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A Refinement Of An Integral Inequality For The Polar Derivative Of A Polynomial*

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Abstract

Certain refinements of a recently obtained integral inequality by Rather and Bhat for the polar derivative of a polynomial with restricted zeros are given.

1 Introduction

Let $\mathcal{P}_n$ be the set of all complex polynomials $P(z)$ of degree n. It was shown by Turan [12] that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le 1$, then

$$n \max_{|z|=1} |P(z)| \le 2 \max_{|z|=1} |P'(z)| . \tag{1}$$

Equality in (1) holds for $P(z) = \alpha z^n + \beta, |\alpha| = |\beta|$.

Govil [4] showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k, k \ge 1$, then

$$n \max_{|z|=1} |P(z)| \le (1 + k^n) \max_{|z|=1} |P'(z)| . \tag{2}$$

The estimate is sharp and equality in (2) holds for $P(z) = (z^n + k^n)$.

Malik [7] obtained an extension of (1) in the sense that the left hand side of of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$ by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \le \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)| .$$

Exremal polynomial is $P(z) = az^n + b, |a| = |b|$.

For the class of polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \le k, k \ge 1$, Aziz [1] proved for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \le \left\{ \int_0^{2\pi} \left| 1 + k^n e^{i\theta}) \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)| . \tag{3}$$

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Equality in (3) holds for $P(z) = z^n + k^n$. In the limiting case when $q \to \infty$, the inequality (3) reduces to inequality (2). In literature there exist other similiar type of results on polynomial approxmation theory(see [5, 9]).

For $\alpha \in \mathbb{C}$, the polar derivative $D_\alpha P(z)$ of a polynomial $P \in \mathcal{P}_n$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$$

(see [6, 8]). The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary $P'(z)$ of $P(z)$ in the sense that

$$Lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \le R$, $R > 0$.

As an extension of inequality (2) to the polar derivative of a polynomial, Aziz and Rather [2] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$ where $k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,

$$n\left(|\alpha| - k\right) \max_{|z|=1} |P(z)| \le (1 + k^n) \max_{|z|=1} |D_\alpha P(z)| . \tag{4}$$

More recently Rather and Bhat [11] extended inequality (3) to the polar derivative of polynomial and obtain a generalization of (4) in the sense that the left hand side of (4) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$ by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$ where $k \ge 1$, then for $|\alpha| \ge k$ and $q > 0$,

$$n\left(|\alpha| - k\right) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} |D_\alpha P(z)| \tag{5}$$

and under the same hypothesis, they [11] also proved that

$$\begin{aligned} n\left(|\alpha| - k\right) &\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta m\right|^q d\theta\right\}^{1/q} \\ &\le \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \left\{\max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1}\right\} \end{aligned} \tag{6}$$

where $|\beta| \le 1$ and $m = \min_{|z|=k} |P(z)|$.

In this paper we first present the following refinement of inequality (5).

THEOREM 1. If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \ge 2$ having all its zeros in $|z| \le k$ where $k \ge 1$, then for every $\alpha \in C$ with $|\alpha| \ge k$ and for each $q > 0$,

$$\begin{aligned} n\left(|\alpha| - k\right) &\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \\ &\le \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} |D_\alpha P(z)| - \phi(k) \left|na_0 + \alpha a_1\right| \end{aligned} \tag{7}$$

where

$$\phi(k) = \left(1 - 1/k^2\right) \ \ or \ \ (1 - 1/k) \ \text{ according as } n > 2 \text{ or } n = 2. \tag{8}$$

Equality in (7) holds in the limiting case when $\alpha \to \infty$ and the extremal polynomial is $P(z) = (z^n + k^n)$.

To see this, we divide the two sides of inequality (7) by $|\alpha|$, let $\alpha \to \infty$ and use the fact that $\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$, we get

$$n \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\} \max_{|z|=1} |P'(z)| - \phi(k)\,|a_1|\,.$$

For the polynomial $P(z) = (z^n + k^n)$, $\max_{|z|=1} |P'(z)| = n$ and $a_1 = 0$. By using property of definite of integrals, the left hand side of above inequality equals

$$n \left\{ \int_0^{2\pi} \left|e^{in\theta} + k^n\right|^q d\theta \right\}^{1/q} = n \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q}$$

whereas the right hand side equals

$$n \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q}.$$

Thus the two sides of above inequality are equal. Therefore, the equality in Theorem 1 holds in limiting case when $\alpha \to \infty$ and the extremal polynomial is $P(z) = (z^n + k^n)$. Further if we let $q \to \infty$ in (7), we get a refinement of inequality (4). We next prove:

THEOREM 2. If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq k,\ |\beta| \leq 1$ and for each $q > 0$,

$$\begin{aligned} n\,(|\alpha| - k) &\left\{ \int_0^{2\pi} \left|P(e^{i\theta}) + \beta m\right|^q d\theta \right\}^{1/q} \\ &\leq \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1} \right\} - \phi(k)\,|na_0 + \alpha a_1| \end{aligned} \tag{9}$$

where $\phi(k)$ is given by (8).

