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On The Volume Of The Trajectory Surface Under The Galilean Motions In The Galilean Space*

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Abstract

In this work, the volumes of the trajectory surfaces which are traced by fixed points during 3-parameter Galilean space motions are studied. Also, the well-known classical Holditch theorem [3] is generalized for the volumes of the trajectory surfaces in the Galilean space.

1 Introduction

In 1958, H. Holditch, [3], came up with the following outstanding classical theorem: If the endpoints A and B of a fixed line segment AB with length $a+b$ are rotated once along an oval k in the Euclidean plane $\mathbb{E}^2$, then a given fixed point X ($\overline{AX}=a, \overline{XB}=b$) of AB describes a closed not necessarily convex curve k_X. The area F of the Holditch-Ring bounded by the curves k and k_X is $F=\pi ab$.

Later, this theorem was studied by different methods [1,2,6–10] and in different spaces [13–15]. One of the generalizations of this theorem is on the volumes of the surfaces of 3-dimensional Euclidean space which are traced by fixed points during 3-parameter motions are given by H. R. Müller [6–8] and W. Blashke [1].

In this paper, the volumes of the trajectory surfaces of fixed points under 3-parameter Galilean space motions are calculated. Also, by the help of a special distance that we have defined, we generalize the well-known classical Holditch theorem for the volumes of the trajectory surfaces of fixed points under 3-parameter Galilean space motions.

2 Preliminaries

Galilean geometry $\mathbb{G}^3$ can be described as the study of properties of 3-dimensional space with coordinates that are invariant under Galilean transformations

$$\begin{cases} x' = x + a, \\ y' = (v\cos\alpha)\,x + (\cos\varphi)\,y + (\sin\varphi)\,z + b, \\ z' = (v\sin\alpha)\,x + (-\sin\varphi)\,y + (\cos\varphi)\,z + d. \end{cases}$$

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The Galilean transformations consist of translation, rotation and shear motions, and are described by I. M. Yaglom in [11]. In the literature, the basic information about Galilean Geometry is firstly given by I. M. Yaglom. Then, the differential geometry of curves and surfaces in the Galilean space $\mathbb{G}^3$ is worked in detail by O. Röshcel in [9]. Also, the quadrics in the Galilean space are examined by Kamenarović in [4]. Now, let's give some basic information about the Galilean space $\mathbb{G}^3$. Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be two vectors in the Galilean space. The scalar product of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$< \mathbf{a}, \mathbf{b} >_{\mathbb{G}} = xx_1.$$

The vectors in the Galilean space are divided into two classes as non-isotropic vectors and isotropic vectors which are of the form $\mathbf{a} = (x, y, z)$, $x \neq 0$ and $\mathbf{p} = (0, y, z)$, respectively. Moreover, the special scalar product of isotropic vectors $\mathbf{p} = (0, y, z)$ and $\mathbf{q} = (0, y_1, z_1)$ is defined by

$$< \mathbf{p}, \mathbf{q} >_{\delta} = yy_1 + zz_1.$$

If $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ are vectors in Galilean space, the vector product of $\mathbf{a}$ and $\mathbf{b}$ is defined as the following in [5]:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 \\ x & y & z \\ x_1 & y_1 & z_1 \end{vmatrix}.$$

Let $\mathbf{g}_1$ be a nonisotropic vector, $\mathbf{g}_2$ and $\mathbf{g}_3$ be isotropic vectors in the Galilean space. If the vectors $\mathbf{g}_1, \mathbf{g}_2$, and $\mathbf{g}_3$ satisfy that $< \mathbf{g}_1, \mathbf{g}_1 >_{\mathbb{G}} = < \mathbf{g}_2, \mathbf{g}_2 >_{\delta} = < \mathbf{g}_3, \mathbf{g}_3 >_{\delta} = 1$ and $< \mathbf{g}_1, \mathbf{g}_2 >_{\mathbb{G}} = < \mathbf{g}_1, \mathbf{g}_3 >_{\mathbb{G}} = < \mathbf{g}_2, \mathbf{g}_3 >_{\delta} = 0$, then the vector system $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is called an orthonormal frame of Galilean space. More information about the Galilean geometry can be found in [9, 11].

