

# Extended Shift-Splitting Iteration Method For Nonsymmetric Generalized Saddle Point Problems\*

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## Abstract

Recently, the extended shift-splitting iteration method is proposed for the nonsingular symmetric saddle point problem [J. Comput. Appl. Math., 313(2017), 70–81]. In this paper, the extended shift-splitting iteration method is applied for solving nonsymmetric generalized saddle point problems. We prove the convergence and semi-convergence of the extended shift-splitting iteration method to solve the nonsingular and singular nonsymmetric generalized saddle point problems, respectively.

## 1 Introduction

In many areas of scientific computing and engineering applications, we need to solve the following large sparse nonsymmetric generalized saddle point problems

$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite,  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) is a rectangular matrix of rank  $r$ ,  $f \in \mathbb{R}^n$  and  $g \in \mathbb{R}^m$  are given vectors. Since the  $(1, 1)$  block matrix  $A$  of  $\mathcal{A}$  is nonsingular, the generalized saddle point matrix  $\mathcal{A}$  can be decomposed into

$$\mathcal{A} = \begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C + BA^{-1}B^T \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}. \quad (2)$$

It readily follows from the block decomposition (2) that the generalized saddle point matrix  $\mathcal{A}$  is nonsingular if and only if  $C + BA^{-1}B^T$  is nonsingular. If  $B$  has full row rank, that is  $r = m$ , then the generalized saddle point matrix  $\mathcal{A}$  is nonsingular and the saddle point problem (1) has a unique solution (see [1]). However, in many applications, the matrix  $B$  is rank deficient, that is  $r < m$ . For such case,  $C + BA^{-1}B^T$  is invertible if and only if  $\text{null}(C) \cap \text{null}(B^T) = \{0\}$  ( $\text{null}(\cdot)$  denotes the null space of the corresponding matrix). Therefore, if  $\text{null}(C) \cap \text{null}(B^T) \neq \{0\}$ , then the generalized

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saddle point matrix  $\mathcal{A}$  is singular, at the moment, we call (1) a singular saddle point problem. Moreover, in such case, we suppose that the singular saddle point problem (1) is consistent, i.e.,  $b \in R(\mathcal{A})$ , the range of  $\mathcal{A}$ . For an overview of its applications, we refer to [4] and references therein.

In view of its property of sparsity, it may be attractive to use iteration methods (see [2,4] for a general introduction to the different solution methods). Recently, based on the shift-splitting (SS) iteration method [3], a class of the shift-splitting type methods received wide attention and obtained considerable achievements. For example, Cao et al. [8,9] applied the shift-splitting iteration method to solve nonsingular saddle point problems with symmetric and nonsymmetric positive definite matrix  $A$ , full row rank matrix  $B$  and zero matrix  $C$ . Shen and Shi generalized the shift-splitting iteration method and proposed the generalized shift-splitting (GSS) iteration method in [17]

$$\frac{1}{2} \begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I + C \end{bmatrix} \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I - C \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \quad (3)$$

