

# One-Iteration Reconstruction Algorithm For Geometric Inverse Problems\*

Maatoug Hassine<sup>†</sup>, Imen Kallel<sup>‡</sup>

Received 28 March 2017

## Abstract

In this work we focus on the detection of objects immersed in anisotropic media from boundary measurements. We propose a one-iteration algorithm based on the Kohn-Vogelius formulation and the topological gradient method. The inverse problem is formulated as a topology optimization one. A topological sensitivity analysis is derived for an energy like functional. The unknown object is reconstructed using a level-set curve of the topological gradient. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results.

## 1 Introduction

There exist many practical problems for which it is necessary to detect the electrical properties of a media from boundary measurements. This kind of studies was realized for the clinical applications such as electrical impedance tomography [10], the geophysical applications such as detection of the mineral deposits location in the earth [14], industrial applications such as non-destructive testing [9], ...etc.

In this work we focus on the detection of objects immersed in an anisotropic media from overdetermined boundary data. More precisely, let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  denote a regular and bounded domain with boundary  $\Gamma$ . Inside the domain  $\Omega$  we assume the existence of an object  $\mathcal{O} \subset \Omega$  with boundary  $\Sigma = \partial\mathcal{O}$ . The geometric inverse problem that we consider can be formulated as:

- Giving two boundary data on  $\Gamma$ ; an imposed flux  $\Phi$  and a measured data  $\psi_m$ .
- Find the unknown location of the object  $\mathcal{O}$  inside the domain  $\Omega$  such that the solution  $\psi$  of the anisotropic Laplace equation satisfies the following overdetermined boundary value problem

$$\begin{cases} -\operatorname{div}(\gamma(x)\nabla\psi) = F & \text{in } \Omega \setminus \overline{\mathcal{O}}, \\ \gamma(x)\nabla\psi \cdot \mathbf{n} = \Phi & \text{on } \Gamma, \\ \psi = \psi_m & \text{on } \Gamma, \\ \gamma(x)\nabla\psi \cdot \mathbf{n} = 0 & \text{on } \Sigma, \end{cases}$$

---

\*Mathematics Subject Classifications: 35N05, 65R32, 65K10.

<sup>†</sup>Department of Mathematics, FSM, Monastir University, Monastir 5019, Tunisia

<sup>‡</sup>FSM, Monastir University, Tunisia

where  $\gamma$  is a scalar smooth function (of class  $\mathcal{C}^1$ ) describing the physical properties of the medium  $\Omega$ ,  $F$  is a given source term. We assume that there exist two constants  $c_0 > 0$  and  $c_1 > 0$  such that  $c_0 \leq \gamma(x) \leq c_1, \forall x \in \Omega$ .

In this formulation the domain  $\Omega \setminus \overline{\mathcal{O}}$  is unknown since the free boundary  $\Sigma$  is unknown. This problem is ill-posed in the sense of Hadamard. The majority of works dealing with this kind of problems fall into the category of shape optimization and based on the shape differentiation techniques. It is proved in [7] that the studied inverse problems, treated as a shape optimization problems, are severely ill-posed (i.e. unstable), for both Dirichlet and Neumann conditions on the boundary  $\Sigma$ . Thus they have to use some regularization methods to solve them numerically.

To solve this inverse problem, we suggest an alternative approach based on the Kohn-Vogelius formulation [6] and the topological sensitivity analysis method [1, 2, 3, 8, 11, 12, 13]. We combine here the advantages of the Kohn-Vogelius formulation as a self regularization technique and the topological sensitivity as an accurate and fast method.

The rest of the paper is organized as follows. In Section 2, we formulate the considered inverse problem as a topological optimization one. This step is based on the Kohn-Vogelius formulation. In Section 3, we present the main theoretical results of our paper. In Section 4, we propose a one-iteration shape reconstruction algorithm. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results.

## 2 Formulation of the Inverse Problem

In this section, we give the main steps of our analysis. Firstly, we introduce the Kohn-Vogelius formulation and we define the cost function to be minimized. Secondly, we present the perturbed problems and we describe the quantity to be estimated.