3 One Parameter Galilean Space Motion

Let R and R' be two $3-$dimensional Galilean spaces. Let

$$\{O; \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} \quad \text{and} \quad \{O'; \mathbf{g}'_1, \mathbf{g}'_2, \mathbf{g}'_3\}$$

are orthonormal frames of spaces R and R', respectively. Assume that the frame $\{O; \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ moves with respect to frame $\{O'; \mathbf{g}'_1, \mathbf{g}'_2, \mathbf{g}'_3\}$. Then, it is accepted that the space R moves according to the space R'. The spaces R and R' are called *moving space* and *fixed space*, respectively. Moreover, the frames $\{O; \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ and $\{O'; \mathbf{g}'_1, \mathbf{g}'_2, \mathbf{g}'_3\}$ are called *moving frame* and *fixed frame*, respectively. This motion is called *one parameter Galilean space motion* and is denoted by $B = R/R'$. Here, the spaces R and R' are orientated in the same direction. During the motion $B = R/R'$, it is clear that

$$\mathbf{x}' = -\mathbf{u} + \mathbf{x},$$

where $\mathbf{x}'$ and $\mathbf{x}$ correspond to the position vectors of any point $X \in R$ according to the rectangular coordinate systems of R', R, respectively, and

$$\mathbf{u} = \mathbf{OO}' = u_1\mathbf{g}_1 + u_2\mathbf{g}_2 + u_3\mathbf{g}_3.$$

Here, the vector $\mathbf{u}$ is called *translation vector*. In the motion $B = R/R'$, the vectors $\mathbf{x}'$, $\mathbf{x}$ and $\mathbf{u}$ are continuously differentiable functions of a real parameter t. If the frames $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ and $\{\mathbf{g}'_1, \mathbf{g}'_2, \mathbf{g}'_3\}$ are written as

$$G = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} \text{ and } G' = \begin{bmatrix} \mathbf{g}'_1 \\ \mathbf{g}'_2 \\ \mathbf{g}'_3 \end{bmatrix}$$

in the matrix form, respectively, then

$$G = AG' \text{ and } G' = A^{-1}G \tag{1}$$

are hold. Here, A is an invertible matrix. In this case, by differentiating both sides of the equation (1) and considering that G' is fixed frame, we get

$$dG = \Omega G, \tag{2}$$

where $\Omega = dAA^{-1}$. If we calculate Ω from above equation (2) and by the necessary operations with the basis vectors $\mathbf{g}_i$, $1 \leq i \leq 3$, we get

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ 0 & 0 & -\omega_1 \\ 0 & \omega_1 & 0 \end{bmatrix}, \tag{3}$$

where $\omega_i, 1 \leq i \leq 3$ are the linear differential forms with respect to t, that is, $\omega_i = f_i(t)\,dt$.

The vector

$$\boldsymbol{\omega} = \omega_1 \mathbf{g}_1 + \omega_2 \mathbf{g}_2 + \omega_3 \mathbf{g}_3$$

which is defined by nonzero components of Ω is called the instantaneous *Pfaffian vector of the motion* $B = R/R'$.

Especially, if $\omega_1 = 0$, then, the motion of B consists of only translation and shear motions. The motion doesn't contain the rotation. We will, therefore, accept $\omega_1 \neq 0$ in this work.

If it is used the equality (3) given for Ω, we get

$$d\mathbf{g}_1 = -\omega_3 \mathbf{g}_2 + \omega_2 \mathbf{g}_3, \; d\mathbf{g}_2 = -\omega_1 \mathbf{g}_3, \; d\mathbf{g}_3 = \omega_1 \mathbf{g}_2. \tag{4}$$

Moreover, if we calculate the exterior derivation of equation (4), by considering that basis vectors $\mathbf{g}_i, i = 1, 2, 3$, are linearly independent, we obtain

$$d\omega_1 = 0, \; d\omega_2 = -\omega_3 \wedge \omega_1, \; d\omega_3 = \omega_2 \wedge \omega_1,$$

where "$\wedge$" is the wedge product of the differential forms. Hence, the conditions of integration for components of the pfaffian vector of the motion R/R' are found as

$$d\omega_1 = 0, \; d\omega_2 = -\omega_3 \wedge \omega_1, \; d\omega_3 = \omega_2 \wedge \omega_1. \tag{5}$$