where  $\alpha$  and  $\beta$  are two positive real parameters, and  $I$  is the identity matrix with appropriate dimension. Theoretical analysis in [17] showed that the GSS iteration method converges to the solution of the nonsymmetric generalized saddle point problem. Beik in [5] investigated the convergence of the MS method for nonsymmetric generalized saddle point problem. For the nonsingular generalized saddle point problems (1), where  $A$  is symmetric positive definite,  $B$  is of full row rank and  $C$  is symmetric positive semi-definite, Salkuyeh et al. [12] proved that the GSS iteration method is convergent unconditionally. Salkuyeh et al. in [13] considered the semi-convergence of the MGSS method for solving singular saddle point problems, where  $A$  is nonsymmetric positive definite,  $B$  is rank deficient and the matrix  $C$  is equal to zero. For a class of singular saddle point problems where  $A$  is symmetric positive definite,  $B$  is rank deficient and  $C = 0$ , Chen and Ma [14] proved that the GSS iteration method is semi-convergent to a solution if the singular linear system is consistent. For singular nonsymmetric saddle point problems where  $A$  is nonsymmetric positive definite,  $B$  is rank deficient and  $C = 0$ , the semi-convergence of the GSS iteration method was proved by Cao and Miao in [10], where a unified analysis was also presented for solving nonsingular nonsymmetric saddle point problems. Cao et al. in [11] studied a preconditioned generalized shift-splitting iteration method for solving saddle point problems and analyzed eigenvalue distribution of the preconditioned saddle point matrix, where  $A$  is symmetric positive definite,  $B$  is of full row rank and  $C = 0$ . Ren et al. in [15] investigated the eigenvalue distribution of the shift-splitting preconditioned saddle point matrix and showed that all eigenvalues having nonzero imaginary parts are located in an intersection of two circles and all real eigenvalues are located in a positive interval, where  $A$  is symmetric positive definite,  $B$  is of full row rank and  $C = 0$ . Shi et al. in [18] provided eigenvalue bounds for the nonzero eigenvalues of shift-splitting preconditioned singular nonsymmetric saddle point matrices, where  $A$  is nonsymmetric positive definite,  $B$  is a rectangular matrix and  $C = 0$ .

By further generalizing the GSS and the SS iteration methods, the extended shift-splitting (ESS) iteration method is proposed for solving the nonsingular symmetric saddle point problem in [19]. However, there is no discussion on the ESS iteration method for the nonsymmetric generalized saddle point problem with nonsymmetric

positive definite  $A$  and nonzero matrix  $C$ . In this paper, the ESS iteration method is further studied for both the singular and the nonsingular nonsymmetric generalized saddle point problems (1). It will be shown that the ESS iteration method is convergent unconditionally for solving the nonsingular nonsymmetric generalized saddle point problems and semi-convergent unconditionally for solving the singular nonsymmetric generalized saddle point problems. It should be noted that our results are quite general, including as particular cases almost all the results given in recent literatures.

The remainder of this paper is organized as follows. In Sections 2 and 3, the convergence and the semi-convergence of the ESS iteration method for solving the nonsingular and singular generalized saddle point problems are studied, respectively. Finally, we end this paper with a few concluding remarks in Section 4.

## 2 The ESS Iteration Method and Convergence Analysis

In this section, we will give the ESS iteration method and study its convergence for solving the nonsingular generalized saddle point problems (1), where  $A$  is nonsymmetric positive definite,  $B$  is of full row rank and  $C$  is symmetric positive semi-definite.

By introducing a block preconditioning matrix

$$\Omega = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},$$

where  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite matrices, we make the following ESS splitting for the generalized saddle point matrix  $\mathcal{A}$

$$\begin{aligned} \mathcal{A} &= \mathcal{P} - \mathcal{Q} = \frac{1}{2}(\Omega + \mathcal{A}) - \frac{1}{2}(\Omega - \mathcal{A}) \\ &= \frac{1}{2} \begin{bmatrix} P + A & B^T \\ -B & Q + C \end{bmatrix} - \frac{1}{2} \begin{bmatrix} P - A & -B^T \\ B & Q - C \end{bmatrix}. \end{aligned}$$

Then the ESS iteration method for solving generalized saddle point problem (1) is defined as follows:

**METHOD 1.** (ESS iteration method) Given an initial guess  $[x^{(0)}; y^{(0)}]$ , for  $k = 1, 2, \dots$ , until  $[x^{(k)}; y^{(k)}]$  converges, compute

$$\frac{1}{2} \begin{bmatrix} P + A & B^T \\ -B & Q + C \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P - A & -B^T \\ B & Q - C \end{bmatrix} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}. \quad (4)$$

