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A Note On A Question Of Lü, Li And Yang*

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Abstract

In this paper we consider the situation when a power of a transcendental meromorphic function shares non-zero polynomials with the derivative of the product of it's shifts and obtain two results. Mainly in this paper we try to solve an open problem posed by Lü, Li, and Yang [10]. Also we exhibit some examples to fortify some conditions of our results.

1 Introduction, Definitions and Results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $|z|<r$, the poles are counted according to their multiplicities. The quantity

$$N(r,\infty;f)=\int\limits_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

is called the integrated counting function or simply the counting function of poles of f. Also

$$m(r,\infty;f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(re^{i\theta})|d\theta$$

is called the proximity function of poles of f, where $\log^+ x=\log x$, if $x\geq 1$ and $\log^+ x=0$, if $0\leq x<1$. The sum $T(r,f)=m(r,\infty;f)+N(r,\infty;f)$ is called the Nevanlinna characteristic function of f. We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to\infty$ except possibly a set of finite linear measure. For $a\in\mathbb{C}$, we put

$$N(r,a;f)=N(r,\infty;\frac{1}{f-a}) \text{ and } m(r,a;f)=m(r,\infty;\frac{1}{f-a}).$$

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Let us denote by $\overline{n}(r, a; f)$ the number of distinct a points of f lying in $|z| < r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} dt + \overline{n}(0, a; f) \log r$$

denotes the reduced counting function of a points of f (see, e.g., [6, 16]). The order of f is defined by

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

Let $k \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{k)}(r, a; f)$ to denote the counting function of a-points of f with multiplicity not greater than k, $N_{(k+1}(r, a; f)$ to denote the counting function of a-points of f with multiplicity greater than k. Similarly $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k+1}(r, a; f)$ are their reduced functions respectively. A meromorphic function a is said to be a small function of f provided that $T(r, a) = S(r, f)$, i.e., $T(r, a) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Rubel and Yang appear to be the first to study the entire functions that share values with their derivatives. In 1977 they proved the following well-known theorem.

THEOREM A ([13]). Let a and b be complex numbers such that $b \neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share a and b CM, then $f \equiv f'$.

From then on, this result has undergone various extensions and improvements (see [16]). In 1980, Gundersen improved Theorem A and obtained the following result.

THEOREM B ([4]). Let f be a non-constant meromorphic function, a and b be two distinct finite values. If f and f' share a and b CM, then $f \equiv f'$.

Mues and Steinmetz [12] generalized Theorem A from sharing values CM to IM and obtained the following result.

THEOREM C ([12]). Let a and b be complex numbers such that $b \neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share a and b IM, then $f \equiv f'$.

In 1996, Brück [1] studied the relation between f and f' if an entire function f shares only one finite value CM with it's derivative f'. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

CONJECTURE A. Let f be a non-constant entire function. Suppose

$$\rho_1(f) := \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a}=c \tag{1}$$

for some $c\in\mathbb{C}\setminus\{0\}$.

The Conjecture for the case $a=0$ and $N(r,0;f')=S(r,f)$ had been proved by Brück [1]. From the differential equations

$$\frac{f'-a}{f-a}=e^{z^n},\quad \frac{f'-a}{f-a}=e^{e^z}, \tag{2}$$

we see that when $\rho_1(f)$ is a positive integer or infinite, the conjecture does not hold. The conjecture for the case that f is of finite order had been proved by Gundersen and Yang [5], the case that f is of infinite order with $\rho_1(f)<\frac{1}{2}$ had been proved by Chen and Shon [2]. But the case $\rho_1(f)\geq\frac{1}{2}$ is still open. However, the corresponding conjecture for meromorphic functions fails in general (see [5]). For example if

$$f(z)=\frac{2e^z+z+1}{e^z+1},$$

then f and f' share the value 1 CM, but (1) does not hold.

It is interesting to ask what happens if f is replaced by f^n in the Brück conjecture. From (2) we see that the conjecture does not hold when $n=1$. Thus we only need to discuss the problem when $n\geq 2$. To the knowledge of authors perhaps Yang and Zhang [15] were the first to consider the uniqueness of a power of an entire function $F=f^n$ and its derivative F' when they share certain value as this type of considerations gives most specific form of the function. Yang and Zhang [15] proved that the Brück conjecture holds for the function f^n and the order restriction on f does not needed if n is relatively large. Actually they proved the following result.

