Applied Mathematics E-Notes, 18(2018), 250-258 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Novel Way To Solve The $\mathcal{D}_q$-Appell Equation*

Daniel Joseph Galiffa†, Boon Wee Ong‡

Received 19 September 2017

Abstract

We develop and implement a novel approach for determining the q-orthogonal polynomial solutions to the $\mathcal{D}_q$-Appell Equation ($\mathcal{D}_q P_n(x) = \gamma_n P_{n-1}(x)$), where $\mathcal{D}_q$ is the Askey-Wilson divided-difference operator, and γ_n is a function of n and q that is independent of x. More specifically, our methodology relies only on the second and third coefficients of $P_n(x)$ and a three-term recurrence relation. Together, these structures lead to various difference equations from which recursion coefficients can be inferred. Moreover, this approach has the potential to be applied to other types of characterization problems as well.

1 Introduction

In this work, we characterize the $\mathcal{D}_q$-*Appell Equation*

$$\mathcal{D}_q P_n(x) = \gamma_n P_{n-1}(x) \tag{1}$$

by determining all of its q-orthogonal polynomial solutions $\{P_n(x)\}_{n=0}^{\infty}$. We do this in a way that is entirely different than what has been done previously, i.e., [1, 7], and hence the title of this paper. We emphasize that in [1, 7], prior knowledge that the Rogers' q-Hermite polynomials were solutions to (1) was necessary for completion. We emphasize that the papers [1, 7] were concerned with showing that the Rogers' q-Hermite polynomials were the only solutions to (1). In our work, we are concerned with obtaining general solutions to (1), as our primary focus is the implementation of our methodology. Furthermore, one of the utilities of our present work is that our method applies to the structure equation (1) directly without using any previous information. Thus, we anticipate that our method can be also be applied to other characterization problems as well.

In relation (1), γ_n is a function of n and q that is independent of x, and $\mathcal{D}_q$ is the Askey-Wilson degree-lowering, divided-difference, linear operator

$$\mathcal{D}_q f(x) := \frac{\breve{f}\left(q^{\frac{1}{2}}z\right) - \breve{f}\left(q^{-\frac{1}{2}}z\right)}{\breve{e}\left(q^{\frac{1}{2}}z\right) - \breve{e}\left(q^{-\frac{1}{2}}z\right)},$$

*Mathematics Subject Classifications: 20F05, 20F10, 20F55, 68Q42.

†Mathematics Department, Penn State Erie, The Behrend College, Erie, PA, 16563, USA

‡Mathematics Department, Penn State Erie, The Behrend College, Erie, PA, 16563, USA

with $z = e^{i\theta}$, $\breve{f}(z) = f(x) = f(\cos\theta)$, for any function f and $e(x) = x$. For further details regarding this operator, consider [1, 3, 8], as well as [6] and the additional references therein.

Over the years, much attention has been paid to the generalized structure equation

$$\pi(x)T(P_n(x)) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x) \tag{2}$$

for different T-operators. In 1972, W.A. Al-Salam and T.S. Chihara [2] determined all of the classical monic orthogonal polynomial sequences (OPS) that solve (2), where $\pi(x)$ is a polynomial of degree at most two[1], i.e.,

$$\pi(x)\frac{d}{dx}P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x). \tag{3}$$

Since $\pi(x) = ax^2 + bx + c$ is at most quadratic, (3) only needed to be analyzed for the following cases:

$$\pi(x) = \begin{cases} 1 \\ x \\ x^2 + c. \end{cases}$$

Any other form of $\pi(x)$ can be achieved via a linear change-of-variables.

In each case above, Al-Salam and Chihara simultaneously analyzed the structure relation (3) and a three-term recurrence relation (a necessary and sufficient condition for orthogonality) of the form

$$\begin{gathered} P_{n+1}(x) = (x + B_n)P_n(x) - C_n P_{n-1}(x), \quad C_n \neq 0 \\ P_{-1}(x) = 0, \quad P_0(x) = 1. \end{gathered} \tag{4}$$

This led to equations relating the coefficients of $\pi(x)$ and the recursion coefficients in (3) and (4). From there, several difference equations were developed from which expressions for B_n and C_n were determined. These expressions contained arbitrary parameters, which when chosen judiciously lead to the sought after (monic) classical OPS.