The above ESS iteration can be written as follows

$$u^{(k+1)} = \Gamma u^{(k)} + c, k = 0, 1, \dots, \quad (5)$$

where

$$\Gamma = \mathcal{P}^{-1} \mathcal{Q} = (\Omega + \mathcal{A})^{-1} (\Omega - \mathcal{A}), \quad (6)$$

is the iteration matrix of the ESS iteration method and  $c = \mathcal{P}^{-1}b$ . The splitting preconditioner that corresponds to the ESS iteration (4) is given by

$$\mathcal{P} = \frac{1}{2} \begin{bmatrix} P + A & B^T \\ -B & Q + C \end{bmatrix}, \quad (7)$$

which is called the ESS preconditioner for the saddle point matrix  $\mathcal{A}$ .

In what follows, we will study the convergence of the ESS iteration method (4) or (5) for solving nonsingular generalized saddle point problem (1). Let  $\sigma(\cdot)$ ,  $\rho(\cdot)$  denote the spectral set and the spectral radius of a given matrix, respectively. Then the ESS iteration method (4) or (5) converges if and only if  $\rho(\Gamma) < 1$  ([6]).

To get the convergence property of the ESS iteration method, we first present a lemma, which describes the eigenvalue distribution of the generalized saddle point matrix  $\mathcal{A}$ .

LEMMA 1 ([7]). Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) has full row rank and  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite. Then the generalized saddle point matrix  $\mathcal{A}$  is positive stable, i.e.,  $Re(\lambda) > 0$  for all  $\lambda \in \sigma(\mathcal{A})$ .

THEOREM 1. Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) has full row rank and  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite,  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite matrices. Let the ESS iteration matrix  $\Gamma$  be defined as in (6). Then it holds that

$$\rho(\Gamma) < 1,$$

i.e., the ESS iteration method converges to the unique solution of the nonsingular nonsymmetric generalized saddle point problem (1).

PROOF. The iteration matrix  $\Gamma$  can be rewritten as

$$\Gamma = (\Omega + \mathcal{A})^{-1}(\Omega - \mathcal{A}) = (I + \Omega^{-1}\mathcal{A})^{-1}(I - \Omega^{-1}\mathcal{A}).$$

Let  $\lambda$  and  $\mu$  be an arbitrary eigenvalue of the matrix  $\Omega^{-1}\mathcal{A}$  and the iteration matrix  $\Gamma$ , respectively, then it holds that

$$\mu = \frac{1 - \lambda}{1 + \lambda}, \quad (8)$$

and

$$\mathcal{A}p = \lambda\Omega p, \quad (9)$$

where  $p$  is the eigenvector corresponding to the eigenvalue  $\lambda$ . The generalized eigenvalue problem (9) is equivalent to

$$\Omega^{-\frac{1}{2}}\mathcal{A}\Omega^{-\frac{1}{2}}\bar{p} = \lambda\bar{p} \quad \text{with} \quad \bar{p} = \Omega^{\frac{1}{2}}p.$$

By direct computation, we have

$$\Omega^{-\frac{1}{2}}\mathcal{A}\Omega^{-\frac{1}{2}} = \begin{pmatrix} P^{-\frac{1}{2}}AP^{-\frac{1}{2}} & P^{-\frac{1}{2}}B^TQ^{-\frac{1}{2}} \\ -Q^{-\frac{1}{2}}BP^{-\frac{1}{2}} & Q^{-\frac{1}{2}}CQ^{-\frac{1}{2}} \end{pmatrix}.$$

The matrix  $\Omega^{-\frac{1}{2}}\mathcal{A}\Omega^{-\frac{1}{2}}$  can also be regarded as a generalized saddle point matrix. Then by Lemma 1, we have  $Re(\lambda) > 0$ . Then from (8) and the fact that the function  $\lambda \rightarrow \mu = \frac{1-\lambda}{1+\lambda}$  maps the half-plane  $\{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$  into the open circle  $\{\mu \in \mathbb{C} : |\mu| < 1\}$ , we know that

$$\rho(\Gamma) < 1,$$

i.e., the ESS iteration method converges to the unique solution of the nonsingular nonsymmetric generalized saddle point problem (1).