THEOREM D ([15]). Let f be a non-constant entire function, $n\in\mathbb{N}$ with $n\geq 7$ and let $F=f^n$. If F and F' share 1 CM, then $F\equiv F'$ and $f(z)=ce^{\frac{1}{n}z}$, where $c\in\mathbb{C}\setminus\{0\}$.

Improving all the results obtained in [15], Zhang [18] proved the following theorem.

THEOREM E ([18]). Let f be a non-constant entire function, $n,k\in\mathbb{N}$ and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose f^n-a and $(f^n)^{(k)}-a$ share 0 CM and $n>k+4$, then $f^n\equiv(f^n)^{(k)}$ and $f(z)=ce^{\frac{\lambda}{n}z}$, where $c\in\mathbb{C}\setminus\{0\}$ and $\lambda^k=1$.

In 2009, Zhang and Yang [19] further improved the above result in the following manner.

THEOREM F ([19]). Let f be a non-constant entire function, $n,k\in\mathbb{N}$ and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose f^n-a and $(f^n)^{(k)}-a$ share 0 CM and $n>k+1$. Then conclusion of Theorem E holds.

In 2010, Zhang and Yang [20] further improved the above result in the following manner.

THEOREM G ([20]). Let f be a non-constant entire function, $n, k \in \mathbb{N}$. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$. Then conclusion of Theorem E holds.

In 2011, Lü and Yi [9] proved the following theorem by using the theory of normal families.

THEOREM H ([9]). Let f be a transcendental entire function, $n, k \in \mathbb{N}$ with $n \geq k+1$, $F = f^n$ and $Q \not\equiv 0$ be a polynomial. If $F - Q$ and $F^{(k)} - Q$ share 0 CM, then $F \equiv F^{(k)}$ and $f(z) = ce^{wz/n}$, where $c, w \in \mathbb{C} \setminus \{0\}$ such that $w^k = 1$.

REMARK 1. By the following example, it is easy to see that the hypothesis of the transcendental of f in Theorem H is necessary.

EXAMPLE 1 ([9]). Let $f(z) = z$ and $n = 2$, $k = 1$. Then

$$\frac{(f^2)' - Q}{f^2 - Q} = 2$$

and $(f^2)' - Q$, $f^2 - Q$ share 0 CM, but $(f^2)'^2$, where $Q(z) = 2z^2 - 2z$.

REMARK 2. It is easy to see that the condition $n \geq k+1$ in Theorem H is sharp by the following example.

EXAMPLE 2. Let $f(z) = e^{e^z} \int\limits_0^z e^{-e^t}(1 - e^t)t \, dt$ and $n = 1$, $k = 1$. Then

$$\frac{f'(z) - z}{f(z) - z} = e^z$$

and $f'(z) - z$, $f(z) - z$ share 0 CM, but $f' \not\equiv f$.

Now observing the above theorem Lü, Li and Yang [10] asked the following question.

QUESTION 1. What can be said "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM" where Q_1 and Q_2 are polynomials, and $Q_1 Q_2 \not\equiv 0$?

In [10] Lü, Li and Yang solved the above question for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{3}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following results.

THEOREM I ([10]). Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (3), $n \in \mathbb{N}$ with $n \geq 2$, then $\frac{Q_1}{Q_2}$ is a polynomial and $f' \equiv \frac{Q_1}{nQ_2} f$.

THEOREM J ([10]). Let f be a transcendental entire function, $n \in \mathbb{N}$ with $n \geq 2$. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, where $Q \not\equiv 0$ is a polynomial, then $f(z) = ce^{z/n}$, where $c \in \mathbb{C} \setminus \{0\}$.

Also at the end of the paper Lü, Li and Yang [10] posed the following conjecture.

CONJECTURE B. Let f be a transcendental entire function, $n \in \mathbb{N}$. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$, where Q_1, Q_2 are non-zero polynomials. Further, if $Q_1 \equiv Q_2$, then $f(z) = ce^{wz/n}$, where $c, w \in \mathbb{C} \setminus \{0\}$ such that $w^k = 1$.