In particular, Al-Salam and Chihara determined that the Hermite, Laguerre and Jacobi polynomials are respectively the only orthogonal polynomial solutions. For each of these polynomial solutions, orthogonality was defined on the real line with respect to a nondecreasing real function. When considering polynomials orthogonal on the real line with respect to a function of bounded variation, the generalized Bessel polynomials were also solutions in the limiting case $c \to 0$ for $\pi(x) = x^2 + c$.

In 2006, S. Datta and J. Griffin [4] discovered all q-orthogonal polynomials solutions to the difference equation

$$\pi(x)D_q P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \tag{5}$$

with the previous restrictions on $\pi(x)$, and with the q-degree-lowering, divided-difference, linear operator, D_q, cf. [6], defined as

$$(D_q f)(x) := \frac{f(x) - f(qx)}{x - qx}.$$

[1] A uniform derivation of (3) appears in [5].

Their work was the q-analogue of [2] because the differential operator d/dx in (3) was replaced by D_q above. Datta and Griffin determined all of the q-orthogonal polynomial solutions to (5) using essentially the same methodology as in [2] and also took into account that (5) does not remain invariant under the linear transformation $x \to ax+b$. Therefore, the following cases and sub-cases were considered:

$$\pi(x) = \begin{cases} 1 \\ x, \quad x+c \\ x^2, \quad x^2+s, \quad x^2+rx, \quad x^2+rx+s. \end{cases} \tag{6}$$

The solutions they obtained were the Al-Salam-Carlitz I, the discrete q-Hermite I, the big and little q-Laguerre polynomials and the big and little q-Jacobi polynomials - the q-Bessel polynomials were achieved by taking appropriate limits.

We also mention that the recent manuscript "q-Orthogonal polynomial solutions to a class of difference equations," by D.J. Galiffa and S.J. Johnston, presents an analysis of the difference equation

$$\pi(x)D_{q^{-1}}P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x)$$

with

$$(D_{q^{-1}}f)(x) := \frac{f(x) - f(x/q)}{x - x/q}.$$

The solutions turned out to be the Al-Salam-Carlitz II, the discrete q-Hermite II, the q-Laguerre and the Stieltjes-Wigert polynomials, as well as q-orthogonal polynomials that are currently not fully characterized.

Interestingly enough, OPS solutions of (2), for each corresponding case of $\pi(x)$, have been determined for the operators $T = d/dx, D_q, D_{q^{-1}}$, but *not* for the case when $T = \mathcal{D}_q$. In this paper, we consider Case 1 of (6) ($\pi(x) = 1$), from which it follows that $\alpha_n = \beta_n = 0$ and hence (1). We also mention that, to the very best of our knowledge, results regarding Cases 2 and 3 of (6) (with $T = \mathcal{D}_q$) do not appear in the literature.

In addition, characterizing Case 1 of [2] is actually equivalent to determining which Appell sets are also orthogonal. Characterizing Case 1 of [4] is the same as determining the q-Appell orthogonal sets. Similarly, (1) defines the $\mathcal{D}_q$-Appell Equation. Therefore, we also determine all of the $\mathcal{D}_q$-Appell sets, as previously discussed.

We now outline the remainder of this paper. In Section 2, we develop some rudimentary results and conventions that are important in our subsequent analysis. In section 3, we derive the general recursion coefficients B_n and C_n of (4) for an OPS satisfying (1). We then select concrete values for the arbitrary parameters contained within our coefficients, which lead to our solution. We conclude the paper by briefly stating some open research that stems from this paper.

2 Preliminaries

In our approach to characterizing (1), we make much use of second and third coefficients of $P_n(x)$, as denoted below:

$$P_n(x) = x^n + S_n x^{n-1} + R_n x^{n-2} + \mathcal{O}\left(x^{n-3}\right), \quad n = 1, 2, 3, \ldots, \tag{7}$$

with $P_1(x) = x + S_1$. From (4) and (7), we immediately obtain expressions for S_n and R_n in terms of B_n and C_n.