Theorem 1 shows the unconditional convergent property of the ESS iteration method for solving the nonsingular generalized saddle point problem (1).

### 3 Semi-Convergence of the Extended Shift-Splitting Method

When the generalized saddle point matrix  $\mathcal{A}$  is nonsingular, from Section 2 we know that the ESS iteration scheme (4) or (5) converges to the exact solution of (1) for any initial vector unconditionally, that is  $\rho(\Gamma) < 1$  holds. But for the singular nonsymmetric generalized saddle point matrix  $\mathcal{A}$ , we have  $1 \in \sigma(\Gamma)$  and  $\rho(\Gamma) \geq 1$ . For such case, only the semi-convergence of the ESS iteration scheme (4) is required.

In this section, we investigate the semi-convergence analysis of the ESS iteration method. Let  $\lambda$  is an eigenvalue of the iteration matrix  $\Gamma$  and  $u = [x^*; y^*]$  be the corresponding eigenvector. Hence, we have  $Qu = \lambda Pu$  or equivalently

$$\begin{cases} (P - A)x - B^T y = \lambda(P + A)x + \lambda B^T y, \\ Bx + (Q - C)y = -\lambda Bx + \lambda(Q + C)y. \end{cases} \quad (10)$$

LEMMA 2. Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite and  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) is rank-deficient,  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite and  $\text{null}(C) \cap \text{null}(B^T) \neq \{0\}$ . If  $\lambda$  is an eigenvalue of the matrix  $\Gamma$ , then  $\lambda \neq -1$ .

PROOF. If  $\lambda = -1$ , from Eq. (10), we obtain  $u = 0$  which is a contradiction, because  $u \neq 0$  and  $\Omega$  is nonsingular.

LEMMA 3. Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite and  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) is rank-deficient,  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite and  $\text{null}(C) \cap \text{null}(B^T) \neq \{0\}$ . then  $\lambda = 1$  if and only if  $x = 0$  and  $Cy = 0$ .

PROOF. If  $\lambda = 1$ , from Eq. (10), we obtain

$$\begin{cases} -2Ax - 2B^T y = 0, \\ 2Bx - 2Cy = 0. \end{cases} \quad (11)$$

Multiplying both sides of the first equality of Eq. (11) by  $x^T$ , we gain  $x^T Ax + (Bx)^T y = x^T Ax + y^T Cy = 0$  that  $x^T Ax = 0$  and  $y^T Cy = 0$ . Since  $A$  is positive definite and  $C$  is positive semi-definite, this implies that  $x = 0$  and  $Cy = 0$ .

If  $x = 0$  and  $Cy = 0$ , from the second equality of (11) we obtain  $(Q - C)y = \lambda(Q + C)y$ , which yields  $\lambda = 1$  since  $y \neq 0$ .

Firstly, we give the following definition and lemma about semi-convergence properties of singular generalized saddle point problems:

**DEFINITION 1** ([16]). The iteration method (4) is semi-convergent for any initial guess  $[x_0^T; y_0^T]$ , if the iteration sequence  $[x_k^T; y_k^T]$  produced by (4) converges to a solution  $[x_*^T; y_*^T]$  of linear systems  $\mathcal{A}x = b$ . Moreover, it holds

$$\begin{pmatrix} x_* \\ y_* \end{pmatrix} = (I - \Gamma)^D c + (I - \gamma) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \text{with } \gamma = (I - \Gamma)(I - \Gamma)^D,$$

where  $I$  is the identity matrix and  $(I - \Gamma)^D$  denotes the Drazin inverse of  $I - \Gamma$ .