On the other hand, by differentiating of translation vector

$$\mathbf{O'O} = -\mathbf{u} = -u_1 \mathbf{g}_1 - u_2 \; \mathbf{g}_2 - u_3 \mathbf{g}_3$$

and by the aid of using the equation (4), we have

$$\sigma' = -d\mathbf{u} = \sigma_1 \mathbf{g}_1 + \sigma_2 \mathbf{g}_2 + \sigma_3 \mathbf{g}_3,$$

where

$$\sigma' = -d\mathbf{u}$$

and

$$\sigma_1 = -du_1,\ \sigma_2 = -du_2 + u_1\omega_3 - u_3\omega_1,\ \sigma_3 = -du_3 - u_1\omega_2 + u_2\omega_1.$$

The equations

$$\begin{cases} d\mathbf{g}_1 = \omega_2 \mathbf{g}_3 - \omega_3 \mathbf{g}_2, \\ d\mathbf{g}_2 = -\omega_1 \mathbf{g}_3, \\ d\mathbf{g}_3 = \omega_1 \mathbf{g}_2, \\ \sigma' = -d\mathbf{u} = \sigma_1 \mathbf{g}_1 + \sigma_2 \mathbf{g}_2 + \sigma_3 \mathbf{g}_3, \end{cases} \tag{6}$$

are called *derivative equations* of motion R/R'. Furthermore,

$$\mathbf{0} = d\sigma' = d\sigma_1 \mathbf{g}_1 + (d\sigma_2 - \sigma_1 \wedge \omega_3 + \sigma_3 \wedge \omega_1)\,\mathbf{g}_2 + (d\sigma_3 + \sigma_1 \wedge \omega_2 - \sigma_2 \wedge \omega_1)\,\mathbf{g}_3$$

and because of the fact that $\{\mathbf{g}_1, \mathbf{g}_2,\ \mathbf{g}_3\}$ are linearly independent, we find

$$d\sigma_1 = 0,\ d\sigma_2 = \sigma_1 \wedge \omega_3 - \sigma_3 \wedge \omega_1,\ d\sigma_3 = -\sigma_1 \wedge \omega_2 + \sigma_2 \wedge \omega_1. \tag{7}$$

So, the conditions of integration obtained for the translation vector of the motion R/R' are equations (7).

Now, let's examine the velocity vectors of the point X under the motion R/R'. Let X be any point in R. So, we can write

$$\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{g}_i$$

with respect to the moving frame $\{O; \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$. Since we have

$$\mathbf{x}' = -\mathbf{u} + \mathbf{x}$$

for position vector $\mathbf{x}'$ of the point X with respect to fixed frame $\{O'; \mathbf{g}'_1,\ \ \mathbf{g}'_2, \mathbf{g}'_3\}$, the differential of position vector $\mathbf{x}'$ can be expressed as

$$d\mathbf{x}' = -d\mathbf{u} + d\mathbf{x}.$$

By (6), it is calculated as

$$d\mathbf{x}' = \sigma_1 \mathbf{g}_1 + (\sigma_2 - x_1\omega_3 + x_3\omega_1)\,\mathbf{g}_2 + (\sigma_3 + x_1\omega_2 - x_2\omega_1)\,\mathbf{g}_3 + dx_1 \mathbf{g}_1 + dx_2 \mathbf{g}_2 + dx_3 \mathbf{g}_3.$$

During the motion R/R', the velocity vector of any point $X \in R$ with respect to fixed space R' and moving space R is called *absolute velocity* $\mathbf{X}_A$ and *relative velocity* $\mathbf{X}_R$, respectively. $\mathbf{X}_A$ and $\mathbf{X}_R$ are expressed by

$$\mathbf{X}_A = d\mathbf{x}'$$

and

$$\mathbf{X}_R = dx_1\mathbf{g}_1 + dx_2\mathbf{g}_2 + dx_3\mathbf{g}_3.$$

The difference between the absolute and the relative velocities is called *sliding velocity* $\mathbf{X}_F$ of the point X and it is stated as

$$\mathbf{X}_F = \sigma_1\mathbf{g}_1 + (\sigma_2 - x_1\omega_3 + x_3\omega_1)\,\mathbf{g}_2 + (\sigma_3 + x_1\omega_2 - x_2\omega_1)\,\mathbf{g}_3.$$

In this way, the following theorem can be given:

