Applied Mathematics E-Notes, 17(2017), 221-230 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Sixth Order Newton-Type Method For Solving System Of Nonlinear Equations And Its Applications*

Kalyanasundaram Madhu†

Received 28 October 2016

Abstract

In this work, we have developed a sixth order Newton-type method for solving system of nonlinear equations. New sixth order method is composed of three steps with only one inverse of Jacobian matrix, namely, Newton iteration as the first step and weighted Newton iteration as the second and third step. As an application, we have tested the present methods on Chandrasekhar's problem and 1-D Bratu problem.

1 Introduction

The problem of finding a real zero of a system of nonlinear equations $F(x) = 0$, where

$$F(x) = (f_1(x), f_2(x), ..., f_n(x))^T, \quad x = (x_1, x_2, ..., x_n)^T, \quad f_i : \mathbb{R}^n \to \mathbb{R},$$

$\forall i = 1, 2, \ldots, n$ and $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and D is an open and convex set, where we assume that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is a zero of the system and $y = (y_1, y_2, ..., y_n)^T$ is an initial guess sufficiently close to α, is an often discussed problem in many applications of science and technology. For example, problems of the above type arises while solving boundary value problems for differential equations. The differential equations is reduced to system of nonlinear equations which are in-turn solved by the familiar Newton's iteration method (NM) having convergence order two [14], which is given by

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, 2, ... \tag{1}$$

where $x^{(0)}$ is initial guess and $F'(x^{(k)})$ is the Jacobian matrix of the function $F(x^{(k)})$ evaluated for the k^{th} iteration.

A lot of iterative methods for solving single non-linear equation are available in the literature [16]. Whereas all these methods cannot be extended to solve nonlinear system involving more than one variable. Even if some methods can be extended to solve nonlinear system, certain decisive factors like efficiency indices, computational

*Mathematics Subject Classifications: 65H10.

†Department of Mathematics, Holy Mary Institute of Technology and Science, Hyderabad - 501301, India

efficiency indices, number of functional evaluation, number of Frechet derivatives evaluation and number of inverse of Frechet derivatives are to be given due importance. Moreover, when extending methods for single equation to solve system of nonlinear equations, due to increase in computational complexity they have no practical value. Chebyshev and Halley [1, 11] extended their methods to system of nonlinear equations and proved cubic convergence where first and second Frechet derivatives are used. Due to evaluation of second Frechet derivative, these methods are considered less practical from a computational point of view. On the other hand, there has been considerable attempts to derive methods free from second derivative having higher order of convergence for single equation [3, 9, 15]. Extensions of these methods for system of nonlinear equations are found in [4, 10, 7].

In this paper, we have presented new sixth order Newton-type method for solving system of nonlinear equations. Rest of this paper as follows. Section 2 presented a new method and discusses its convergence analysis in Section 3. In Section 4, Numerical experiments including the applications on Chandrasekhar's problem and 1-D bratu problem are given to illustrate the efficiency of the new methods. A brief conclusion is given in section 5.

2 Description of the Methods

In this section, we display a new method for solving nonlinear systems that we call $M6$

$$y(x^{(k)}) = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \qquad z^{(k)} = y(x^{(k)}) - \tau\ [F'(x^{(k)})]^{-1}F(y(x^{(k)})),$$
$$x^{(k+1)} = z^{(k)} - \tau\ [F'(x^{(k)}))]^{-1}F(z^{(k)}), \quad \tau = 2I - [F'(x^{(k)})]^{-1}F'(y^{(k)}), \tag{2}$$

where I is the $n \times n$ identity matrix.