Again in the same paper Lü, Li and Yang [10] asked the following question.

QUESTION 2. What can be said if the condition in the Conjecture B "$(f^n)^{(k)}$" be replaced by "$\big(f(z+c_1)f(z+c_2)\ldots f(z+c_n)\big)^{(k)}$" where $c_j \in \mathbb{C} (j = 1, 2, \ldots, n)$.

In 2015, present first author [11] proved that the Conjecture B is true and obtained the following result.

THEOREM K ([11]). Let f be a transcendental entire function, $n, k \in \mathbb{N}$. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$, where Q_1, Q_2 are non-zero polynomials. Further, if $Q_1 \equiv Q_2$, then $f(z) = ce^{\frac{\lambda}{n} z}$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$.

Now taking the possible answer of Question 2 into background we obtain the following results.

THEOREM 1. Let $f(z)$ be a transcendental meromorphic function of finite order with finitely many poles and $n, k \in \mathbb{N}$. Suppose $f^n(z) - Q_1(z)$, $(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)} - Q_2(z)$ share 0 IM and $f^n(z)$, $f(z+c_1)f(z+c_2)\ldots f(z+c_n)$ share 0 CM, where $c_j \in \mathbb{C}\setminus\{0\} (j = 1, 2, \ldots, n)$ and Q_1, Q_2 are two polynomials with $Q_1 Q_2 \not\equiv 0$. If $n \geq k+2$, then $(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$. Furthermore, if $Q_1 \equiv Q_2$, then $f(z) = d\, e^{\frac{\lambda}{n} z}$, where $d, \lambda \in \mathbb{C} \setminus \{0\}$ such that $e^{\frac{\lambda}{n}(c_1 + c_2 + \ldots + c_n)} = 1$ and $\lambda^k = 1$.

THEOREM 2. Let $f(z)$ be a transcendental meromorphic function of finite order with finitely many poles and $n, k \in \mathbb{N}$ such that $n \geq k$. Suppose $f^n(z) - Q_1(z)$, $(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)} - Q_2(z)$ share 0 IM and $f^n(z)$, $f(z+c_1)f(z+c_2)\ldots f(z+c_n)$ share 0 CM, where $c_j \in \mathbb{C} \setminus \{0\} (j = 1, 2, \ldots, n)$ and Q_1, Q_2 are two polynomials with $Q_1 Q_2 \not\equiv 0$. If $\overline{N}_{2)}(r, 0; f) = S(r, f)$, then $(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$. Furthermore, if $Q_1 \equiv Q_2$, then $f(z) = d\, e^{\frac{\lambda}{n} z}$, where $d, \lambda \in \mathbb{C} \setminus \{0\}$ such that $e^{\frac{\lambda}{n}(c_1 + c_2 + \ldots + c_n)} = 1$ and $\lambda^k = 1$.

NOTE 1. If $k \geq 2$, then in THEOREM 2 instead of $\overline{N}_{2)}(r, 0; f) = S(r, f)$ we can assume $N_{1)}(r, 0; f) = S(r, f)$.

REMARK 3. It is easy to see that the conditions "$f^n(z)$ and $f(z+c_1)f(z+c_2)\dots f(z+c_n)$ share 0 CM" in THEOREM 1 is sharp by the following example.

EXAMPLE 3. Let $f(z) = e^{cz}+1$, $c \in \mathbb{C}\setminus\{0\}$ and $e^{cc_1}, e^{cc_2}, e^{cc_3}$ are the roots of the equation $6z^3 - 18z^2 + 9z - 2 = 0$. Clearly $f^3(z)$ and $f(z+c_1)f(z+c_2)f(z+c_3)$ do not share 0 CM. Also $f^3(z) - Q_1(z)$ and $\big(f(z+c_1)f(z+c_2)f(z+c_3)\big)' - Q_2(z)$ share 0 CM, but $\big(f(z+c_1)f(z+c_2)f(z+c_3)\big)' \not\equiv \frac{Q_2(z)}{Q_1(z)} f^2(z)$, where $Q_1(z) = -2$ and $Q_2(z) = c$.