### 2.1 The Kohn-Vogelius Formulation

The Kohn-Vogelius formulation rephrases the considered inverse problem into a topological optimization one. In fact, the Kohn-Vogelius formulation leads to define for any given domain  $\mathcal{O} \subset \Omega$  two forward problems. The first one is associated to the Neumann datum  $\Phi$ , it will be named the ‘‘Neumann problem’’:

$$(\mathcal{P}_N) \left\{ \begin{array}{l} \text{Find } \psi_N \in H^1(\Omega \setminus \overline{\mathcal{O}}) \text{ solving ,} \\ - \operatorname{div} (\gamma(x) \nabla \psi_N) = F \quad \text{in } \Omega \setminus \overline{\mathcal{O}}, \\ \gamma(x) \nabla \psi_N \cdot \mathbf{n} = \Phi \quad \text{on } \Gamma, \\ \gamma(x) \nabla \psi_N \cdot \mathbf{n} = 0 \quad \text{on } \Sigma. \end{array} \right.$$

The second one is associated to the Dirichlet datum (measured)  $\psi_m$

$$(\mathcal{P}_D) \left\{ \begin{array}{l} \text{Find } \psi_D \in H^1(\Omega \setminus \overline{\mathcal{O}}) \text{ solving ,} \\ - \operatorname{div} (\gamma(x) \nabla \psi_D) = F \quad \text{in } \Omega \setminus \overline{\mathcal{O}}, \\ \psi_D = \psi_m \quad \text{on } \Gamma, \\ \gamma(x) \nabla \psi_D \cdot \mathbf{n} = 0 \quad \text{on } \Sigma. \end{array} \right.$$

One can see that if  $\Sigma$  coincides with the actual boundary  $\Sigma^*$  then the misfit between the solutions vanishes,  $\psi_D = \psi_N$ . According to this observation, we propose an identification process based on the minimization of the following energy like functional [6]

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}) = \int_{\Omega \setminus \overline{\mathcal{O}}} \gamma(x) |\nabla \psi_D - \nabla \psi_N|^2 dx.$$

The inverse problem can be formulated as a topological optimization one

$$\min_{\mathcal{O} \subset \Omega} \mathcal{K}(\Omega \setminus \overline{\mathcal{O}}).$$

To solve this problem we will use the topological sensitivity analysis method.

## 2.2 The Sensitivity Analysis Method

This method consists in studying the variation of the function  $\mathcal{K}$  with respect to the presence of a small object inside the background domain  $\Omega$ .

To present the main idea of this method, we consider the case in which  $\Omega$  contains a small object  $\mathcal{O}_{z,\varepsilon}$  that is centred at  $z \in \Omega$  and has the shape  $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\omega \subset \Omega$ , where  $\varepsilon > 0$  and  $\omega \subset \mathbb{R}^d$  is a given, fixed and bounded domain containing the origin, whose boundary  $\partial\omega$  is of  $\mathcal{C}^1$ . The topological sensitivity analysis leads to an asymptotic expansion of the variation  $\mathcal{K}(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) - \mathcal{K}(\Omega)$  with respect to  $\varepsilon$ .

Using the Kohn-Vogelius formulation, one can define for each arbitrary location of  $\mathcal{O}_{z,\varepsilon}$  in the domain  $\Omega$ , two forward problems  $(\mathcal{P}_N^\varepsilon)$  and  $(\mathcal{P}_D^\varepsilon)$ :