Another important aspect of this work is the comparative study of the efficiency of the proposed method with well known high-order methods, such as Jarratt's method [12] and the one recently introduced by Wang et al. [17] which are given below,

Jarratt's method (JM):

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - [6F'(y(x^{(k)}))) - 2F'(x^{(k)}))]^{-1}[3F'(y(x^{(k)}))) + F'(x^{(k)})] \\ &\quad \times [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ y(x^{(k)}) &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}). \end{aligned}$$

Method of Wang et al. (Wang):

$$\begin{aligned} x^{(k+1)} &= z(x^{(k)}) - \left[\frac{3}{2}F'(y(x^{(k)}))^{-1} - \frac{1}{2}F'(x^{(k)})^{-1}\right] F(z(x^{(k)})), \\ z(x^{(k)}) &= x^{(k)} - [6F'(y(x^{(k)})) - 2F'(x^{(k)})]^{-1}[3F'(y(x^{(k)})) + F'(x^{(k)})] \\ &\quad \times [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ y(x^{(k)}) &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}). \end{aligned}$$

3 Convergence Analysis

THEOREM 1. Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Frechet differentiable at each point of an open convex neighborhood D of $\alpha \in \mathbb{R}^n$, that is a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular in x^*, and $x^{(0)}$ close enough to x^*. Then the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained using the iterative expression (2) converges to x^* with sixth order.

PROOF. Using Taylor's expansion about x^* we have

$$F(x^{(k)}) = F'(x^*)\left[e^{(k)} + C_2{e^{(k)}}^2 + C_3{e^{(k)}}^3 + C_4{e^{(k)}}^4 + C_5{e^{(k)}}^5 + C_6{e^{(k)}}^6\right] + O({e^{(k)}}^7),$$

and

$$F'(x^{(k)}) = F'(x^*)\left[I + 2C_2e^{(k)} + 3C_3{e^{(k)}}^2 + 4C_4{e^{(k)}}^3 + 5C_5{e^{(k)}}^4 + 6C_6{e^{(k)}}^5\right] + O({e^{(k)}}^6),$$

where $C_k = (1/k!)[F'(x^*))]^{-1}F^{(k)}(x^*)$, $k = 2, 3, \ldots$, and $e^{(k)} = x^{(k)} - x^*$. We have

$$[F'(x^{(k)})]^{-1} = \left[I + X_1e^{(k)} + X_2{e^{(k)}}^2 + X_3{e^{(k)}}^3 + X_4{e^{(k)}}^4 + X_5{e^{(k)}}^5\right][F'(x^*))]^{-1}, \tag{3}$$

where

$$X_1 = -2C_2, \quad X_2 = 4C_2^2 - 3C_3, \quad X_3 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4,$$

$$X_4 = -5C_5 + 9C_3^2 + 8C_2C_4 + 8C_4C_2 + 16C_2^4 - 12C_2^2C_3 - 12C_3C_2^2 - 12C_2C_3C_2$$

and

$$\begin{aligned} X_5 \ = \ & -32C_2^5 + 24C_3C_2^3 + 24C_2C_3C_2^2 - 16C_4C_2^2 + 24C_2^2C_3C_2 - 18C_3^2C_2 \\ & -16C_2C_4C_2 + 10C_5C_2 + 24C_2^3C_3 - 18C_3C_2C_3 - 18C_2C_3^2 + 12C_4C_3 \\ & -16C_2^2C_4 + 12C_3C_4 + 10C_2C_4 - 6C_6. \end{aligned}$$

Then

$$\begin{aligned} [F'(x^{(k)})]^{-1}F(x^{(k)}) \ = \ & e^{(k)} - C_2{e^{(k)}}^2 + 2(C_2^2 - C_3){e^{(k)}}^3 + (-3C_4 - 4C_2^3 + 4C_2C_3 \\ & +3C_3C_2){e^{(k)}}^4 + (6C_3^2 + 8C_2^4 - 8C_2^2C_3 - 6C_2C_3C_2 \\ & -6C_3C_2^2 + 6C_2C_4 + 4C_4C_2 - 4C_5){e^{(k)}}^5 + (-5C_6 - 2C_2C_5 \\ & -14C_2^2C_4 + 9C_3C_4 + 16C_2^3C_3 - 12C_3C_2C_3 - 12C_2C_3^2 \\ & +8C_4C_3 - 16C_2^5 + 12C_3C_2^3 + 12C_2C_3C_2^2 - 8C_4C_2^2 \\ & +12C_2^2C_3C_2 - 9C_3^2C_2 - 8C_2C_4C_2 + 5C_5C_2 + 10C_2C_4){e^{(k)}}^6 \\ & +O({e^{(k)}}^7), \end{aligned}$$