THEOREM 1. Let X be a point in R and $\mathbf{X}_A, \mathbf{X}_F$, and $\mathbf{X}_R$ be the absolute, the sliding and the relative velocity of the point X under the motion R/R', respectively. Then, the relation between the velocities is given by

$$\mathbf{X}_A = \mathbf{X}_F + \mathbf{X}_R.$$

If any point X in R is fixed, then the relative velocity $\mathbf{X}_R = \mathbf{0}$ and

$$\mathbf{X}_A = \mathbf{X}_F$$

under the motion R/R'. Furthermore, by considering derivative equations (6), the following equation holds:

$$d\mathbf{x} = (-x_1\omega_3 + x_3\omega_1)\,\mathbf{g}_2 + (x_1\omega_2 - x_2 w_1)\,\mathbf{g}_3$$

or

$$d\mathbf{x} = \mathbf{x} \times \boldsymbol{\omega}$$

for any fixed point X in R. So, for any fixed point X during the motion R/R', one can state

$$d\mathbf{x}' = \sigma' + \mathbf{x} \times \boldsymbol{\omega}.$$

Also, one can rewrite

$$d\mathbf{x}' = \tau_1\mathbf{g}_1 + \tau_2\mathbf{g}_2 + \tau_3\mathbf{g}_3,$$

where

$$\tau_1 = \sigma_1,\ \tau_2 = \sigma_2 + x_3\omega_1 - \omega_3 x_1,\ \tau_3 = \sigma_3 + x_1\omega_2 - \omega_1 x_2. \tag{8}$$

4 The Volume of the Trajectory Surface in $\mathbb{G}^3$

I. Until now, we have considered that the translation vector $\mathbf{u}$ and basis vectors $\mathbf{g}_i$ for $1 \le i \le 3$ of the motion B are functions of a real parameter t. From now on, we assume that the translation vector $\mathbf{u}$ and basis vectors $\mathbf{g}_i$ for $1 \le i \le 3$ of the motion B are functions of real parameters t_1, t_2 and t_3. And this motion is called *3-parameter Galilean space motion* and we shall denote this 3-parameter motion by B_3. During the motion B_3, ω_i and σ_i are the linear differential forms with respect to t_1, t_2 and t_3. So, the equations (5), (7) and (8) are not changed for the motion B_3.

Under the motion B_3, any fixed point X in the moving space R determines a volumetric trajectory surface in R'. The volume element of the trajectory surface of X under the motion B_3, is defined by

$$dJ_X = \tau_1 \wedge \tau_2 \wedge \tau_3. \tag{9}$$

Thus, the integration of the volume element over a region G determined by the fixed point X of the parameter space during the motion B_3 yields the volume of the trajectory surface, i.e.,

$$J_X = \int_G dJ_X.$$

By putting equation (8) into (9) and after making necessary arrangement, the volume of the trajectory surface of fixed point X in R during the motion B_3 is calculated as

$$J_X = J_O + ax_1^2 + \sum_{i=2}^{3} b_i x_i x_1 + \sum_{i=1}^{3} c_i x_i, \tag{10}$$

where

$$J_O = \int_G \sigma_1 \wedge \sigma_2 \wedge \sigma_3,\ a = \int_G \sigma_1 \wedge \omega_2 \wedge \omega_3,\quad b_2 = \int_G (\sigma_1 \wedge \omega_1 \wedge \omega_3),\ b_3 = \int_G (\sigma_1 \wedge \omega_1 \wedge \omega_2),$$

$$c_1 = \int_G \sigma_1 \wedge \sigma_2 \wedge \omega_2 - \sigma_1 \wedge \omega_3 \wedge \sigma_3,\quad c_2 = -\int_G (\sigma_1 \wedge \sigma_2 \wedge \omega_1),\ c_3 = \int_G \sigma_1 \wedge \omega_1 \wedge \sigma_3.$$

J_O is the volume of trajectory surface of origin point O. So, the volume J_X of trajectory surface is a quadratic polynomial of x_i.

THEOREM 2. All fixed points in R whose trajectory surfaces have equal volume under the motion B_3 lie on the same quadric.