LEMMA 1. In regard to (4) and (7), the following recurrence relations hold

$$S_{n+1} = S_n + B_n, \quad n = 1, 2, 3, \ldots, \tag{8}$$

$$R_{n+1} = R_n + B_n S_n - C_n, \quad n = 2, 3, 4, \ldots, \tag{9}$$

with $S_1 = B_0$ for $n = 0$, and $R_2 = B_0 B_1 - C_1$ for $n = 1$.

PROOF. From expanding (4) in terms of (7) we see that

$$\begin{aligned} & x^{n+1} + S_{n+1}x^n + R_{n+1}x^{n-1} + \mathcal{O}\left(x^{n-2}\right) \\ = \;& x\left(x^n + S_n x^{n-1} + R_n x^{n-2} + \mathcal{O}\left(x^{n-3}\right)\right) \\ & + B_n\left(x^n + S_n x^{n-1} + R_n x^{n-2} + \mathcal{O}\left(x^{n-3}\right)\right) \\ & - C_n\left(x^{n-1} + S_{n-1}x^{n-2} + R_{n-1}x^{n-3} + \mathcal{O}\left(x^{n-4}\right)\right). \end{aligned}$$

We obtain our results by comparing coefficients of x^n and x^{n-1}.

We can readily see that (8) has the solution

$$S_{n+1} = \sum_{k=0}^{n} B_k.$$

The solution to (9) is established in Section 3 of the paper.

We conclude this section with some fundamental results regarding $\mathcal{D}_q$ that are used in Section 3. For these, we call upon the Chebyshev polynomials of the first and second kind, respectively denoted $\{T_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$. It then follows that

$$\mathcal{D}_q(1) = 0, \quad {}_q(x) = 1,$$

and

$$\mathcal{D}_q T_n(x) = \nu_n U_{n-1}(x), \tag{10}$$

where

$$\nu_n := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{11}$$

3 The Main Result

In this section, we derive the general recursion coefficients B_n and C_n of (4) for an OPS satisfying (1), by utilizing the results of the previous section. From there, we select concrete values for the arbitrary parameters contained within our coefficients, which lead to our solution. We conclude this section by briefly summarizing some relevant open research.

To begin, we note that since $\mathcal{D}_q$ is linear, in order to evaluate $\mathcal{D}_q P_n(x)$ we only need to know $\mathcal{D}_q(x^m)$ for $m = 0, 1, 2, \ldots$. This leads to the proceeding statement.

THEOREM 1. For any natural number m,

$$\begin{aligned}2^{m-1}\mathcal{D}_q(x^m) &= \nu_m U_{m-1}(x) + m\nu_{m-2}U_{m-3}(x) \\ &\quad + \binom{m}{2}\nu_{m-4}U_{m-5}(x) + \mathcal{O}\left(x^{m-7}\right).\end{aligned}$$

PROOF. With $z = e^{i\theta}$, we have $2x = z + z^{-1}$ so that

$$\begin{aligned}(2x)^m &= \left(z + z^{-1}\right)^m \\ &= z^m + mz^{m-2} + \binom{m}{2}z^{m-4} + \cdots + mz^{2-m} + z^{-m} \\ &= \left(z^m + z^{-m}\right) + m\left(z^{m-2} + z^{2-m}\right) + \binom{m}{2}\left(z^{m-4} + z^{4-m}\right) + \cdots \\ &= 2\left(T_m(x) + mT_{m-2}(x) + \binom{m}{2}T_{m-4}(x) + \cdots\right).\end{aligned}$$

Using (10) we have

$$\mathcal{D}_q(2^m x^m) = 2\left(\nu_m U_{m-1}(x) + m\nu_{m-2}U_{m-3}(x) + \binom{m}{2}\nu_{m-4}U_{m-5}(x) + \mathcal{O}\left(x^{m-7}\right)\right),$$

where we have used the fact that all negatively indexed $U_n(x)$-terms must be zero. Since $\mathcal{D}_q$ is linear, we complete the proof by dividing both sides of the above equation by 2.