we have

$$
\begin{aligned}
y^{(k)} &= x^* + C_2 e^{(k)^2} + 2(-C_2^2 + C_3)e^{(k)^3} + (3C_4 + 4C_2^3 - 4C_2C_3 - 3C_3C_2)e^{(k)^4} \\
&\quad +(-6C_3^2 - 8C_2^4 + 8C_2^2C_3 + 6C_2C_3C_2 + 6C_3C_2^2 - 6C_2C_4 - 4C_4C_2 \\
&\quad +4C_5)e^{(k)^5} + (5C_6 + 2C_2C_5 + 14C_2^2C_4 - 9C_3C_4 - 16C_2^3C_3 \\
&\quad +12C_3C_2C_3 + 12C_2C_3^2 - 8C_4C_3 + 16C_2^5 - 12C_3C_2^3 - 12C_2C_3C_2^2 \\
&\quad +8C_4C_2^2 - 12C_2^2C_3C_2 + 9C_3^2C_2 + 8C_2C_4C_2 - 5C_5C_2 - 10C_2C_4)e^{(k)^6} \\
&\quad +O(e^{(k)^7}).
\end{aligned} \tag{4}
$$

Using equ. (4) we have

$$
\begin{aligned}
F(y^{(k)}) &= F'(x^*)\Big[C_2 e^{(k)^2} + 2(-C_2^2 + C_3)e^{(k)^3} + (3C_4 + 5C_2^3 - 4C_2C_3 \\
&\quad -3C_3C_2)e^{(k)^4} + (-6C_3^2 - 12C_2^4 + 12C_2^2C_3 + 6C_2C_3C_2 \\
&\quad +6C_3C_2^2 - 6C_2C_4 - 4C_4C_2 + 4C_5)e^{(k)^5} + (5C_6 + 2C_2C_5 \\
&\quad +14C_2^2C_4 - 9C_3C_4 - 16C_2^3C_3 + 12C_3C_2C_3 + 12C_2C_3^2 \\
&\quad -8C_4C_3 + 16C_2^5 - 12C_3C_2^3 - 12C_2C_3C_2^2 + 8C_4C_2^2 - 12C_2^2C_3C_2 \\
&\quad +9C_3^2C_2 + 8C_2C_4C_2 - 5C_5C_2 - 10C_2C_4)e^{(k)^6}\Big] + O(e^{(k)^7}),
\end{aligned} \tag{5}
$$

$$
F'(y^{(k)}) = [F'(x^*)]\left[I + P_1 e^{(k)^2} + P_2 e^{(k)^3} + P_3 e^{(k)^4}\right] + O(e^{(k)^5}), \tag{6}
$$

where $P_1 = 2C_2^2$, $P_2 = 4C_2C_3 - 4C_2^3$ and $P_3 = 4C_2^3 - 4C_2C_3 - 3C_3C_2 + 3C_4 + 3C_3C_2^2$. Again, using eqs. (3) and (6) we have

$$
\begin{aligned}
\tau(x^{(k)}) &= I + 2C_2 e^{(k)} - (6C_2^2 - 3C_3)e^{(k)^2} - (10C_2C_3 + 6C_3C_2 - 16C_2^3 \\
&\quad -4C_4)e^{(k)^3} - (4C_2^3 - 4C_2C_3 - 3C_3C_2 + 3C_4 - 15C_3C_2^2 \\
&\quad -20C_2^2C_3 + 32C_2^4 - 5C_5 + 9C_3^2 + 8C_2C_4 + 8C_4C_2 \\
&\quad -12C_2C_3C_2)e^{(k)^4} + O(e^{(k)^5}).
\end{aligned} \tag{7}
$$

