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Duration Of Play In The Gambler's Ruin Problem: A Novel Derivation*

Greg Orosi†

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Abstract

In this paper, a novel derivation of the duration of play of the classical gambler's ruin problem is presented. Instead of solving the main difference equation the traditional way, our result is obtained by the application of elementary linear algebra. Moreover, the advantage of the derivation is briefly discussed.

1 Introduction

In this work, we present a novel derivation of the duration of play of the classical gambler's ruin problem. Our approach is similar to that introduced in [1], and the difference equations are solved by the application of eigenvalues and eigenvectors. We also point out that besides its pedagogical value, the method is applicable to finding solutions for higher moments in the duration of play.

2 Background and the Formulation of the Problem

The gambler's ruin problem is one of the classical problems in the field of probability. Besides its relevance for academic research, the problem has a substantial pedagogical value. In the classical formulation, a player starts with a positive initial wealth of n units, and wins or loses one unit on each turn of a game with respective probabilities p and $q = 1 - p$. The player's objective is to accumulate a wealth of N units (with $n < N$), but the game can also end with gambler being ruined (that implies a final wealth of 0 units).

Two of the basic questions are determining the probability of ruin and the expected duration of the game. Both of these problems are solved in [2] by finding the appropriate solutions to the difference equations. Although the traditional derivation is conceptually simple, it requires students to be familiar with difference equations.

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†Associate Professor, Department of Mathematics and Statistics, American University of Sharjah, Office: Nab 254, P.O. Box 26666, Sharjah, UAE

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4 The Alternative Derivation

Using conditional probabilities, one can obtain the following relationship:

$$E(n) = pE(n+1) + qE(n-1) + 1, \tag{1}$$

where $E(n)$ denotes the expected duration of the game as a function of the player's wealth. Rearranging the above yields

$$E(n+1) = E(n)\left(\frac{1}{p}\right) - \left(\frac{q}{p}\right)E(n-1) - \frac{1}{p},$$

that can be expressed as

$$\begin{pmatrix} E(n+1) \\ E(n) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{p} & -\frac{q}{p} & -\frac{1}{p} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E(n) \\ E(n-1) \\ 1 \end{pmatrix}, \tag{2}$$

and

$$\begin{pmatrix} E(n+1) \\ E(n) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{p} & -\frac{q}{p} & -\frac{1}{p} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} E(1) \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

The next step in the solution is to find a matrix decomposition of the matrix

$$A = \begin{pmatrix} \frac{1}{p} & -\frac{q}{p} & -\frac{1}{p} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

to find a simple form for A^n. Subsequently, the eigenvalues are found from the characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} \frac{1}{p} - \lambda & -\frac{1-p}{p} & -\frac{1}{p} \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\left(\lambda^2 - \lambda\frac{1}{p} + \frac{1-p}{p}\right) \\ &= (1-\lambda)\left(\lambda^2 - \lambda\frac{1}{p} + \frac{1-p}{p}\right) = -(\lambda-1)^2\left(\lambda - \frac{1-p}{p}\right). \end{aligned} \tag{5}$$

4.1 The Expected Duration of Play if $p \neq q$

Assuming that $p \neq q$, there are two distinct eigenvalues of (5): $\lambda_1 = \frac{1-p}{p}$ and $\lambda_2 = 1$. Then, $A - \lambda_1$ is

$$\begin{pmatrix} 1 & -\frac{1-p}{p} & -\frac{1}{p} \\ 1 & -\frac{1-p}{p} & 0 \\ 0 & 0 & 1-\frac{1-p}{p} \end{pmatrix}$$

that reduces to

$$\begin{pmatrix} 1 & -\frac{1-p}{p} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvector corresponding to λ_1 is given by

$$\overrightarrow{v}_1 = \begin{pmatrix} \frac{1}{p}-1 \\ 1 \\ 0 \end{pmatrix}.$$

