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Generalized Order Statistics From Power Lomax Distribution And Characterization*

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Abstract

In this paper, recurrence relations for single and product moments of generalized order statistics (gos) from the Power Lomax (POLO) distribution have been established. These relations are deduced for moments of order statistics and upper record values. Further, this distribution has been characterized through the recurrence relations for moments of generalized order statistics.

1 Introduction

Kamps [6] has introduced the concept of generalized order statistics (gos) as a unifications of several models of ascendingly ordered random variables with different interpretations such as ordinary order statistics, record values, sequential order statistics, progressive type II censor order statistics and Pfeifer's records.

Let $X_1, X_2, ...X_n$ be a sequence of independent and identically distributed (iid) random variables (rv) with the cumulative distribution function (cdf) $F(x)$ and the probability density (pdf) $f(x)$. Let $n \in N,\ n \geq 2,\ k > 0,\ \widetilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=1}^{n-1} m_j$,such that $\gamma_i = k + n - i + M_r > 0$ for all $r \in (1, 2, ..., n-1)$. Then $X(r, n, \widetilde{m}, k)$, $r = 1, 2, ..., n$ are called (gos) if their joint pdf is given by

$$k\left(\prod_{j=1}^{n=1}\gamma_j\right)\left(\prod_{i=1}^{n=1}[1-F(x_i)]^{m_i}f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n),$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$ of $\Re^n$. Here $\overline{F}(x) = 1 - F(x)$. The pdf of r-th, m-gos is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)]f(x), \tag{1}$$

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and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, is given by

$$\begin{aligned} & f_{X(r,n,m,k),X(s,n,m,k)}(x,y) \\ = \; & \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}[F(x)][h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ & \times[\bar{F}(y)]^{\gamma_s - 1} f(x)f(y), \quad x < y, \end{aligned} \tag{2}$$

where

$$\overline{F}(x) = 1 - F(x), \quad \gamma_i = k + (n-i) + (m+1), \quad C_{s-1} = \prod_{i=1}^{s} \gamma_i$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & m \neq -1, \\ -\log(1-x) & m = -1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) \;\text{ for } x \in [0,1).$$

The probability density function (*pdf*) of the POLO distribution is defined as follows,

$$f(x) = \alpha\beta\lambda^{-1}x^{\beta-1}\left(1 + \frac{x^\beta}{\lambda}\right)^{-(\alpha+1)} \quad ; x \geq 0, \quad (\alpha, \beta, \lambda > 0). \tag{3}$$

The corresponding cumulative distribution function (*cdf*) of the POLO distribution is given by

$$F(x) = 1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha} \quad ; x \geq 0, \quad (\alpha, \beta, \lambda > 0). \tag{4}$$

The pdf of the POLO distribution $f(x)$ has the following properties:

(i) It is a unimodal if $(\alpha > 0, \beta > 1, \lambda > 0)$.

(ii) It is decreasing if $(\alpha > 0, \beta \leq 1, \lambda > 0)$.

(iii) At $\beta = 1$,the POLO distribution is reduced for Lomax distribution.

It can be easily noted that from (3) and (4),

$$\alpha\beta\overline{F}(x) = (x + \lambda x^{1-\beta})f(x). \tag{5}$$

The relation in (5) will be used to derive some recurrence relations for the moments of *gos* for the POLO distribution. Further probabilistic properties of this distribution and its applications are given, see for example, in (Rady et al. [5]).

Recurrence relations based on generalized order statistics have received considerable attention in recent years. Many authors have derived the recurrence relations for generalized order statistics for different distributions. See, Ahmad and Fawzy [3], AL-Hussaini et al. [4], Kumar and Khan [7], Khan et al. [8, 9] and Khan and Khan [10, 11] among others.

It is of interest to note that the recurrence relations for moments of *gos* have not been studied in context of the POLO distribution.

In this paper, some recurrence relations for single and product moments of *gos* for the POLO distribution have been established and provide special cases. Recurrence relation reduced the amount of direct computation and hence reduce the time and labour.

The contents of this paper are organised as follows. First, the recurrence relations for single moments is presented in Section 2 and its special cases for order statistic, upper record values are discussed. Recurrence relations for product moments and its reduced cases are presented in Section 3. Characterization result is given in Section 4. Finally, a conclusion is discussed in Section 5.

2 Relations for Single Moments

In this section, we present Theorem 1.