From eqs. (3) and (5), we have

$$
\begin{aligned}
[F'(x^{(k)})]^{-1}F(y^{(k)}) &= C_2 e^{(k)^2} + (2C_3 - 4C_2^2)e^{(k)^3} + (13C_2^3 - 8C_2C_3 - 6C_3C_2 \\
&\quad +3C_4)e^{(k)^4} + (-12C_3^2 - 38C_2^4 + 28C_2^2C_3 + 18C_3C_2^2 \\
&\quad -12C_2C_4 - 8C_4C_2 + 4C_5)e^{(k)^5} + (5C_6 - 6C_2C_5 \\
&\quad +38C_2^2C_4 - 18C_3C_4 - 72C_2^3C_3 + 36C_3C_2C_3 + 36C_2C_3^2 \\
&\quad -16C_4C_3 + 76C_2^5 - 39C_3C_2^3 - 36C_2C_3C_2^2 + 16C_4C_2^2 \\
&\quad -36C_2^3C_3C_2 + 18C_3^2C_2 + 16C_2C_4C_2 - 5C_5C_2 - 10C_2C_4)e^{(k)^6} \\
&\quad +O(e^{(k)^7}).
\end{aligned} \tag{8}
$$

Using eqs. (4), (7) and (8), we have

$$\begin{aligned} z^{(k)} &= x^* + 5C_2^3 {e^{(k)}}^4 + (-36C_2^4 + 8C_2^2C_3 + 28C_2C_3C_2 + 6C_3C_2^2){e^{(k)}}^5 \\ &+(18C_2^2C_4 - 80C_2^3C_3 + 12C_3C_2C_3 + 20C_2C_3^2 + 190C_2^5 - 51C_3C_2^3 \\ &-64C_2C_3C_2^2 + 16C_4C_2^2 - 48C_2^2C_3C_2 + 18C_3^2C_2 + 16C_2C_4C_2 \\ &-5C_5C_2 + 36C_2^3C_3C_2 + 4C_2^4 - 4C_2C_3C_2 - 3C_3C_2^2 + 3C_4C_2 \\ &-20C_2^2C_3C_2){e^{(k)}}^6 + \cdots . \end{aligned} \tag{9}$$

Then

$$\begin{aligned} F(z^{(k)}) &= F'(x^*)\Big[5C_2^3 {e^{(k)}}^4 + (-36C_2^4 + 8C_2^2C_3 + 28C_2C_3C_2 + 6C_3C_2^2){e^{(k)}}^5 \\ &+(18C_2^2C_4 - 80C_2^3C_3 + 12C_3C_2C_3 + 20C_2C_3^2 + 190C_2^5 \\ &-51C_3C_2^3 - 64C_2C_3C_2^2 + 16C_4C_2^2 - 48C_2^2C_3C_2 + 18C_3^2C_2 \\ &+16C_2C_4C_2 - 5C_5C_2 + 36C_2^3C_3C_2 + 4C_2^4 - 4C_2C_3C_2 - 3C_3C_2^2 \\ &+3C_4C_2 - 20C_2^2C_3C_2){e^{(k)}}^6\Big] + \cdots . \end{aligned} \tag{10}$$