Also, $A - \lambda_2$ is

$$\begin{pmatrix} \frac{1}{p}-1 & -\frac{1-p}{p} & -\frac{1}{p} \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

that reduces to

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvector corresponding λ_2 is given by

$$\overrightarrow{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Since there is only one eigenvector corresponding to the repeated eigenvalue $\lambda_2 = 1$, a generalized eigenvector must be obtained from

$$(A - I)\,\overrightarrow{v}_3 = \overrightarrow{v}_2.$$

A straightforward calculation shows that

$$\overrightarrow{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1-2p \end{pmatrix}$$

satisfies the equation

$$\begin{pmatrix} \frac{1}{p}-1 & -\frac{1-p}{p} & -\frac{1}{p} \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overrightarrow{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, the Jordan decomposition of A is

$$
\begin{aligned}
A &= \begin{pmatrix} \frac{1}{p} & -\frac{1-p}{p} & -\frac{1}{p} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = S \cdot J \cdot S^{-1} \\
&= [\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3] \begin{pmatrix} \frac{1-p}{p} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} [\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3]^{-1} \\
&= \begin{pmatrix} \frac{1}{p}-1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1-2p \end{pmatrix} \begin{pmatrix} \frac{1-p}{p} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{p}{1-2p} & \frac{p}{2p-1} & -\frac{p}{(1-2p)^2} \\ \frac{p}{2p-1} & \frac{p-1}{2p-1} & \frac{p}{(1-2p)^2} \\ 0 & 0 & \frac{1}{1-2p} \end{pmatrix}.
\end{aligned}
$$

It can be noted from (3) that it is not necessary to compute all the entries of A^n. Specifically, it is enough to compute the first and third entries of the matrix product $S \cdot J^n \cdot S^{-1}$. Taking advantage of this observation, we obtain

$$
\begin{aligned}
A^n &= \begin{pmatrix} \frac{1}{p}-1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1-2p \end{pmatrix} \begin{pmatrix} \frac{1-p}{p} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} \frac{p}{1-2p} & \frac{p}{2p-1} & -\frac{p}{(1-2p)^2} \\ \frac{p}{2p-1} & \frac{p-1}{2p-1} & \frac{p}{(1-2p)^2} \\ 0 & 0 & \frac{1}{1-2p} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{p}-1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1-2p \end{pmatrix} \begin{pmatrix} \left(\frac{1-p}{p}\right)^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{p}{1-2p} & \frac{p}{2p-1} & -\frac{p}{(1-2p)^2} \\ \frac{p}{2p-1} & \frac{p-1}{2p-1} & \frac{p}{(1-2p)^2} \\ 0 & 0 & \frac{1}{1-2p} \end{pmatrix} \\
&= \begin{pmatrix} * & * & * \\ \left(\frac{1-p}{p}\right)^n & 1 & n \\ * & * & * \end{pmatrix} \begin{pmatrix} \frac{p}{1-2p} & \frac{p}{2p-1} & -\frac{p}{(1-2p)^2} \\ \frac{p}{2p-1} & \frac{p-1}{2p-1} & \frac{p}{(1-2p)^2} \\ 0 & 0 & \frac{1}{1-2p} \end{pmatrix} \\
&= \begin{pmatrix} * & * & * \\ \left(\frac{\left(\left(\frac{1-p}{p}\right)^n - 1\right)p}{1-2p}\right) & * & -\left(\frac{1-p}{p}\right)^n \left(\frac{p}{(1-2p)^2}\right) + \frac{p}{(1-2p)^2} + \frac{n}{1-2p} \\ * & * & * \end{pmatrix},
\end{aligned}
$$

where the $*'$s are elements that do not have to be computed explicitly. Therefore,

$$
\begin{aligned}
E(n) &= \begin{pmatrix} \left(\frac{\left(\left(\frac{1-p}{p}\right)^n - 1\right)p}{1-2p}\right) & * & -\left(\frac{1-p}{p}\right)^n \left(\frac{p}{(1-2p)^2}\right) + \frac{p}{(1-2p)^2} + \frac{n}{1-2p} \end{pmatrix} \begin{pmatrix} E(1) \\ 0 \\ 1 \end{pmatrix} \\
&= \left(\frac{\left(\left(\frac{1-p}{p}\right)^n - 1\right)p}{1-2p}\right) E(1) + \left(1 - \left(\frac{1-p}{p}\right)^n\right) \frac{p}{(1-2p)^2} + \frac{n}{1-2p}. \qquad (6)
\end{aligned}
$$