II. Let $X = (x_i)$ and $Y = (y_i)$ be two fixed points in R and $Z = (z_i)$ be another point on the line segment XY, that is, $z_i = \lambda x_i + \mu y_i,\quad \lambda + \mu = 1$ in barycentric coordinates. Then, the volume of the region in R' determined by the fixed point Z under the motion B_3, by using equation (10), is obtained as

$$J_Z = \lambda^2 J_X + 2\lambda\mu J_{XY} + \mu^2 J_Y,$$

where

$$J_{XY} = J_O + ax_1y_1 + \frac{1}{2}\sum_{i=2}^{3} b_i (x_1y_i + y_1x_i) + \frac{1}{2}\sum_{i=1}^{3} c_i (x_i + y_i). \tag{11}$$

Also, J_{XY} is called the *mixture trajectory surface volume.* It is clearly seen that $J_{XX} = J_X$ and $J_{XY} = J_{YX}$. Since

$$J_X - 2J_{XY} + J_Y = a(x_1 - y_1)^2 + \sum_{i=2}^{3} b_i (x_1 - y_1)(x_i - y_i), \tag{12}$$

we can restate the volume trajectory surface of the fixed point Z as $J_Z = \lambda J_X + \mu J_Y - \lambda\mu\left(a\left(x_1 - y_1\right)^2 + \sum_{i=2}^{3} b_i\left(x_1 - y_1\right)\left(x_i - y_i\right)\right)$. So, we may give following theorem:

THEOREM 3. Let X and Y be two different fixed points in R, and Z be another point on the segment XY. During the motion B_3, the relation between the volumes of the trajectory surfaces of fixed points X, Y and Z is as follows:

$$J_Z = \lambda J_X + \mu J_Y - \lambda\mu\left(a\left(x_1 - y_1\right)^2 + \sum_{i=2}^{3} b_i\left(x_1 - y_1\right)\left(x_i - y_i\right)\right). \tag{13}$$

We will define the distance $D(X,Y)$ between the fixed points $X, Y \in R$, by

$$D^2(X,Y) = \varepsilon\left(a\left(x_1 - y_1\right)^2 + \sum_{i=2}^{3} b_i\left(x_1 - y_1\right)\left(x_i - y_i\right)\right), \quad \varepsilon = \pm 1. \tag{14}$$

Therefore, by the help of the distance (14), the equation (13) can be restated as follows:

$$J_Z = \lambda J_X + \mu J_Y - \varepsilon\lambda\mu D^2(X,Y). \tag{15}$$

Since X, Y and Z are collinear, we can write $D(X,Z) + D(Z,Y) = D(X,Y)$. Hence, if we represent $\lambda = \frac{D(Z,Y)}{D(X,Y)}, \mu = \frac{D(X,Z)}{D(X,Y)}$, where $D(X,Y) \neq 0$, i.e., $x_1 \neq y_1$, then from equation (15), we have

$$J_Z = \frac{1}{D(X,Y)}\left[D(Z,Y) J_X + D(X,Z) J_Y\right] - \varepsilon D(Z,Y) D(X,Z). \tag{16}$$

Now, we consider that the fixed points X and Y trace the same trajectory surface. In this case, we get $J_X = J_Y$. Then, from the above equation (16), we get

$$J_X - J_Z = \varepsilon D(Z,Y) D(X,Z).$$

So, we may give the following theorem:

THEOREM 4. *Let* $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, where $x_1 \neq y_1$, *be two different fixed points in* R *and* Z *be another point on the segment* XY. Let the fixed points X and Y trace the same trajectory surface and the fixed point Z trace the different trajectory surface during the motion B_3. Then, the difference between the volumes of these two trajectory surfaces depends on the distances of Z from the endpoints which are defined in (14) with respect to the motion B_3.

In case of $x_1 = y_1$, then the equation (13) is arranged as $J_Z = \lambda J_X + \mu J_Y$. If the fixed points X and Y trace the same trajectory surface during the motion B_3, we obtain

$$J_X - J_Z = 0.$$

COROLLARY. *Let* $X = (x_1, x_2, x_3)$ *and* $Y = (y_1, y_2, y_3)$, where $x_1 = y_1$ *be two different fixed points in* R *and* Z *be another point on the segment* XY. If the fixed points X and Y trace the same trajectory surface, the fixed point Z trace the trajectory surface with same volume during the motion B_3.

III.

THEOREM 5. Let us consider a triangle in R whose vertices are points $X_1 = (x_1, x_2, x_3)$, $X_2 = (y_1, y_2, y_3)$ *and* $X_3 = (z_1, z_2, z_3)$, where $x_1 \neq y_1$, $x_1 \neq z_1$ and $y_1 \neq z_1$. If the vertices of this triangle trace the same trajectory surface in R', then a different point Q on the plane which is determined by X_1, X_2 *and* X_3 traces another surface. The difference between the volumes of these two trajectory surfaces depends on the distances $D(X_k, Q), D(X_k, Q_k), D(Q_k, X_j)$ and $D(X_i, Q_k)$ which are measured with respect to the motion B_3. Here, the point Q_i is a intersection point of the segments X_iQ and X_jX_k, $i, j, k = 1,2,3; 2,3,1; 3,1,2$.