Finally, using eqs. (3), (7), (9) and (10) in (2), we obtained

$$e^{(k+1)} = 30C_2^5 {e^{(k)}}^6 + O({e^{(k)}}^7).$$

4 Applications

4.1 Chandrasekhar's Equation

Consider the quadratic integral equation related with Chandrasekhar's work [6]

$$x(s) = f(s) + \lambda x(s) \int_0^1 k(s,t)x(t)dt, \tag{11}$$

which arises in the study of the radiative transfer theory, the transport of neutrons and the kinetic theory of the gases. Equation (11) is also studied by Argyros [2] and along with some conditions for the kernel $k(s,t)$ in [8]. We consider the maximum norm for the kernel $k(s,t)$ as a continuous function in $s,t \in [0,1]$ such that $0 < k(s,t) < 1$ and $k(s,t) + k(t,s) = 1$. Moreover, we assume that $f(s) \in C[0,1]$ is a given function and λ is a real constant. Note that finding a solution for (11) is equivalent to solving the equation $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$F(x)(s) = x(s) - f(s) - \lambda x(s) \int_0^1 k(s,t)x(t)dt, \ x \in C[0,1], \ s \in [0,1].$$

In particular, we consider

$$F(x)(s) = x(s) - 1 - \frac{x(s)}{4} \int_0^1 \frac{s}{s+t} x(t)dt, \ x \in C[0,1], \ s \in [0,1], \tag{12}$$

Finally, we approximate numerically a solution for $F(x) = 0$, where $F(x)$ is given in (12) by means of a discretization procedure. We solve the integral equation (12) by the Gauss-Legendre quadrature formula:

$$\int_0^1 f(t)dt \approx \frac{1}{2}\sum_{j=1}^{m} \beta_j f(t_j),$$

where β_j are the weights and t_j are the knots tabulated in Table 1 for $m = 8$. Denote x_i for the approximations of $x(t_i), i = 1, 2, ...8$, we obtain the following nonlinear system:

$$x_i \approx 1 + \frac{1}{8} x_i \sum_{j=1}^{8} a_{ij} x_j, \; where \; a_{ij} = \frac{t_i \beta_j}{8(t_i + t_j)}, \; i = 1, ...8. \tag{13}$$

We use the following stopping criterion for this problem $err_{min} = \|x^{(k+1)} - x^{(k)}\|_2 < 10^{-13}$, the initial approximation assumed is $x^{(0)} = \{1, 1, ..., 1\}^t$ for obtaining the solution of this problem, $x^* = \{1.02171973146..., 1.07318638173..., 1.12572489365..., 1.16975331216..., 1.20307175130..., 1.22649087463..., 1.24152460059..., 1.24944851669...\}^t$. Table 2 shows that the proposed method $M6$ is better than NM.

Table 1: Weights and knots for the Gauss-Legendre formula (m = 8)

j	t_j	β_j
1	0.0198550717512...	0.101228536290...
2	0.101666761293...	0.222381034453...
3	0.237233795041...	0.313706645877...
4	0.408282678752...	0.362683783378...
5	0.591717321247...	0.362683783378...
6	0.762766204958...	0.31370664587...
7	0.898333238706...	0.222381034453...
8	0.980144928248...	0.101228536290...

Table 2: Comparison of iteration and error for Chandrasekhar's problem

Methods	M	$error$
NM	5	3.1408e-016
JM	3	3.1218e-016
$Wang$	3	3.2401e-016
$M6$	3	2.2204e-016

4.2 1-D Bratu Problem

The 1-D Bratu problem [5] is given by

$$\frac{d^2U}{dx^2} + \lambda \exp U(x) = 0,\ \lambda > 0,\ 0 < x < 1, \tag{14}$$

with the boundary conditions $U(0) = U(1) = 0$. The 1-D Planar Bratu problem has two known, bifurcated, exact solutions for values of $\lambda < \lambda_c$, one solution for $\lambda = \lambda_c$ and no solutions for $\lambda > \lambda_c$. The critical value of λ_c is simply $8(\eta^2 - 1)$, where η is the fixed point of the hyperbolic cotangent function $\coth(x)$. The exact solution to eq. (14) is known and can be presented here as

$$U(x) = -2\ln\left[\frac{\cosh\,(x - \frac{1}{2})\frac{\theta}{2}}{\cosh\left(\frac{\theta}{4}\right)}\right], \tag{15}$$