Equality in (9) holds in the limiting case when $|\alpha| \to \infty$ and the extremal polynomial is $P(z) = (z^n + k^n)$ as can be verified as before since $m = 0$. For $\beta = 0$, Theorem 2 gives the following refinement of Theorem 1.

COROLLARY 1. If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha, \beta \in C$ with

$|\alpha| \geq k,\ |\beta| \leq 1$ and for each $q > 0$,

$$n\left(|\alpha| - k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}$$
$$\leq \left\{\int_0^{2\pi}\left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}|D_\alpha P(z)| - nm/k^{n-1}\right\}$$
$$- \phi(k)\left|na_0 + \alpha a_1\right|$$

where $\phi(k)$ is same as defined in Theorem 1.

Letting $q \to \infty$ in (9) and chosing the argument of β with $|\beta| = 1$ suitably, we obtain the following refinement of inequality (4).

COROLLARY 2. If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k}|P(z)|$, then for every $\alpha \in C$ with $|\alpha| \geq k$,

$$n\left(|\alpha| - k\right)\max_{|z|=1}|P(z)| + n\left(|\alpha| + 1/k^{n-1}\right)m + \phi(k)\left|na_0 + \alpha a_1\right|$$
$$\leq \left(1 + k^n\right)\max_{|z|=1}|D_\alpha P(z)|$$

where $\phi(k)$ is given by (8).

2 Lemmas

For the proofs of these theorems we need the following results. The first result is due to Frappier, Rahman and Ruscheweyh [3].

LEMMA 1. If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$,

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)|,\ if\ n > 1$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)|,\ if\ n = 1.$$

Next result is due to Rahman and Schmeisser [10].

LEMMA 2. If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$ and $q > 0$,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leq C_q\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}$$

where

$$C_q = \frac{\left\{\int_0^{2\pi}\left|1 + R^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi}\left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q}}.$$

3 Proofs of the Theorems

PROOF OF THEOREM 1. By hypothesis all the zeros of $P(z)$ lie in $|z| \leq k$, therefore, all the zeros of $f(z) = P(kz)$ lie in $|z| \leq 1$. Applying inequality (5) with $k = 1$ to the polynomial $f(z)$, we get for each $q > 0$ and $|\beta| \geq 1$,

$$n(|\beta| - 1)\left\{\int_0^{2\pi} \left|f(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} |D_\beta f(z)|.$$

Setting $\beta = \frac{\alpha}{k}$ in above inequality and noting that $|\beta| = \left|\frac{\alpha}{k}\right| \geq 1$, we have

$$n\left(\frac{|\alpha|}{k} - 1\right)\left\{\int_0^{2\pi} \left|f(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|D_{\frac{\alpha}{k}} f(z)\right| \tag{10}$$

Let $g(z) = z^n \overline{f(1/\bar{z})}$. Then

$$|g(z)| = |f(z)| \quad \text{for} \quad |z| = 1$$

and $f(z) \neq 0$ in $|z| < 1$. By Lemma 2 applied to the polynomial $g(z)$ with $R = k \geq 1$, it follows that for each $q > 0$,

$$\int_0^{2\pi} \left|g(ke^{i\theta})\right|^q \leq B_q^q \int_0^{2\pi} \left|g(e^{i\theta})\right|^q d\theta = B_q^q \int_0^{2\pi} \left|f(e^{i\theta})\right|^q d\theta, \tag{11}$$

where

$$B_q = \frac{\left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q d\theta\right\}^{1/q}}. \tag{12}$$

Combining (10) and (11), we get for each $q > 0$,

$$\begin{aligned} n(|\alpha| - k)&\left\{\int_0^{2\pi} \left|g(ke^{i\theta})\right|^q d\theta\right\}^{1/q} \\ &\leq kB_q \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|D_{\frac{\alpha}{k}} f(z)\right| \\ &= k\left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|D_{\frac{\alpha}{k}} f(z)\right|. \end{aligned} \tag{13}$$

Also,

$$g(z) = z^n \overline{f(1/\bar{z})} = z^n \overline{P(k/\bar{z})},$$

gives for $0 \leq \theta < 2\pi$,

$$\left|g(ke^{i\theta}\right| = \left|k^n e^{in\theta} \overline{P(e^{i\theta})}\right| = k^n \left|P(e^{i\theta})\right|.$$

Using this in (13), we get

$$nk^n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le k\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right|. \tag{14}$$

Again, noting that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right| = \max_{|z|=k}\left|D_\alpha P(z)\right|,$$

by Lemma 1 for $R=k\ge 1$, we have

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right| = \max_{|z|=k}\left|D_\alpha P(z)\right| \le k^{n-1}\max_{|z|=1}\left|D_\alpha P(z)\right| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1|, \tag{15}$$

if $n>2$ and

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right| = \max_{|z|=k}\left|D_\alpha P(z)\right| \le k\max_{|z|=1}\left|D_\alpha P(z)\right| - (k-1)|na_0+\alpha a_1|, \tag{16}$$

if $n=2$. Combining (14), (15) and (16), we immediately get the desired result. This completes the proof of Theorem 1.