THEOREM 2. Equation (1) gives rise to the following difference equations in S_n and R_n, as in Lemma 1:

$$\nu_{n-1}S_n = \nu_n S_{n-1}, \quad n = 2, 3, 4, \ldots, \tag{12}$$

$$\nu_{n-2}\left(R_n + \frac{n}{4}\right) = \nu_n\left(R_{n-1} + \frac{n-2}{4}\right), \quad n = 3, 4, 5, \ldots. \tag{13}$$

PROOF. The right-hand side of (1) is

$$\gamma_n P_{n-1}(x) = \gamma_n\left(x^{n-1} + S_{n-1}x^{n-2} + R_{n-1}x^{n-3} + \mathcal{O}\left(x^{n-4}\right)\right)$$

and the left-hand side is

$$\begin{aligned}\mathcal{D}_q P_n(x) &= \mathcal{D}_q\left(x^n + S_n x^{n-1} + R_n x^{n-2} + \mathcal{O}\left(x^{n-3}\right)\right) \\ &= \frac{1}{2^{n-1}}\left(\nu_n U_{n-1}(x) + n\nu_{n-2}U_{n-3}(x) + \mathcal{O}\left(x^{n-5}\right)\right) \\ &\quad + \frac{S_n}{2^{n-2}}\left(\nu_{n-1}U_{n-2}(x) + (n-1)\nu_{n-3}U_{n-4}(x) + \mathcal{O}\left(x^{n-6}\right)\right) \\ &\quad + \frac{R_n}{2^{n-3}}\left(\nu_{n-2}U_{n-3}(x) + (n-2)\nu_{n-4}U_{n-5}(x) + \mathcal{O}\left(x^{n-7}\right)\right) + \cdots.\end{aligned}$$

After comparing the coefficients of x^{n-1} and noting that the Chebyshev polynomials of the second kind, $\{U_n(x)\}_{n=0}^{\infty}$, have a leading coefficient of 2^n, we obtain

$$\gamma_n = \nu_n.$$

Similarly, comparing coefficients of x^{n-2} establishes (12).

In order to compare the coefficients of x^{n-3}, we call upon the formula

$$U_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r}.$$

We then have

$$U_{n-1}(x) = 2^{n-1}x^{n-1} - (n-2)2^{n-3}x^{n-3} + \mathcal{O}\left(x^{n-5}\right),$$

which implies that

$$\nu_n R_{n-1} = \frac{n}{4}\nu_{n-2} + \nu_{n-2}R_n - \frac{(n-2)}{4}\nu_n,$$

from which (13) follows.

The relation (12), together with the solution to equation (8), imply that $S_n = \nu_n B_0$ and that

$$B_n = \left(\nu_{n+1} - \nu_n\right) B_0, \quad n = 0, 1, 2, \ldots. \tag{14}$$

We now derive a more explicit form for R_n in terms of R_2 by iterating (13), beginning with $n = 3$.

LEMMA 2. For $n \geq 3$, the solution to equation (13) is

$$R_n = \frac{\nu_n \nu_{n-1}}{\nu_2}\left(R_2 + \frac{1}{4} - \sum_{k=2}^{n-2} \frac{\nu_2}{4\nu_k \nu_{k+1}}\right) - \frac{n}{4}. \tag{15}$$

PROOF. We prove this statement via induction on n. For $n = 3$, equation (13) yields

$$\nu_1\left(R_3 + \frac{3}{4}\right) = \nu_3\left(R_2 + \frac{1}{4}\right) \Rightarrow R_3 = \nu_3\left(R_2 + \frac{1}{4}\right) - \frac{3}{4}.$$

Substituting $n = 3$ into formula (15), we obtain

$$R_3 = \frac{\nu_3 \nu_{3-1}}{\nu_2}\left(R_2 + \frac{1}{4} - \sum_{k=2}^{1} \frac{\nu_2}{4\nu_k \nu_{k+1}}\right) - \frac{3}{4} = \nu_3\left(R_2 + \frac{1}{4}\right) - \frac{3}{4}$$

and the base case is secured.