where θ is a constant to be determined, which satisfies the boundary conditions and is carefully chosen and assumed to be the solution of the differential equation (14). Using a similar procedure as in ([13]), we show how to obtain the critical value of λ. Substituting eq. (15) in (14), simplifying and collocating at the point $x = \frac{1}{2}$ because it is the midpoint of the interval. Another point could be chosen, but low-order approximations are likely to be better if the collocation points are distributed somewhat evenly throughout the region. Then, we have

$$\theta^2 = 2\lambda\cosh^2\left(\frac{\theta}{4}\right). \tag{16}$$

Differentiating eq. (16) with respect to θ and setting $\frac{d\lambda}{d\theta} = 0$, the critical value λ_c satisfies

$$\theta = \frac{1}{2}\lambda_c\cosh\left(\frac{\theta}{4}\right)\sinh\left(\frac{\theta}{4}\right). \tag{17}$$

By eliminating λ from eqs.(16) and (17), we have the value of θ_c for the critical λ_c satisfying

$$\frac{\theta_c}{4} = \coth\left(\frac{\theta_c}{4}\right) \tag{18}$$

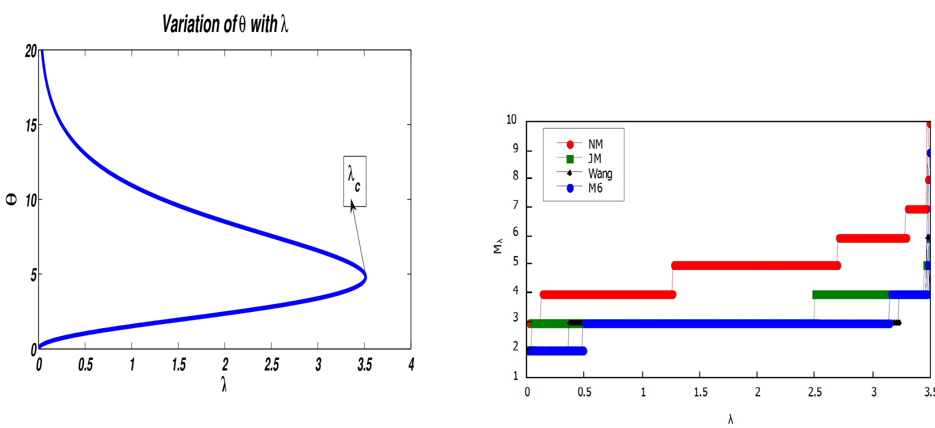
for which $\theta_c = 4.798714560$ can be obtained using an iterative method. We then get $\lambda_c = 3.513830720$ from eq. (16). Figure 1 (a) illustrates this critical value of λ. The finite dimensional problem using standard finite difference scheme is given by

$$F_j(U_j) = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \lambda \exp U_j = 0,\ j = 1..N-1 \tag{19}$$

with discrete boundary conditions $U_0 = U_N = 0$ and the stepsize $h = 1/N$. There are $N-1$ unknowns ($n = N-1$). The Jacobian is a sparse matrix and its typical number of nonzero per row is three. It is known that the finite difference scheme converges to the lower solution of the 1-D Bratu using the starting vector $U^{(0)} = (0, 0, .., 0)^T$. We use $N = 101$ ($n = 100$) and test for 350 λ's in the interval $(0, 3.5]$ (interval width

Table 3: Comparison of number of λ's in different methods for 1-D Bratu problem

Method	$M=2$	$M=3$	$M=4$	$M=5$	$M>5$	$\overline{M_\lambda}$
NM	0	12	114	143	81	4.92
JM	4	245	95	4	2	3.30
$Wang$	36	286	26	0	2	2.99
$M6$	48	266	33	2	1	2.98



(a) (b)