THEOREM 1. Let X be a non- negative continuous random variable that follows the POLO distribution given in (4). Suppose that $j>0$ and $1\leq r\leq n$. Then

$$\begin{aligned}&E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)]\\ =\ &\frac{j}{\alpha\beta\gamma_r}(E[X^j(r,n,m,k)]+\lambda E[X^{j-\beta}(r,n,m,k)]).\end{aligned}$$

PROOF. From (1), we have

$$E[X^j(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx.$$

Integrating by parts taking $[\overline{F}(x)]^{\gamma_r-1}f(x)$ as the part to be integrated, we get

$$\begin{aligned}E[X^j(r,n,m,k)]\ =\ &\frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]f(x)dx\\ &+\frac{\gamma_rC_{r-2}}{(r-1)!\gamma_r}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_{r-1}-1}g_m^{r-2}[F(x)]dx,\end{aligned}$$

which implies that

$$E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)]=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\overline{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx.$$

Now in view of equation (5), we have,

$$\begin{aligned}&E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)]\\ =\ &\frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)](x+\lambda x^{1-\beta})f(x)dx\end{aligned}$$

and

$$\begin{aligned}&E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)]\\ =\ &\frac{j}{\alpha\beta\gamma_r}\left(E[X^j(r,n,m,k)]+\lambda E[X^{j-\beta}(r,n,m,k)]\right).\end{aligned}\tag{6}$$

Hence the proof is complete.

REMARK 2.1. Setting $m=0$ and $k=1$ in (6), the result reduces for order statistic as follows

$$E[X^j_{r:n}]-E[X^j_{r-1:n}]=\frac{j}{\alpha\beta(n-r+1)}(E[X^j_{r:n}]+\lambda E[X^{j-\beta}_{r:n}]).$$

Putting $j=b$ and $r=a$ in Remark 2.1, confirms the result obtained by Abdul-Moniem [1].

REMARK 2.2. Setting $m=-1$ and $k\geq 1$ in (6), the result reduces for upper k^{th} record values as follows

$$\begin{aligned} & E[X^j(r,n,-1,k)]-E[X^j(r-1,n,-1,k)] \\ = \; & \frac{j}{\alpha\beta k}(E[X^j(r,n,-1,k)]+\lambda E[X^{j-\beta}(r,n,-1,k)]) \end{aligned}$$

Putting $j=s+1$ in Remark 2.2, our result agrees with that obtained by Abdul-Moniem [2].

REMARK 2.3. Recurrence relations for single moments of *gos* for the Lomax distribution can be easily obtained from (6) by using $\beta=1$.

3 Relation for Product Moments

In this section, we present Theorem 2.

THEOREM 2. Let X be a non- negative continuous random variable that follows the POLO distribution given in (3). Suppose that $i,j>0$ and $1\leq r<s\leq n$. then,

$$\begin{aligned} & E[X^i(r,n,m,k),X^j(s,n,m,k)]-E[X^i(r,n,m,k),X^j(s-1,n,m,k)] \\ = \; & \frac{j}{\alpha\beta\gamma_s}(E[X^i(r,n,m,k),X^j(s,n,m,k)] \\ & \qquad +\lambda E[X^i(r,n,m,k),X^{j-\beta}(s,n,m,k)]. \end{aligned}$$

PROOF. From (2), we have

$$\begin{aligned} & E[X^i(r,n,m,k),X^j(s,n,m,k)] \\ = \; & \frac{C_{r-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i[\overline{F}(x)]^m g_m^{r-1}[F(x)]f(x)I(x)dx, \end{aligned} \tag{7}$$

where

$$I(x)=\int_x^\infty y^j[\overline{F}(y)]^{\gamma_s-1}[h_m(F(y))-h_m(F(x))]^{s-r-1}f(y)dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (7), we get

$$\begin{aligned}
& E[X^i(r,n,m,k),X^j(s,n,m,k)]-E[X^i(r,n,m,k),X^j(s-1,n,m,k)] \\
= \quad & \frac{jC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[\overline{F}(x)]^m g_m^{r-1}[F(x)]f(x) \\
& \times[h_m(F(y))-h_m(F(x))]^{s-r-1}[y+\lambda y^{1-\beta}]f(y)[F(y)]^{\gamma_{s-1}}dydx,
\end{aligned}$$