$$\begin{aligned} (\mathcal{P}_N^\varepsilon) \left\{ \begin{array}{ll} \text{find } \psi_N^\varepsilon \in H^1(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}), & \text{such that} \\ -\operatorname{div}(\gamma(x)\nabla\psi_N^\varepsilon) = F & \text{in } \Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}, \\ \gamma(x)\nabla\psi_N^\varepsilon \mathbf{n} = \Phi & \text{on } \Gamma, \\ \gamma(x)\nabla\psi_N^\varepsilon \mathbf{n} = 0 & \text{on } \partial\mathcal{O}_{z,\varepsilon}, \end{array} \right. \\ (\mathcal{P}_D^\varepsilon) \left\{ \begin{array}{ll} \text{find } \psi_D^\varepsilon \in H^1(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}), & \text{such that} \\ -\operatorname{div}(\gamma(x)\nabla\psi_D^\varepsilon) = F & \text{in } \Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}, \\ \psi_D^\varepsilon = \psi_m & \text{on } \Gamma, \\ \gamma(x)\nabla\psi_D^\varepsilon \mathbf{n} = 0 & \text{on } \partial\mathcal{O}_{z,\varepsilon}. \end{array} \right. \end{aligned}$$

In order to describe the presence of the object  $\mathcal{O}_{z,\varepsilon}$  inside the domain  $\Omega$ , we will use the shape function

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}} \gamma(x) |\nabla\psi_D^\varepsilon - \nabla\psi_N^\varepsilon|^2 dx.$$

Next, we will derive a topological sensitivity analysis for the function  $\mathcal{K}$  with respect to the insertion of a small object  $\mathcal{O}_{z,\varepsilon}$  in  $\Omega$ . It leads to an asymptotic expansion of the form

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \rho(\varepsilon)\delta\mathcal{K}(z) + o(\rho(\varepsilon)), \quad \forall z \in \Omega,$$

where  $\varepsilon \mapsto \rho(\varepsilon)$  is a scalar positive function going to zero with  $\varepsilon$ . The function  $z \mapsto \delta\mathcal{K}(z)$  is called the topological gradient and play the role of leading term of the variation  $\mathcal{K}(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) - \mathcal{K}(\Omega)$ . In order to minimize the shape function  $\mathcal{K}$ , the best location  $z$  of the object  $\mathcal{O}_{z,\varepsilon}$  in  $\Omega$  is where  $\delta\mathcal{K}$  is most negative.

### 3 Main Results

This section is devoted to the main theoretical results of the paper. We start our analysis by estimating the perturbation caused by the presence of the small object  $\mathcal{O}_{z,\varepsilon}$ . The established estimates lead to a simplified mathematical analysis of the topological sensitivity. The topological asymptotic expansion for the anisotropic laplace operator is summarized in Theorem 1.

#### 3.1 Estimate of the Perturbed Solutions

In this paragraph, we establish two estimates describing the perturbation caused by the presence of the geometry modification  $\mathcal{O}_{z,\varepsilon}$  on the solutions of the Dirichlet and Neumann problems. To this end, we introduce two auxiliaries problems.

The first one is related to the Neumann problem:

$$\begin{cases} \text{Find } \varphi_N \in W^1(\mathbb{R}^d \setminus \bar{\omega}) \text{ such that ,} \\ -\Delta \varphi_N = 0 & \text{in } \mathbb{R}^d \setminus \bar{\omega}, \\ \varphi_N = 0 & \text{at } \infty, \\ \nabla \varphi_N \cdot \mathbf{n} = -\nabla \psi_N(z) \cdot \mathbf{n} & \text{on } \partial\omega. \end{cases} \quad (1)$$

The second one is related to the Dirichlet problem:

$$\begin{cases} \text{Find } \varphi_D \in W^1(\mathbb{R}^d \setminus \bar{\omega}) \text{ such that ,} \\ -\Delta \varphi_D = 0 & \text{in } \mathbb{R}^d \setminus \bar{\omega}, \\ \varphi_D = 0 & \text{at } \infty, \\ \nabla \varphi_D \cdot \mathbf{n} = -\nabla \psi_D(z) \cdot \mathbf{n} & \text{on } \partial\omega. \end{cases} \quad (2)$$

The functions  $\varphi_N$  and  $\varphi_D$  can be expressed by a single layer potential on  $\partial\omega$  on the following way

$$\varphi_N(y) = \int_{\partial\omega} E(y-x)\eta_N(x)ds(x), \quad \varphi_D(y) = \int_{\partial\omega} E(y-x)\eta_D(x)ds(x), \quad \forall y \in \mathbb{R}^d \setminus \bar{\omega},$$

where  $E$  is the fundamental solution of the Laplace problem in  $\mathbb{R}^d$ ;  $E(y) = -\frac{1}{2\pi} \log(|y|)$  if  $d = 2$ , and  $E(y) = \frac{1}{4\pi} \frac{1}{|y|}$  if  $d = 3$ .