**LEMMA 4** ([16]). The iteration scheme (5) is semi-convergent if and only if the following two conditions are satisfied:

- (1). The elementary divisors of the iteration matrix  $\Gamma$  associated with  $\lambda = 1 \in \sigma(\Gamma)$  are linear, i.e.,  $\text{rank}(I - \Gamma)^2 = \text{rank}(I - \Gamma)$ , or equivalently, the index of matrix  $I - \Gamma$  is 1;
- (2). The pseudo-spectral radius satisfies

$$\vartheta(\Gamma) = \max\{|\lambda|, \lambda \in \sigma(\Gamma), \lambda \neq 1\} < 1.$$

In what follows, we investigate the two conditions in Lemma 2 for the ESS iteration method (4) or (5).

**THEOREM 2.** Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite and  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) is rank-deficient,  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite and  $\text{null}(C) \cap \text{null}(B^T) \neq \{0\}$ . Let  $\Gamma$  be the iteration matrix of the ESS iteration method, then  $\text{rank}(I - \Gamma)^2 = \text{rank}(I - \Gamma)$ .

**PROOF.** Since  $\Gamma = \mathcal{P}^{-1}\mathcal{Q} = I - \mathcal{P}^{-1}\mathcal{A}$ ,  $\text{rank}(I - \Gamma)^2 = \text{rank}(I - \Gamma)$  holds if

$$\text{null}((\mathcal{P}^{-1}\mathcal{A})^2) = \text{null}(\mathcal{P}^{-1}\mathcal{A}).$$

It is obvious that  $\text{null}((\mathcal{P}^{-1}\mathcal{A})^2) \supseteq \text{null}(\mathcal{P}^{-1}\mathcal{A})$ . Now, we only need to show

$$\text{null}((\mathcal{P}^{-1}\mathcal{A})^2) \subseteq \text{null}(\mathcal{P}^{-1}\mathcal{A}). \quad (12)$$

Without loss of generality, we assume that  $p = [p_1^T; p_2^T] \in \text{null}((\mathcal{P}^{-1}\mathcal{A})^2)$  with  $p_1 \in \mathbb{R}^n$ , and  $p_2 \in \mathbb{R}^m$ . Then, it must be satisfied  $(\mathcal{P}^{-1}\mathcal{A})^2 p = 0$ , which is equivalent to

$\mathcal{A}\mathcal{P}^{-1}\mathcal{A}p = 0$ . Assuming  $q = [q_1^T; q_2^T] = \mathcal{P}^{-1}\mathcal{A}p$ , with  $q_1 \in \mathbb{R}^n$ , and  $q_2 \in \mathbb{R}^m$ , we have  $\mathcal{A}q = 0$ . which is equivalent to

$$\begin{cases} Aq_1 + B^T q_2 = 0, \\ -Bq_1 + Cq_2 = 0. \end{cases} \quad (13)$$

Multiplying both sides of the first equation in (13) by  $q_1^T$ , we get  $q_1^T Aq_1 + q_1^T B^T q_2 = q_1^T Aq_1 + (Bq_1)^T q_2 = 0$ . Thus, using the second equality of (13), it follows from  $q_1^T Aq_1 + q_2^T Cq_2 = 0$  that  $q_1^T Aq_1 = 0$  and  $q_2^T Cq_2 = 0$ . Since  $A$  is positive definite and  $C$  is positive semidefinite, this implies that  $q_1 = 0$  and  $Cq_2 = 0$ . Substituting  $q_1 = 0$  into first relation of Eq. (13) gives  $B^T q_2 = 0$ .

On the other hand, from  $q = \mathcal{P}^{-1}\mathcal{A}p$ , we have  $\mathcal{P}q = \mathcal{A}p$  which equivalent to

$$\begin{bmatrix} P + A & B^T \\ -B & Q + C \end{bmatrix} \begin{bmatrix} 0 \\ q_2 \end{bmatrix} = 2 \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

which can be written as

$$\begin{cases} B^T q_2 = 2Ap_1 + 2B^T p_2, \\ (Q + C)q_2 = -2Bp_1 + 2Cp_2. \end{cases} \quad (14)$$