$$\begin{aligned}
& E[X^i(r,n,m,k),X^j(s,n,m,k)]-E[X^i(r,n,m,k),X^j(s-1,n,m,k)] \\
= \quad & \frac{jC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^j[\overline{F}(x)]^m g_m^{r-1}[F(x)]f(x) \\
& \times[h_m(F(y))-h_m(F(x))]^{s-r-1}f(y)[F(y)]^{\gamma_{s-1}}dydx \\
& +\frac{j\lambda C_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-\beta}[\overline{F}(x)]^m g_m^{r-1}[F(x)]f(x) \\
& \times[h_m(F(y))-h_m(F(x))]^{s-r-1}f(y)[F(y)]^{\gamma_{s-1}}dydx
\end{aligned}$$

and

$$\begin{aligned}
& E[X^i(r,n,m,k),X^j(s,n,m,k)]-E[X^i(r,n,m,k),X^j(s-1,n,m,k)] \\
= \quad & \frac{j}{\alpha\beta\gamma_s}(E[X^i(r,n,m,k),X^j(s,n,m,k)] \\
& \qquad +\lambda E[X^i(r,n,m,k),X^{j-\beta}(s,n,m,k)]). \qquad (8)
\end{aligned}$$

The proof is complete.

REMARK 3.1. Setting $m=0$ and $k=1$ in (8), recurrence relations for product moments of order statistic from the POLO distribution follows

$$E[X^{ij}_{r,s:n}]-E[X^{ij}_{r,s-1:n}]=\frac{j}{\alpha\beta(n-s+1)}\left(E[X^{ij}_{r,s:n}]+\lambda E[X^{ij-\beta}_{r,s:n}]\right).$$

REMARK 3.2. Setting $m=-1$ and $k\geq 1$ in (8), recurrence relations for product moments of upper k^{th} record from the POLO distribution follows

$$E[X^{i,j}_{r,s,n,-1,k}]-E[X^{i,j}_{r,s-1,n,-1,k}]=\frac{j}{\alpha\beta k}\left(E[X^{i,j}_{r,s,n,-1,k}]+\lambda E[X^{i,j-\beta}_{r,sn,-1,k}\right).$$

Putting $j=s+1$ in Remark 3.2, our result agrees with that obtained by Abdul-Moniem [2]

REMARK 3.3. Putting $\beta=1$ in (8), recurrence relations for product moments of *gos* for the Lomax distribution is reduced.

4 Characterization

This section discusses the characterization result of the POLO distribution. Characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various method. In recent years, there has been a great interest in the characterizations of probability distributions through recurrence relations based on *gos*.

Theorem 3 is characterized based on the following result of Lin [12], which is given in proposition 1.

PROPOSITION 1. Let n_0 be any fixed non-negative integer and let a, b be real numbers such that $-\infty < a < b < \infty$. Let $g(x) \geq 0$ be an absolutely continuous function with $g'(x) \neq 0$ almost everywhere on (a, b). Then the sequence of functions $\{[g(x)]^n \exp^{-g(x)}, n \geq n_0\}$ is complete in $L(a, b)$ if and only if $g(x)$ is strictly monotone on (a, b).

THEOREM 3. The necessary and sufficient condition for a random variable to be distributed with (3) is that,

$$\begin{aligned} & E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] \\ = \; & \frac{j}{\alpha\beta\gamma_r}(E[X^j(r,n,m,k)] + \lambda E[X^{j-\beta}(r,n,m,k)]) \end{aligned} \tag{9}$$

if and only if

$$F(x) = 1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha} \qquad ; x \geq 0, \quad (\alpha, \beta, \lambda > 0).$$

PROOF. The necessary part follows immediately from (9). On the other hand if the recurrence relation (9) is satisfied, then on rearranging the terms in (9),

$$\begin{aligned} & \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx \\ & \qquad - \frac{C_{r-1}(r-1)}{(r-1)!\gamma_r}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r+m} g_m^{r-2}[F(x)]f(x)dx \\ = \; & \frac{j}{\alpha\beta}\frac{C_{r-1}}{\gamma_r(r-1)!}\left\{\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx \right. \\ & \qquad \left. + \lambda\int_0^\infty x^{j-\beta}[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx\right\} \end{aligned}$$

and

$$\begin{aligned} & \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)]f(x)\left[\frac{g_m[F(x)]}{[\overline{F}(x)]} - \frac{(r-1)[\overline{F}(x)^m]}{\gamma_r}\right]dx \\ = \; & \frac{j}{\alpha\beta}\frac{C_{r-1}}{\gamma_r(r-1)!}\left\{\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx \right. \\ & \qquad \left. + \lambda\int_0^\infty x^{j-\beta}[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx\right\}. \end{aligned}$$