Incorporating the boundary condition $E(N) = 0$ in the above gives

$$E(N) = 0 = \left(\frac{\left(\left(\frac{1-p}{p} \right)^N - 1 \right) p}{1-2p} \right) E(1) + \left(1 - \left(\frac{1-p}{p} \right)^N \right) \frac{p}{(1-2p)^2} + \frac{N}{1-2p},$$

that can be rearranged as

$$E(1) = \frac{\left(\left(\frac{1-p}{p} \right)^N - 1 \right) \frac{p}{(1-2p)^2} - \frac{N}{1-2p}}{\left(\frac{\left(\left(\frac{1-p}{p} \right)^N - 1 \right) p}{1-2p} \right)} = \frac{1}{(1-2p)} - \frac{N}{\left(\left(\left(\frac{1-p}{p} \right)^N - 1 \right) p \right)}.$$

Finally, substituting the above expression into (6) gives the result

$$E(n) = -\frac{N}{1-2p} \frac{\left(\left(\frac{1-p}{p} \right)^n - 1 \right)}{\left(\left(\frac{1-p}{p} \right)^N - 1 \right)} + \frac{n}{1-2p}$$

which agrees with the formula presented in [2].

4.2 The Expected Duration of Play if $p = q$

Assuming that $p = q$, there is only one eigenvalue of (5): $\lambda_4 = 1$. Then

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A - \lambda_4 = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The above reduces to

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvector corresponding to λ_4 is given by

$$\overrightarrow{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then, it can be easily shown that

$$\overrightarrow{v}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

satisfies the equation

$$(A - I)\, \overrightarrow{v}_5 = \overrightarrow{v}_4,$$

and another generalized eigenvector $\overrightarrow{v}_6$ can be found from

$$(A - I)\,\overrightarrow{v}_6 = \overrightarrow{v}_5,$$

and is given by

$$\overrightarrow{v}_6 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}.$$

Therefore, the Jordan decomposition in this case is given by

$$\begin{aligned} A &= \begin{pmatrix} 2 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [\overrightarrow{v}_4, \overrightarrow{v}_5, \overrightarrow{v}_6] \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} [\overrightarrow{v}_4, \overrightarrow{v}_5, \overrightarrow{v}_6]^{-1} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ 1 & 0 & 0 \\ * & * & * \end{pmatrix} \begin{pmatrix} n & -n+1 & -n(n-1) \\ 1 & -1 & -2n \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ n & * & -n(n-1) \\ * & * & * \end{pmatrix}. \end{aligned}$$

Hence,

$$E(n) = \begin{pmatrix} n & * & -n(n-1) \end{pmatrix} \begin{pmatrix} E(1) \\ 0 \\ 1 \end{pmatrix} = nE(1) - n(n-1), \tag{7}$$

and incorporating the boundary condition $E(N) = 0$ in the above gives

$$E(N) = 0 = N \cdot E(1) - N(N-1),$$

from which one obtains

$$E(1) = (N-1).$$

Finally, the required result presented in [2] can be obtained substituting the above expression into (7) and is given by

$$E(n) = n(N-1) - n(n-1) = n\,(N-1)\,. \tag{8}$$

5 Conclusions

In this work, we present a novel derivation for the expected duration of play of the classical gambler's ruin problem. The primary advantage of the novel approach is that it is conceptually simpler for students familiar with linear algebra. Furthermore, besides its education value, the method could be useful to derive expressions for higher moments of the duration of play. For example, our approach is applicable to the problem studied in [3] and to further generalizations.

References

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