PROOF. Let $X_1 = (x_i)$, $X_2 = (y_i)$ and $X_3 = (z_i)$, $x_1 \neq y_1$, $x_1 \neq z_1$ and $y_1 \neq z_1$ be three non- collinear fixed points in R, and $Q = (q_i)$ be another fixed point on the plane which is determined by $X_1 = (x_i)$, $X_2 = (y_i)$ and $X_3 = (z_i)$. Then, for the point Q, we can write

$$q_i = \lambda_1 x_1 + \lambda_2 y_i + \lambda_3 z_i, \lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{17}$$

Under the motion B_3, by the help of equations (10), (11) and (17), the volume of the region determined by the point Q can be calculated as

$$J_Q = \lambda_1^2 J_{X_1} + \lambda_2^2 J_{X_2} + \lambda_3^2 J_{X_3} + 2\lambda_1\lambda_2 J_{X_1X_2} + 2\lambda_1\lambda_3 J_{X_1X_3} + 2\lambda_2\lambda_3 J_{X_2X_3}.$$

Also, by considering the equations (12) and (14), we get

$$\begin{aligned} J_Q &= \lambda_1 J_{X_1} + \lambda_2 J_{X_2} + \lambda_3 J_{X_3} \\ &\quad - \left\{\varepsilon_{12}\lambda_1\lambda_2 D^2(X_1, X_2) + \varepsilon_{13}\lambda_1\lambda_3 D^2(X_1, X_3) + \varepsilon_{23}\lambda_2\lambda_3 D^2(X_2, X_3)\right\}. \end{aligned}$$

Let the points $Q_1 = (q_{1i})$ are intersection point of the segments X_1Q and X_2X_3. Then, we can write

$$q_{1i} = \xi_1 y_i + \xi_2 z_i, \quad q_i = \xi_3 x_i + \xi_4 a_i,$$

where $\xi_1 + \xi_2 = \xi_3 + \xi_4 = 1$. Hence, we have $\lambda_1 = \xi_3$, $\lambda_2 = \xi_1\xi_4$, $\lambda_2 = \xi_2\xi_4$ i.e.,

$$\lambda_1 = \frac{D(Q, Q_1)}{D(X_1, Q_1)}, \quad \lambda_2 = \frac{D(Q_1, X_3)}{D(X_2, X_3)}\frac{D(X_1, Q)}{D(X_1, Q_1)}, \quad \lambda_3 = \frac{D(X_2, Q_1)}{D(X_2, X_3)}\frac{D(X_1, Q)}{D(X_1, Q_1)}.$$

Similarly, for Q_2 and Q_3, we calculate

$$\lambda_i = \frac{D(Q, Q_i)}{D(X_i, Q_i)} = \frac{D(X_j, Q)}{D(X_j, Q_j)}\frac{D(X_k, Q_j)}{D(X_k, X_i)} = \frac{D(X_k, Q)}{D(X_k, Q_k)}\frac{D(Q_k, X_j)}{D(X_i, X_j)}$$

for $i, j, k = 1,2,3; 2,3,1; 3,1,2$. So, the volume of the trajectory surface determined by the point Q can be rearranged as

$$J_Q = \sum_{i=1}^{3} \frac{D(Q, Q_i)}{D(X, Q_i)} J_{X_i} - \sum_{i=1}^{3} \varepsilon_{ij} \left(\frac{D(X_k, Q)}{D(X_k, Q_k)}\right)^2 D(Q_k, X_j) D(X_i, Q_k). \tag{18}$$

During the motion B_3, since the points X_1, X_2 and X_3 trace the same trajectory surface in R', we can write $J_{X_1} = J_{X_2} = J_{X_3}$. So, from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and equation (18), we can get

$$J_{X_i} - J_Q = \sum_{i=1}^{3} \varepsilon_{ij} \left(\frac{D(X_k, Q)}{D(X_k, Q_k)} \right)^2 D(Q_k, X_j) D(X_i, Q_k),$$

for i, j, k (*cyclic*).

Finally, the difference between the volumes J_{X_1} and J_Q only depends on distances on triangle $X_1 \overset{\triangle}{X_2} X_3$ defined in (14) according to the motion B_3.

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