Now assume the formula holds for $n = N - 1$, i.e.,

$$R_{N-1} = \frac{\nu_{N-1}\nu_{N-2}}{\nu_2}\left(R_2 + \frac{1}{4} - \sum_{k=2}^{N-3} \frac{\nu_2}{4\nu_k \nu_{k+1}}\right) - \frac{N-1}{4}.$$

Thus, using the above relation and (13), with $n = N$, we have

$$\begin{aligned} R_N + \frac{N}{4} &= \frac{\nu_N}{\nu_{N-2}} \left(\frac{\nu_{N-1}\nu_{N-2}}{\nu_2} \left(R_2 + \frac{1}{4} - \sum_{k=2}^{N-3} \frac{\nu_2}{4\nu_k\nu_{k+1}} \right) - \frac{1}{4} \right) \\ &= \frac{\nu_N\nu_{N-1}}{\nu_2} \left(R_2 + \frac{1}{4} - \sum_{k=2}^{N-3} \frac{\nu_2}{4\nu_k\nu_{k+1}} - \frac{\nu_2}{4\nu_{N-2}\nu_{N-1}} \right) \\ &= \frac{\nu_N\nu_{N-1}}{\nu_2} \left(R_2 + \frac{1}{4} - \sum_{k=2}^{N-2} \frac{\nu_2}{4\nu_k\nu_{k+1}} \right). \end{aligned}$$

This evidently establishes our result.

We now evaluate C_n. Equation (9) of Lemma 1 implies that

$$\begin{aligned} C_n &= B_n S_n - (R_{n+1} - R_n) \\ &= \nu_n \left(\nu_{n+1} - \nu_n\right) B_0^2 - (R_{n+1} - R_n). \end{aligned} \tag{16}$$

We can further simplify the difference $R_{n+1} - R_n$ via Lemma 2 as follows

$$\begin{aligned} R_{n+1} - R_n &= \frac{\nu_{n+1}}{\nu_{n-1}} R_n - R_n + \frac{\nu_{n+1}}{\nu_{n-1}} \left(\frac{n-1}{4} \right) - \frac{n+1}{4} \\ &= \left(\frac{\nu_{n+1}}{\nu_{n-1}} - 1 \right) \left(R_n + \frac{n}{4} \right) - \frac{1}{4} \left(\frac{\nu_{n+1}}{\nu_{n-1}} + 1 \right) \\ &= \frac{\mu_n\nu_n}{\nu_2} \left(R_2 + \frac{1}{4} - \sum_{k=2}^{n-2} \frac{\nu_2}{4\nu_k\nu_{k+1}} \right) - \frac{\mu_1\nu_n}{4\nu_{n-1}}, \end{aligned} \tag{17}$$

with

$$\mu_n := q^{n/2} + q^{-n/2}.$$

The above sum can be written in closed form using a *telescoping* technique, as seen in the next statement.

LEMMA 3. With ν_n as defined in (11), we have

$$\sum_{k=2}^{n-2} \frac{\nu_2}{4\nu_k\nu_{k+1}} = \frac{q(1-q^{n-3})}{4(1-q^{n-1})}.$$