Figure 1: (a) Variation of θ for λ, (b) Variation of number of iteration with λ

= 0.01). For each λ, we let M_λ be the minimum number of iterations for which $\|U_j^{(k+1)} - U_j^{(k)}\|_2 < 1e-13$, where the approximation $U_j^{(k)}$ is calculated correct to 14 decimal places. Let $\overline{M_\lambda}$ be the mean of iteration number for the 350 λ's. Figure 1 (b) and table 3 give the results for 1-D Bratu problem, where M represents number of iterations for convergence. It can be observed from the methods considered in table 3, as λ increases to its critical value, the number of iterations required for convergence increase. However, as the order of method increases, the mean of iteration number decreases. The $M6$ is the most efficient method among the compared methods because it has the lowest mean iteration number and the highest number of λ's points are converging only in 2 iterations.

5 Conclusion

In this work, we have proposed an efficient new iterative method of order six for solving system of nonlinear equations. The main advantage of the proposed schemes are: they do not use second Frechet derivative, evaluate only one inverse of first Frechet derivative, evaluate less number of linear systems per iteration. To illustrate the proposed new methods and to check the validity of the theoretical results we have tabulated numerical results. The performance is compared with Newton's method and some recently developed methods and proposed method is to be superior over some existing methods.

Acknowledgement. The authors would like to thank the referee for his helpful comments and suggestions.

References

[1] S. Amat, S. Busquier and J. M. Gutierrez, Geometric constructions of iterative methods to solve nonlinear equations, Comput. Appl. Math., 157(2003), 197–205.

[2] I. K. Argyros, Quadratic equations and applications to chandrasekhars and related equations, Bull. Austral. Math. Soc., 32(1985), 275–292.

[3] D. K. R. Babajee, K. Madhu and J. Jayaraman, A family of higher order multipoint iterative methods based on power mean for solving nonlinear equations, Afr. Mat., 27(2016), 865–876.

[4] D. K. R. Babajee, K. Madhu and J. Jayaraman, On some improved harmonic mean newton-like methods for solving systems of nonlinear equations, Algorithms, 8(2015), 895–909.

[5] R. Buckmire, Investigations of nonstandard Mickens-type finite-difference schemes for singular boundary value problems in cylindrical or spherical coordinates, Num. Meth. P. Diff. Eqns., 19(2003), 380–398.

[6] D. Chandrasekhar, Radiative Transfer, Dover, New York, 1960.

[7] A. Cordero and J. R. Torregrosa, Variants of Newtons method for functions of several variables, Appl. Math. Comp., 183(2006), 199–208.

[8] J. A. Ezquerro, J. M. Gutierrez, M. A. Hernandez and M. A. Salanova, Solving nonlinear integral equations arising in radiative transfer, Numer. Funct. Anal. Optim., 20(1999), 661–673.

[9] M. Frontini and E. Sormani, Some variants of Newtons method with third-order convergence, Appl. Math. Comp., 140(2003), 419–426.

[10] M. Frontini and E. Sormani, Third-order methods from Quadrature Formulae for solving systems of nonlinear equations, Appl. Math. Comp., 149(2004), 771–782.

[11] J. M. Gutierrez and M. A. Hernandez, A family of Chebyshev-Halley type methods in banach spaces, Bull. Austral. Math. Soc., 55(1997), 113–130.

[12] P. Jarratt, Some fourth order multipoint iterative methods for solving equations. Math. Comp., 20(1966), 434–437.

[13] S.A. Odejide and Y.A.S. Aregbesola. A note on two dimensional Bratu problem. Kragujevac J. Math., 29(2006), 49–56.

[14] A. M. Ostrowski, Solutions of Equations and System of Equations, Academic Press, New York, 1960.

[15] A. Ozban, Some new variants of Newtons method, Appl. Math. Lett., 17(2004), 677–682.

[16] J. F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, New Jersey, 1964.

[17] X. Wang, J. Kou and C. Gu, Semilocal convergence of a sixth-order Jarrat method in Banach spaces, Numer. Algorithms, 57(2011), 441–456.