The proof of Theorem 2 follows on the lines of proof of Theorem 2 of [11]. However, for the sake of completeness we present a proof.

PROOF OF THEOREM 2. Since $f(z)=P(kz)$ has all its zeros in $|z|\le 1$, therefore, applying the inequality (6) to the polynomial $f(z)$ (with $k=1$ and α replaced by α/k), we get for each $q>0, |\beta|\le 1$ and $|\alpha|\ge k$,

$$\begin{aligned} n\left(\frac{|\alpha|}{k}-1\right)&\left\{\int_0^{2\pi}\left|f(e^{i\theta})+\beta\min_{|z|=1}|f(z)|\right|^q d\theta\right\}^{1/q} \\ &\le \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right| - n\min_{|z|=1}|f(z)|\right\}. \end{aligned} \tag{17}$$

Also since

$$m=\min_{|z|=k}|P(z)| = \min_{|z|=1}|P(kz)| = \min_{|z|=1}|f(z)|,$$

therefore, from (17), we obtain for each $q>0, |\beta|\le 1$ and $|\alpha|\ge k$,

$$\begin{aligned} n\left(|\alpha|-k\right)&\left\{\int_0^{2\pi}\left|f(e^{i\theta})+\beta m\right|^q d\theta\right\}^{1/q} \\ &\le k\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}f(z)\right| - nm\right\}. \end{aligned} \tag{18}$$

Moreover, $f(z)=0$ in $|z|\le 1$ and

$$m\le |f(z)| \quad \text{for } |z|=1,$$

it follows by the maximum modulus theorem,

$$m|z|^n < |f(z)| \quad \text{for } |z| > 1. \tag{19}$$

We show all the zeros of polynomial $g(z) = f(z) + \beta m$ lie in $|z| \leq 1$ for every β with $|\beta| \leq 1$. This is obvious if $m = 0$, that is, if $f(z)$ has a zero on $|z| = 1$. Assume that $f(z)$ has no zero on $|z| = 1$ so that $m \neq 0$. If there is a point $z = z_0$ with $|z_0| > 1$ such that $g(z_0) = f(z_0) + \beta m = 0$, then we have

$$|f(z_0)| = |\beta| m < m|z_0|^n, \quad |z_0| > 1,$$

a contradiction to (19). Hence, the polynomial $g(z)$ has all its zeros in $|z| \leq 1$ and therefore, the polynomial $h(z) = z^n\overline{g(1/\bar{z})} \neq 0$ in $|z| < 1$. Applying Lemma 2 to the polynomial $h(z)$ with $R = k \geq 1$, it follows that for each $q > 0$,

$$\begin{aligned}\int_0^{2\pi} \left|h(ke^{i\theta})\right|^q d\theta \leq B_q^q \int_0^{2\pi} \left|h(e^{i\theta})\right|^q d\theta &= B_q^q \int_0^{2\pi} \left|g(e^{i\theta})\right|^q d\theta \\ &= B_q^q \int_0^{2\pi} \left|f(e^{i\theta}) + \beta m\right|^q d\theta \end{aligned} \tag{20}$$

where B_q is the same as given by (12). Using (18) in (20), we obtain for each $q > 0$,

$$\begin{aligned} n\left(|\alpha| - k\right) &\left\{\int_0^{2\pi} \left|h(ke^{i\theta})\right|^q d\theta\right\}^{1/q} \\ &\leq k \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \left\{\max_{|z|=1} \left|D_{\frac{\alpha}{k}} f(z)\right| - nm\right\}. \end{aligned} \tag{21}$$

But

$$h(z) = z^n\overline{g(1/\bar{z})} = z^n\overline{f(1/\bar{z})} + \bar{\beta} z^n m,$$

therefore, for $|z| = 1$, we get

$$|h(kz)| = \left|k^n z^n \overline{f(1/k\bar{z})} + \bar{\beta} z^n m k^n\right| = k^n \left|f(z/k) + \beta m\right| = k^n \left|P(z) + \beta m\right|. \tag{22}$$

From (15), (16), (21) and (22), we deduce after short simplication for each $q > 0, |\beta| \leq 1$ and $|\alpha| \geq k$,

$$\begin{aligned} n\left(|\alpha| - k\right) &\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta m\right|^q d\theta\right\}^{1/q} \\ &\leq \left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \left\{\max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1}\right\} \\ &\quad - \phi(k) \left|na_0 + \alpha a_1\right|. \end{aligned}$$

This proves Theorem 2.

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