Here  $\eta_N$  and  $\eta_D$  are the solutions to the following integral equations:

$$-\frac{\eta_N(y)}{2} + \int_{\partial\omega} \nabla E(y-x) \cdot \mathbf{n} \eta_N(x) ds(x) = -\nabla \psi_N^0(z) \cdot \mathbf{n}, \quad y \in \partial\omega, \quad (3)$$

$$-\frac{\eta_D(y)}{2} + \int_{\partial\omega} \nabla E(y-x) \cdot \mathbf{n} \eta_D(x) ds(x) = -\nabla \psi_D^0(z) \cdot \mathbf{n}, \quad y \in \partial\omega. \quad (4)$$

The perturbed solutions  $\psi_N^\varepsilon, \psi_D^\varepsilon \in H^{-1/2}(\partial\omega)$  satisfy the following estimates.

**PROPOSITION 1.** There exists positive constant  $c > 0$ , independent of  $\varepsilon$ , such that

$$\|\psi_N^\varepsilon - \psi_N^0 - \varepsilon \varphi_N((x-z)/\varepsilon)\|_{1,\Omega_{z,\varepsilon}} \leq c\varepsilon^{d/2},$$

$$\|\psi_D^\varepsilon - \psi_D^0 - \varepsilon\varphi_D((x-z)/\varepsilon)\|_{1,\Omega_{z,\varepsilon}} \leq c\varepsilon^{d/2}.$$

### 3.2 Asymptotic Expansion

In this section, we derive a topological asymptotic expansion for the Kohn-Vogelius function  $\mathcal{K}$ . The mathematical analysis is general and can be adapted for various partial differential equations.

To this end, we introduce the polarization matrix  $\mathcal{M}_\omega$ . Thanks to the linearity of the integral equations (3) and (4), there exists a  $d \times d$  matrix  $\mathcal{M}_\omega$  such that

$$\int_{\partial\omega} \eta_N(y)y^T ds(y) = \mathcal{M}_\omega \nabla \psi_N^0(z) \text{ and } \int_{\partial\omega} \eta_D(y)y^T ds(y) = \mathcal{M}_\omega \nabla \psi_D^0(z).$$

The matrix  $\mathcal{M}_\omega$  can be defined as (see [4] for similar work)

$$(\mathcal{M}_\omega)_{ij} = \int_{\partial\omega} \eta_i y_j ds(y), \quad 1 \leq i, j \leq d,$$

where  $y_j$  is the  $j$ th component of  $y \in \mathbb{R}^d$  and  $\eta_i$  is the solution to

$$-\frac{\eta_i(y)}{2} + \int_{\partial\omega} \nabla E(y-x) \cdot \mathbf{n} \eta_i(x) ds(x) = -e_i \cdot \mathbf{n}, \quad y \in \partial\omega$$

with  $\{e_i\}_{1 \leq i \leq d}$  is the canonical basis in  $\mathbb{R}^d$ .

The topological sensitivity analysis with respect to the presence of an arbitrary shaped object is described by the following Theorem.

**THEOREM 1.** Let  $\mathcal{O}_{z,\varepsilon}$  be an arbitrary shaped object inserted inside the background domain  $\Omega$ . The function  $\mathcal{K}$  admits the asymptotic expansion

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \varepsilon^d \delta\mathcal{K}(z) + o(\varepsilon^d),$$

with  $\delta\mathcal{K}$  is the topological gradient

$$\begin{aligned} \delta\mathcal{K}(x) &= \gamma(x) \{ \nabla \psi_N^0(x) \mathcal{M}_\omega \nabla \psi_N^0(z) - \nabla \psi_D^0(x) \mathcal{M}_\omega \nabla \psi_D^0(z) \} \\ &\quad + 2|\omega| F(x) (\psi_D^0(x) - \psi_N^0(x)), \quad \forall x \in \Omega. \end{aligned}$$

The polarization matrix  $\mathcal{M}_\omega$  can be determined analytically in some cases. Otherwise, it can be approximated numerically.