Let

$$h(x) = -\frac{[\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]}{\gamma_r}. \tag{10}$$

Differentiating both sides of (10), we get

$$h^{'}(x) = [\overline{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)]f(x)\left\{\frac{g_m[F(x)]}{[\overline{F}(x)]} - \frac{(r-1)[\overline{F}(x)]^m}{\gamma_r}\right\}.$$

Thus,

$$\begin{aligned}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j h^{'}(x)dx \quad &= \quad \frac{j}{\alpha\beta}\frac{C_{r-1}}{\gamma_r(r-1)!}\left\{\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx\right.\\ &\left.+\lambda\int_0^\infty x^{j-\beta}[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx\right\}. \end{aligned} \tag{11}$$

Integrating LHS in (11) by parts and using the value of $h(x)$ from (10),

$$\begin{aligned}&\frac{C_{r-1}}{(r-1)!}\int_0^\infty jx^{j-1}[\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]f(x)dx\\ = \quad &\frac{j}{\alpha\beta}\frac{C_{r-1}}{\gamma_r-1(r-1)!}\left\{\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]f(x)dx\right.\\ &\left.+\lambda\int_0^\infty x^{j-\beta}[\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx\right\},\end{aligned}$$

which reduces to,

$$\frac{jC_{r-1}}{(r-1)!}\int_0^\infty [\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]\left[x^{j-1} - \frac{1}{\alpha\beta}(x^j+\lambda x^{j-\beta})\frac{f(x)}{\overline{F}(x)}\right]dx = 0.$$

It follows from the above proposition

$$\alpha\beta\overline{F}(x) = (x+\lambda x^{1-\beta})f(x),$$

which proves that $f(x)$ has the form as in (3).i.e,

$$F(x) = 1 - \left(1+\frac{x^\beta}{\lambda}\right)^{-\alpha} \qquad ; x \geq 0, \quad (\alpha, \beta, \lambda > 0).$$

5 Conclusion

Characterization of probability distribution plays an important role in probability and statistics. A particular probability distribution model is applied to fit the real data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various method. In this paper, characterization result based on recurrence relations of single moments of generalized order statistics for the POLO distribution has been established.

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References

[1] I. B. Abdul-Moniem, Order statistics from Power Lomax distribution, Int. J. Innov. Sci. Eng. and Tech., 4(2017), 1–4.

[2] I. B. Abdul-Moniem, K-th upper record values from Power Lomax distribution, Int. J. Math. Stats. Invention, 4(2016), 41–43.

[3] A. A. Ahmad, and M. Fawzy, Recurrence relations for single and product moments of generalized order statistics from doubly truncated distributions, J. Stat. Plann. Infer., 17(2002), 241–249.

[4] E. K. AL-Hussaini, A. A. Ahmad and M. A. AL-Kashif, Recurrence relations for moment and conditional moment generating functions of generalized order statistics, Metrika, 61(2005), 199–220.

[5] E. A. Rady, W. A. Hassanein, and T. A. Elhaddad, The Power Lomax distribution with an application to bladder cancer data, Springer Plus, 5(2016), 18-38.

[6] U. Kamps, A Concept of Generalized Order Statistics, B.G. Teubner Stuttgart, Germany, (1995).

[7] D. Kumar and M. I. Khan, Relations for generalized order statistics from doubly truncated generalized exponential distribution and its characterization, Math. Sci. Lett., 2(2013), 9–18.

[8] M. I. Khan, M. A. R. Khan, and M. A. Khan, Generalized order statistics from q- exponential type -1 distribution and its characterization, Int. J. Comp. Theo. Stat., 2(2015), 1–6.

[9] M. I. Khan, M. A. R. Khan, and M. A. Khan, Generalized order statistics from q-exponential type -II distribution and its characterization, J. Math. Comput. Sci., 5(2015), 507–516.

[10] M. I. Khan and M. A. R. Khan, Moments of generalized order statistics from new Weibull- Pareto distribution and its characterization, Int. J. Appl. Math., 29(2016), 641–653.

[11] M. I. Khan and M. A. R. Khan, Recurrence relations of moments of generalized order statistics in a compound Rayleigh distribution, Thailand Statistician, 15(2016), 11–16.

[12] G. D. Lin, On a moment problem, Tohoku Math. J., 38(1986), 595–598.