REMARK 4. It is easy to see that the conditions "$f^n(z)$ and $f(z+c_1)f(z+c_2)\dots f(z+c_n)$ share 0 CM" and "$\overline{N}_{2)}(r, 0; f) = S(r, f)$" in THEOREM 2 are sharp by the following examples.

EXAMPLE 4. Let $f(z) = (e^z+1)^2$ and $e^c = \frac{1}{4}$. Clearly $f(z)$ and $f(z+c)$ do not share 0 CM. Also $f(z) - Q_1(z)$ and $(f(z+c))' - Q_2(z)$ share 0 IM and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$, but $(f(z+c))' \not\equiv \frac{Q_2(z)}{Q_1(z)} f(z)$, where $Q_1(z) = 1$ and $Q_2(z) = -\frac{1}{2}$.

EXAMPLE 5. Let $f(z) = (e^z+1)^2$ and $e^c = \frac{1}{2}$. Clearly $f(z)$ and $f(z+c)$ do not share 0 CM. Also $f(z) - Q_1(z)$ and $(f(z+c))' - Q_2(z)$ share 0 CM and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$, but $(f(z+c))' \not\equiv \frac{Q_2(z)}{Q_1(z)} f(z)$, where $Q_1(z) = 3$ and $Q_2(z) = 1$.

EXAMPLE 6. Let $f(z) = e^{\frac{1}{2}z}+1$, $e^{\frac{1}{2}c_1} = 1$ and $e^{\frac{1}{2}c_2} = \frac{5}{3}$. Clearly $f^2(z)$ and $f(z+c_1)f(z+c_2)$ do not share 0 CM and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$. Also $f^2(z) - Q_1(z)$ and $\big(f(z+c_1)f(z+c_2)\big)' - Q_2(z)$ share 0 IM, but $\big(f(z+c_1)f(z+c_2)\big)' \not\equiv \frac{Q_2(z)}{Q_1(z)} f^2(z)$, where $Q_1(z) = 1$ and $Q_2(z) = -\frac{4}{3}$.

EXAMPLE 7. Let $f(z) = e^z - e^{-z}$, $e^{c_1} = -1$ and $e^{c_2} = i$. Clearly $f^2(z)$ and $f(z+c_1)f(z+c_2)$ do not share 0 CM and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$. Also $f^2(z) - Q_1(z)$ and $\big(f(z+c_1)f(z+c_2)\big)' - Q_2(z)$ share 0 CM, but $\big(f(z+c_1)f(z+c_2)\big)' \not\equiv \frac{Q_2(z)}{Q_1(z)} f^2(z)$, where $Q_1(z) = 2$ and $Q_2(z) = -8i$.

EXAMPLE 8. Let $f(z) = e^{2z}+1$, $c_1 = \pi i$. Then $f(z)$ and $f(z+c_1)$ share 0 CM and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$. Also $f(z) - Q_1(z)$ and $f'(z+c) - Q_2(z)$ share 0 CM, but $f'(z+c) \not\equiv \frac{Q_2(z)}{Q_1(z)} f(z)$, where $Q_1(z) = 3$ and $Q_2(z) = 4$.

REMARK 5. It is natural to ask whether THEOREM 2 holds if $f(z)$ has infinitely many poles. The answer is negative. We give the following.

EXAMPLE 9. Let $f(z) = \frac{2}{1-e^{-2z}}$ and $e^{-2c} = -2$. Clearly $f'(z) = -\frac{4e^{-2z}}{(1-e^{-2z})^2}$. Note that $f(z) - 4 = \frac{2(-1+e^{-2z})}{1-e^{-2z}}$ and $f'(z+c) - 1 = -\frac{(e^{-2z}-1)^2}{(1+2e^{-2z})^2}$. clearly $f(z) - 4$ and $f'(z+c) - 1$ share 0 IM and $\overline{N}_{2)}(r, 0; f) = 0$, but $f'(z+c) \not\equiv \frac{1}{4} f(z)$.

EXAMPLE 10. Let $f(z) = \frac{1}{2}\frac{e^{8z}}{e^{8z}+1}$ and $e^{8c} = -\frac{1}{2}$. Then $f(z)$ and $f'(z+c)$ share the value 1 IM and $N_{2)}(r,0;f) = 0$, but $f(z) \not\equiv f'(z+c)$.