In particular, in the case of circular or spherical object (i.e.  $\omega = B(0,1)$ ), the matrix  $\mathcal{M}_\omega$  is given by

$$\mathcal{M}_\omega = 2\pi\mathcal{I} \text{ if } d = 2 \text{ or } d = 3,$$

where  $\mathcal{I}$  is the  $d \times d$  identity matrix.

**COROLLARY 1.** If  $\omega = B(0,1)$ , the function  $\mathcal{K}$  satisfies the following asymptotic expansion

$$\mathcal{K}(\Omega_{z,\varepsilon}) - \mathcal{K}(\Omega) = 2\pi \varepsilon^d \delta\mathcal{K}(z) + o(\varepsilon^d),$$

and the topological gradient  $\delta\mathcal{K}$  admits the expression

$$\delta\mathcal{K}(x) = \begin{cases} \gamma(x) \left( |\nabla\psi_N^0(x)|^2 - |\nabla\psi_D^0(x)|^2 \right) + F(x) (\psi_D^0(x) - \psi_N^0(x)), & \text{if } d = 2, \\ \gamma(x) \left( |\nabla\psi_N^0(x)|^2 - |\nabla\psi_D^0(x)|^2 \right) + \frac{4}{3}F(x) (\psi_D^0(x) - \psi_N^0(x)), & \text{if } d = 3. \end{cases}$$

## 4 Algorithm and Numerical Results

In this section we consider the bidimensional case and we present a fast and simple one-iteration identification algorithm. Our numerical procedure is based on the formula described by Corollary 1. The unknown object  $\mathcal{O}$  is identified using a level set curve of the topological gradient  $\delta\mathcal{K}$ . More precisely, the unknown object  $\mathcal{O}$  is likely to be located at zone where the topological gradient  $\delta\mathcal{K}$  is negative.

### One-iteration algorithm:

- Solve the two problems  $(\mathcal{P}_N^0)$  and  $(\mathcal{P}_D^0)$ .
- Compute the topological gradient  $\delta\mathcal{K}(x)$ ,  $x \in \Omega$ .
- Determine the unknown object

$$\mathcal{O} = \{x \in \Omega; \text{ such that } \delta\mathcal{K}(x) < c < 0\},$$

where  $c$  is a constant chosen in such a way that the cost function  $\mathcal{K}$  decreases as much as possible.

This one-iteration procedure has already been illustrated in [5] for the identification of cracks from overdetermined boundary data and in [8] for the detection of small gas bubbles in Stokes flow.

Next, we present some numerical results showing the efficiency and accuracy of our proposed one-iteration algorithm. In Figure 1, we test our algorithm on circular shape.

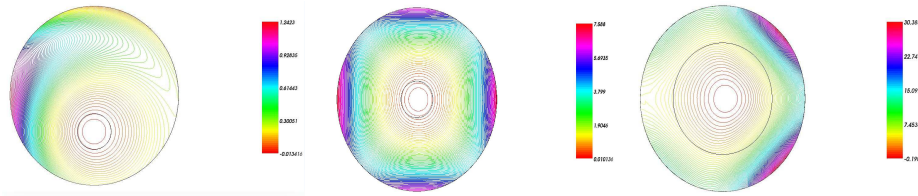


Figure 1: Reconstruction of circle shapes.

In Figure 2, we consider the case of an elliptical shape. As one can observe, the domain to be detected is located at zone where the topological gradient is negative and it is approximated by a level set curve of the topological gradient  $\delta\mathcal{K}$ . The result is quite efficient. In Figure 3, we obtain an interesting reconstruction result for a non trivial shape.