PROOF. From considering the expansion of the rational function below

$$\frac{1-x}{(1-x^{k+1})(1-x^k)} = \frac{1}{1-x^k} - \frac{x}{1-x^{k+1}},$$

we see that

$$\begin{aligned} \frac{1}{\nu_k\nu_{k+1}} &= \frac{1-q}{1-q^k} q^{\frac{k}{2}-\frac{1}{2}} \frac{1-q}{1-q^{k+1}} q^{\frac{k+1}{2}-\frac{1}{2}} \\ &= (1-q) \left(\frac{q^{k-\frac{1}{2}}}{1-q^k} - \frac{q^{k+\frac{1}{2}}}{1-q^{k+1}} \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=2}^{n-2} \frac{\nu_2}{4\nu_k \nu_{k+1}} &= \frac{\nu_2}{4}(1-q) \sum_{k=2}^{n-2} \left(\frac{q^{k-\frac{1}{2}}}{1-q^k} - \frac{q^{k+\frac{1}{2}}}{1-q^{k+1}} \right) \\ &= \frac{1}{4q^{1/2}}(1-q^2) \left(\frac{q^{3/2}}{1-q^2} - \frac{q^{n-\frac{3}{2}}}{1-q^{n-1}} \right) \\ &= \frac{q(1-q^{n-3})}{4(1-q^{n-1})}.\end{aligned}$$

Recalling that $R_2 = B_0 B_1 - C_1$ via Lemma 1, we can expand (17) as follows:

$$R_{n+1} - R_n = \frac{(1-q)(1-q^n)(q-q^n) + 4q\left(1-q^{2n}\right)(B_0 B_1 - C_1)}{4q^n (1-q^2)}.$$

Therefore, from expanding (16), we achieve

$$\begin{aligned} C_n = &\frac{\left(1-\sqrt{q}\right) q^{\frac{1}{2}-n} (1-q^n) \left(1+q^{\frac{1}{2}+n}\right)}{(1-q)^2} B_0^2 \\ &- \frac{(1-q)(1-q^n)(q-q^n) + 4q\left(1-q^{2n}\right)(B_0 B_1 - C_1)}{4q^n (1-q^2)}. \end{aligned} \tag{18}$$

We now have the general recursion coefficients B_n and C_n, as respectively in (14) and (18) that an OPS, $\{P_n(x)\}_{n=0}^{\infty}$, must satisfy in order to solve (1). These coefficients contain the arbitrary parameters B_0 and C_1. In other words, all q-orthogonal polynomial solutions satisfying the structure equation (1) and the three-term recurrence relation (4) must have recursion coefficients of the form (14) and (18).

We can obtain the recursion coefficients for the continuous Rogers' q-Hermite polynomials as a *special case* of our recursion coefficients in (14) and (18) by concretely selecting $B_0 = 0$ and $C_1 = (1-q)/4$. Namely, with these choices we obtain

$$B_n \equiv 0 \quad \text{and} \quad C_n = \frac{1}{4}(1-q^n),$$

which are the desired coefficients.

We leave open for consideration the problem of determining all q-OPS that satisfy the structure equation

$$\pi(x)\mathcal{D}_q P_n(x) = (\alpha_n x + \beta_n) P_n(x) + \gamma_n P_{n-1}(x)$$

for $\pi(x)$ as in Cases 2 and 3 of (6). M. E. H. Ismail also discusses this problem in [6], cf. eq. (24.7.7). As stated in Section 1, these completed characterizations do not appear in the literature.

References

[1] W. A. Al-Salam, A characterization of the Rogers q-Hermite polynomials, Int. J. Math. Math. Sci., 18(1995), 641–647.

[2] W. A. Al-Salam and T. S. Chihara, Another characterization of the classical orthogonal polynomials, SIAM J. Math. Anal., 3(1972), 65–70.

[3] B. M. Brown, W. D. Evans and M. E. H. Ismail, The Askey-Wilson polynomials and q-Sturm-Liouville problems, Math. Proc. Cambridge Philos. Soc., 119(1996), 1–16.

[4] S. Datta and J. Griffin, A characterization of some q-orthogonal polynomials, Ramanujan J., 12(2006), 425–437.

[5] A. Eldélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol. 2 (McGraw-Hill, New York, 1953).

[6] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia of Mathematics and its Applications, vol. 98 (Cambridge University Press, Cambridge, 2005).

[7] M. E. H. Ismail and R. Zhang, Diagonalization of Certain Integral Operators, Adv. Math., 109(1994), 1–33.

[8] A. P. Magnus, Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials, in M. Alfaro et al. (Ed.), Orthogonal Polynomials and Their Applications, (Lecture Notes in Mathematics, Vol. 1329, Springer-Verlag, Berlin, 1988), pp. 261–278.