We now explain following definition and notation which will be used in the paper.

DEFINITION 1. For $a \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + \ldots + \overline{N}_{(p}(r,a;f)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

2 Lemmas

In this section we present the lemmas which will be needed in the sequel.

LEMMA 1 ([14]). Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic small functions. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

LEMMA 2 ([3]). Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have

$$m\Big(r,\infty;\frac{f(z+c)}{f(z)}\Big) + m\Big(r,\infty;\frac{f(z)}{f(z+c)}\Big) = O(r^{\sigma-1+\varepsilon}).$$

LEMMA 3 ([7]). Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$N(r,0;f(z+c)) \le N(r,0;f(z)) + S(r,f), \quad N(r,\infty;f(z+c)) \le N(r,\infty;f) + S(r,f),$$

$$\overline{N}(r,0;f(z+c)) \le \overline{N}(r,0;f(z)) + S(r,f), \quad \overline{N}(r,\infty;f(z+c)) \le \overline{N}(r,\infty;f) + S(r,f).$$

LEMMA 4 ([6], Lemma 3.5). Suppose that F is meromorphic in a domain D and set $f = \frac{F'}{F}$. Then for $n \ge 1$,

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f'^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \le 3$ and has degree $n-3$ when $n > 3$.

3 Proofs of the Theorems

PROOF OF THEOREM 1. Let $F_1(z) = \frac{f^n(z)}{Q_1(z)}$ and $G_1(z) = \frac{(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)}}{Q_2(z)}$. Clearly F_1 and G_1 share 1 IM except for the zeros of $Q_i(z)$, where $i = 1, 2$ and so $\overline{N}(r, 1; F_1) = \overline{N}(r, 1; G_1) + S(r, f)$. Let

$$F(z) = f^n(z) \quad and \quad G(z) = f(z+c_1)f(z+c_2)\ldots f(z+c_n).$$

Therefore by LEMMA 1 we have $S(r, F) = S(r, f)$. Also by LEMMA 2 we have

$$\begin{aligned} T(r, G) = N(r, \infty; G) + m(r, \infty; G) &\leq O(\log r) + m\big(r, \infty; \frac{G}{F}\big) + m(r, \infty; F) \\ &\leq m(r, \infty; F) + S(r, f) = T(r, F) + S(r, F). \end{aligned}$$

Similarly we have $T(r, F) \leq T(r, G) + S(r, G)$. Therefore $S(r, G) = S(r, F) = S(r, f)$. Consequently by LEMMA 2 we get $m\big(r, \infty; \frac{G^{(k)}}{F}\big) = S(r, f)$. Note that $N(r, \infty; G^{(k)}) = N(r, \infty; G) + k\,\overline{N}(r, \infty; G)$ and so by LEMMA 3 we have $N(r, \infty; G^{(k)}) = O(\log r)$. Also we have $N(r, \infty; F) = O(\log r)$. We now consider following two cases.

Case 1. Let $F_1 \not\equiv G_1$. Note that

$$\begin{aligned} \overline{N}(r, 1; F_1) &\leq \overline{N}\big(r, 1; \frac{G_1}{F_1}\big) + S(r, f) \\ &\leq T\big(r, \frac{G_1}{F_1}\big) + S(r, f) \\ &\leq N\big(r, \infty; \frac{G_1}{F_1}\big) + m\big(r, \infty; \frac{G_1}{F_1}\big) + S(r, f) \\ &= N\Big(r, \infty; \frac{Q_1}{Q_2}\frac{G^{(k)}}{F}\Big) + m\Big(r, \infty; \frac{Q_1}{Q_2}\frac{G^{(k)}}{F}\Big) + S(r, f) \\ &\leq N(r, \infty; G^{(k)}) + N(r, \infty; F) + N_k(r, 0; f^n) + S(r, f) \\ &\leq k\,\overline{N}(r, 0; f) + S(r, f). \end{aligned} \tag{4}$$