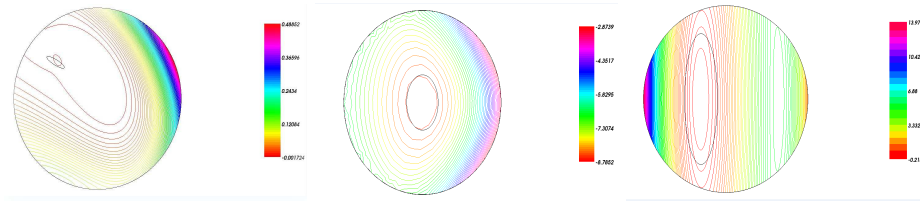


Figure 2: Reconstruction of an ellipse shapes.

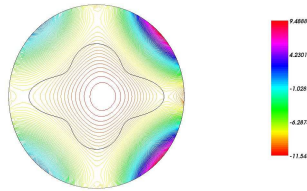


Figure 3: Reconstruction of a non trivial shape.

## 5 Conclusion

This work concerns the detection of objects immersed in anisotropic media from boundary measurements. The presented approach is based on the Kohn-Vogelius formulation and the topological gradient method. A topological sensitivity analysis is derived for an energy like functional. An accurate and fast reconstruction algorithm is proposed. The efficiency and accuracy of the suggested algorithm are illustrated by some numerical results. The considered model can be viewed as a prototype of a geometric inverse problem arising in many applications. The presented approach is general and can be adapted for various partial differential equations.

## References

- [1] M. Abdelwahed, M. Hassine and M. Masmoudi, Optimal shape design for fluid flow using topological perturbation technique, *Journal of Mathematical Analysis and Applications*, 356(2009), 548–563.
- [2] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities From Boundary Measurements*, *Lecture Notes in Mathematics*, 1846, Springer, 2004.
- [3] H. Ammari and H. Kang, *Polarization and Moment Tensors: with applications to inverse problems and effective medium theory*, *Applied Mathematical Sciences*, 162, Springer, New York, 2007.
- [4] H. Ammari, H. Kang and K. Kim, Polarization tensors and effective properties of anisotropic composite materials, *J. Differential Equations*, 215(2005), 401–428.

- [5] S. Amstutz, I. Horchani and M. Masmoudi, Crack detection by the topological gradient method, *Control and Cybernetics*, 34(2005), 81–101.
- [6] S. Andrieux, T. N. Baranger and A. BenAbda, Solving Cauchy problems by minimizing an energy-like functional, *Inverse Problems*, 22(2006), 115–134.
- [7] M. Badra, F. Caubet and M. Dambrine, Detecting an obstacle immersed in a fluid by shape optimization methods, *M3AS*, 21(2011), 2069–2101.
- [8] A. Ben Abda, M. Hassine, M. Jaoua and M. Masmoudi, Topological sensitivity analysis for the location of small cavities in Stokes flow, *SIAM J. Contr. Optim.*, 48(2009), 2871–2900.
- [9] J. Bowler, Thin-skin eddy-current inversion for the determination of cracks shapes, *Inverse Problems*, 18(2002), 1891–1905.
- [10] M. Cheney, D. Isaacson and J.C. Newell, Electrical impedance tomography, *SIAM Review*, 40(1999), 85–101.
- [11] Ph. Guillaume and M. Hassine, Removing holes in topological shape optimization, *ESAIM, Control, Optimisation and Calculus of Variations*, 14(2008), 160–191.
- [12] B. B. Guzina and M. Bonnet, Small-inclusion asymptotic of misfit functionals for inverse problems in acoustics, *Inverse Problems*, 22(2006), 1761–1785.
- [13] M. Hassine, S. Jan and M. Masmoudi, From differential calculus to 0–1 topological optimization, *SIAM J. Cont. Optim.*, 45(6)(2007), 1965–1987.
- [14] R. L. Parker, The inverse problem of resistivity sounding, *Geophysics*, 42(1984), 2143–2158.