Solving  $p_1$  from the first equality of (14) and substituting it into the second equality of (14) gives

$$q_2 = 2(Q + Q^{-1}C)^{-1}(C + BA^{-1}B^T)p_2.$$

Owing to the positive definiteness of the matrix  $A^{-1}$  and  $q_2 \in \text{null}(C) \cap \text{null}(B^T)$ , we have  $(C + BA^{-1}B^T)^T q_2 = 0$  and

$$2p_2^T (C + BA^{-1}B^T)^T (Q + Q^{-1}C)^{-1} (C + BA^{-1}B^T) p_2 = 0,$$

which implies  $(C + BA^{-1}B^T)p_2 = 0$ . Therefore  $q_2 = 0$ .

In summary, we obtain  $q = [q_1^T; q_2^T] = 0$ . Thus, the proof is completed.

**THEOREM 3.** Assume that  $A \in \mathbb{R}^{n \times n}$  is nonsymmetric positive definite,  $C \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite and  $B \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) is rank-deficient,  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}^{m \times m}$  are symmetric positive definite and  $\text{null}(C) \cap \text{null}(B^T) \neq \{0\}$ . Let  $\vartheta(\Gamma)$  be the pseudo-spectral radius of the ESS iteration matrix  $\Gamma$ . Then it holds that

$$\vartheta(\Gamma) < 1.$$

**PROOF.** Without loss of generality let  $\|x\|_P^2 = x^* P x = 1$ . Multiplying both sides of the first equation in (10) by  $x^*$  yields

$$\frac{1 - \lambda}{1 + \lambda} = x^* A x + (Bx)^* y. \quad (15)$$

Also from the second equation in (10) we have

$$Bx = \omega Q y + C y, \quad \text{with } \omega = \frac{\lambda - 1}{\lambda + 1}. \quad (16)$$

Substituting Eq. (16) in (15) yields

$$\omega = -x^*Ax - \bar{\omega}y^*Qy - y^*Cy. \quad (17)$$

Therefore, we have  $\Re(\omega) = -\Re(x^*Ax) - \Re(\omega)y^*Qy - y^*Cy$ . Hence, we deduce that

$$\Re(\omega) = \frac{-\Re(x^*Ax) - y^*Cy}{1 + y^*Qy} \leq 0. \quad (18)$$

On the other hand, we have  $\omega = \frac{\lambda-1}{\lambda+1}$ , which is equivalent to

$$\lambda = \frac{1 + \omega}{1 - \omega} = \frac{1 + \Re(\omega) + i\Im(\omega)}{1 - \Re(\omega) - i\Im(\omega)}.$$

Then

$$|\lambda| = \sqrt{\frac{(1 + \Re(\omega))^2 + (\Im(\omega))^2}{(1 - \Re(\omega))^2 + (\Im(\omega))^2}}. \quad (19)$$

From Eqs. (18) and (19), we get  $|\lambda| \leq 1$ . To complete the proof we need to prove that if  $|\lambda| = 1$ , then  $\lambda = 1$ . If  $|\lambda| = 1$ , then it follows from Eq. (19) that  $\Re(\omega) = 0$ . This, together with Eq. (18) gives  $\Re(x^*Ax) = 0$  and  $y^*Cy = 0$ . Since  $A$  is nonsymmetric positive definite and  $C$  is symmetric positive semi-definite, it eventuates  $x = 0$  and  $Cy = 0$ . Therefore, from Lemma 3 we conclude that  $\lambda = 1$ . Thus, the proof is completed.

Theorems 2 and 3 show that the two conditions in Lemma 4 are satisfied naturally for the ESS iteration method (4) or (5). Thus, the ESS iteration method is semi-convergent unconditionally for solving the singular generalized saddle point problem (1).

## 4 Conclusion

In this paper, we discuss the ESS method for solving the nonsymmetric generalized saddle point problem (1). The convergence properties of the ESS method for nonsingular case and the semi-convergence property of the ESS method for singular case are studied.

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