Now using (4) and LEMMA 1 we get from the second fundamental theorem that

$$\begin{aligned} n\,T(r, f) &= T(r, F) + S(r, f) \\ &\leq T(r, F_1) + S(r, f) \\ &\leq \overline{N}(r, \infty; F_1) + \overline{N}(r, 0; F_1) + \overline{N}(r, 1; F_1) + S(r, F) \\ &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f^n) + \overline{N}(r, 1; F_1) + S(r, f) \\ &\leq (k+1)\,\overline{N}(r, 0; f) + S(r, f) \\ &\leq (k+1)\,T(r, f) + S(r, f). \end{aligned} \tag{5}$$

Since $n > k + 1$, (5) leads to a contradiction.

Case 2. $F_1 \equiv G_1$. Then

$$(f(z+c_1)f(z+c_2)\ldots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z).$$

Furthermore if $Q_1 \equiv Q_2$, then we have

$$f^n(z) \equiv (G(z))^{(k)}. \tag{6}$$

Let z_1 be a zero of $f(z)$ of multiplicity t. Then z_1 is a zero of $f^n(z)$ of multiplicity nt. Since $f^n(z)$ and $G(z)$ share 0 CM, it follows that z_1 must be a zero of $G(z)$ of multiplicity nt. Consequently z_1 will be a zero of $(G(z))^{(k)}$ of multiplicities $nt-k$. Therefore from (6), we arrive at a contradiction. As a result we have $f(z) \neq 0$, $G(z) \neq 0$ and $(G(z))^{(k)} \neq 0$. Since $f(z)$ is a transcendental meromorphic function with finitely many poles and $f(z) \neq 0$, $f(z)$ must take the form $f(z) = \frac{1}{P_1(z)} e^{P_2(z)}$, where $P_1(z)$ is a non-zero polynomial and $P_2(z)$ is a non-constant polynomial. Therefore $G(z) = \frac{1}{P_3(z)} e^{P_4(z)}$, where $P_3(z) = \prod\limits_{i=1}^{n} P_1(z+c_i)$ and $P_4(z) = \sum\limits_{i=1}^{n} P_2(z+c_i)$. Let

$$g(z) = \frac{G'(z)}{G(z)} = P_4'(z) - \frac{P_3'(z)}{P_3(z)}.$$

Therefore by LEMMA 4 we have

$$\frac{G^{(k)}}{G} = g^k + Q_{k-1}(g),$$

where $Q_{k-1}(g)$ is a polynomial of degree $k-1$ in g and its derivatives.
If P_4' is not a constant, we see that $\frac{G^{(k)}}{G} \sim g^k \sim (P_4'^k \to \infty$ as $z \to \infty$. We know that every non-constant rational function assumes every value in the closed complex plane. Consequently $\frac{G^{(k)}}{G} = 0$ somewhere in the open complex plane, i.e. $G^{(k)} = 0$ somewhere in the open complex plane, which is a contradiction.
Next we suppose $P_4' = \lambda \neq 0$. If $g(z)$ is non-constant, then we see that $g(z) = \lambda$, $g' = g'' = \ldots = 0$ at ∞. Also we observe that $\frac{G^{(k)}}{G} = \lambda^k$ at ∞. Again $\frac{G^{(k)}}{G}$ and so $G^{(k)}$ must have a zero in the open complex plane, which is a contradiction. Consequently $g(z)$ is constant. Therefore if $P_4' \neq 0$, we must have $P_4' = \lambda = g(z)$ and so $G(z) = e^{\lambda z + d}$. Finally $f(z)$ assumes the form $f(z) = d\ e^{\frac{\lambda}{n} z}$, where $d \in \mathbb{C} \setminus \{0\}$, $e^{\frac{\lambda}{n}(c_1+c_2+\ldots+c_n)} = 1$ and $\lambda^k = 1$.

PROOF OF THEOREM 2. We omit the proof since the proof of Theorem 2 can be carried out in the line of proof of Theorem 1.

4 Concluding Remark and an Open Question

EXAMPLE 8 shows that THEOREM 1 does not hold for $n = k$. But we do not know whether THEOREM 1 holds for $n = k+1$. Finally we pose the following natural question.

QUESTION. Is THEOREM 1 true for $n = k+1$?

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