha;\delta\mid_k$ (see [8]). For any sequence (λ_n) we write that $\Delta^2\lambda_n=\Delta\lambda_n-\Delta\lambda_{n+1}$ and $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$. Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by (see [1])

$$\theta_n^{\alpha,\beta}=\begin{cases}\left|t_n^{\alpha,\beta}\right|, & \text{for } \alpha=1,\beta>-1,\\ \max_{1\leq v\leq n}\left|t_v^{\alpha,\beta}\right|, & \text{for } 0<\alpha<1,\beta>-1.\end{cases} \tag{2}$$

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2 Known Result

The following theorem is known dealing with the generalized absolute Cesàro summability factors of infinite series.

THEOREM 1 ([4]). Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (2). If (λ_n) is a non-negative and non-increasing sequence such that the series $\sum \frac{\lambda_n}{n}$ is convergent,

$$n\Delta\lambda_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3}$$

$$\sum_{n=1}^{\infty}(n+1)\Delta^2\lambda_n \tag{4}$$

is convergent and the condition

$$\sum_{n=1}^{m}(n^{\delta}\theta_n^{\alpha,\beta})^k = O(m) \quad \text{as} \quad m \to \infty \tag{5}$$

holds, then the series $\sum a_n\lambda_n$ is summable $|C,\alpha,\beta;\delta|_k$, $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, $\delta \geq 0$, and $(\alpha+\beta-\delta) > 0$.

3 Main Result

The aim of this paper is to generalize Theorem 1 for the $|C,\alpha,\beta,\sigma;\delta|_k$ summability method. Now, we shall prove the following theorem.

THEOREM 2. Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (2). If (λ_n) is a non-negative and non-increasing sequence such that the series $\sum \frac{\lambda_n}{n}$ is convergent, the conditions (3), (4), and

$$\sum_{n=1}^{m} n^{\sigma(\delta k+k-1)}\frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}} = O(m) \quad \text{as} \quad m \to \infty \tag{6}$$

hold, then the series $\sum a_n\lambda_n$ is summable $|C,\alpha,\beta,\sigma;\delta|_k$, $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, $\sigma \in R$, and $(\alpha+\beta+1)k - \sigma(\delta k+k-1) > 1$.

We need the following lemma for the proof of our theorem.

LEMMA 1 ([1]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^{\beta}a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^{\beta}a_p\right|.$$

4 Proof of Theorem 2

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have that

$$T_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v.$$

First applying Abel's transformation and then using Lemma 1, we have that

$$T_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

$$\begin{aligned}\left|T_n^{\alpha,\beta}\right| &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\right|+\frac{|\lambda_n|}{A_n^{\alpha+\beta}}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\right|\\ &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha}A_v^{\beta}\theta_v^{\alpha,\beta}\left|\Delta\lambda_v\right|+\left|\lambda_n\right|\theta_n^{\alpha,\beta}\\ &= T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.\end{aligned}$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}n^{\sigma(\delta k+k-1)-k}\left|T_{n,r}^{\alpha,\beta}\right|^k<\infty,\quad \text{for}\quad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k' where

$$\frac{1}{k}+\frac{1}{k'}=1,$$

we get that

$$\begin{aligned}&\sum_{n=2}^{m+1}n^{\sigma(\delta k+k-1)-k}\left|T_{n,1}^{\alpha,\beta}\right|^k\\ \leq\ &\sum_{n=2}^{m+1}n^{\sigma(\delta k+k-1)-k}\left|\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha}A_v^{\beta}\theta_v^{\alpha,\beta}\Delta\lambda_v\right|^k\\ =\ &O(1)\sum_{n=2}^{m+1}\frac{1}{n^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}}\left\{\sum_{v=1}^{n-1}v^{\alpha k}v^{\beta k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\right\}\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}\\ =\ &O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\sum_{n=v+1}^{m+1}\frac{1}{n^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}}\\ =\ &O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\int_v^{\infty}\frac{dx}{x^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}}\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v v^{\sigma(\delta k+k-1)}\frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(\Delta\lambda_v)\sum_{p=1}^{v}p^{\sigma(\delta k+k-1)}\frac{(\theta_p^{\alpha,\beta})^k}{p^{k-1}}+O(1)\Delta\lambda_m\sum_{v=1}^{m}v^{\sigma(\delta k+k-1)}\frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}}\\
&= O(1)\sum_{v=1}^{m}v\Delta^2\lambda_v+O(1)m\Delta\lambda_m\\
&= O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}
$$

in view of hypotheses of Theorem 2.

Similarly, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}n^{\sigma(\delta k+k-1)-k}\mid \lambda_n\theta_n^{\alpha,\beta}\mid^k &= O(1)\sum_{n=1}^{m}\frac{\lambda_n}{n}n^{\sigma(\delta k+k-1)}\frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\Delta(\frac{\lambda_n}{n})\sum_{v=1}^{n}v^{\sigma(\delta k+k-1)}\frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}}\\
&\quad +O(1)\frac{\lambda_m}{m}\sum_{n=1}^{m}n^{\sigma(\delta k+k-1)}\frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\Delta\lambda_n+O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1}+O(1)\lambda_m\\
&= O(1)\sum_{n=1}^{m-1}\Delta\lambda_n+O(1)\sum_{n=2}^{m-1}\frac{\lambda_n}{n}+O(1)\lambda_m\\
&= O(1)(\lambda_1-\lambda_m)+O(1)\sum_{n=1}^{m-1}\frac{\lambda_n}{n}+O(1)\lambda_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by virtue of hypotheses of Theorem 2. This completes the proof of Theorem 2.

5 Conclusions

If we take $\beta=0$ and $\sigma=1$, then we get a new result for $|C,\alpha;\delta|_k$ summability factors of infinite series. If we set $\sigma=1$, then we get Theorem 1. Because in this case condition (6) reduces to condition (5). Also, if we take $\beta=0$ and δ=0, then we get a result concerning the $|C,\alpha|_k$ summability. Furthermore, if we take $\sigma=1$, $\beta=0$, $\alpha=1$, and $\delta=0$, then we obtain a new result for the $|C,1|_k$ summability factors. Finally, if we take $\delta=0$, $\beta=0$, $\sigma=1$, and $k=1$, then we get the known result of Pati dealing with $|C,\alpha|$ summability factors of infinite series (see [9]).

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