Applied Mathematics E-Notes, 17(2017), 91-96 © ISSN 1607-2510
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Existence Of Solutions For Elliptic BVPs On A Bounded Domain of $\mathbb{R}^N$ Via Some Fixed Point Theorems*

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Received 23 October 2016

Abstract

In this paper we present some existence results of solutions for elliptic boundary value problems on bounded domains of $\mathbb{R}^N (N \geq 1)$ depending on the behavior of the nonlinear term. We mainly use fixed point arguments by means of Krasnosel'skii's compression-expansion and Schauder's fixed point theorems.

1 Introduction

Let us consider the problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary and $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Existence and uniqueness of solutions of this problem has been investigated in many papers where different type of arguments were used (see [5] and references therein). In [4, 5] the compression-expansion fixed point theorem of Krasnosel'skii has been used as a basic tool to prove existence and localisation of positive solutions of Problem (1). We mention that the role of Harnack type inequalities for the application of Krasnosel'skii's theorem in cones was first shown in [2]. In [4], the authors reformulate the Problem (1) as a fixed point problem $Tu = u$ on the space of continuous functions where the operator is of the form $T = (-\Delta)^{-1}F$ with $(-\Delta)^{-1}$ the reverse of the Laplace's operator and F the Nemytskii operator. Inspired by [3] and [4], we will prove existence theorems for the given elliptic boundary value problem in the space of continuous functions according to the behavior of the nonlinear term. In section 2, we prove existence of a classical positive solution under some assumptions on f by using Krasnosel'skii's compression-expansion fixed point theorem. In section 3, we will use Schauder's fixed point theorem to prove existence of classical solutions

*Mathematics Subject Classifications: 47H10, 35J25.

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when f satisfies suitable general growth conditions. Let us recall some definitions and tools which are used in the sequel.

DEFINITION 1. By a *positive* solution of Problem (1) we mean a function $u \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ which satisfies (1) (with Δu in the sense of distributions), and $u(x) > 0$ for all $x \in \Omega$.

DEFINITION 2. A *superharmonic* function in a domain $\Omega \subset \mathbb{R}^N$ is a function $u \in C^1(\Omega, \mathbb{R})$ with $\Delta u \leq 0$ in the sense of distributions, i.e.,

$$\int_\Omega \nabla u \cdot \nabla v \geq 0 \quad \text{for every } v \in C_0^\infty(\Omega, \mathbb{R}) \quad \text{satisfying } v(x) \geq 0 \quad \text{on } \Omega.$$

DEFINITION 3. By a cone in a Banach space E we mean a closed convex subset $\mathcal{C}$ of E such that $\mathcal{C} \neq \{0\}$, $\lambda\mathcal{C} \subset \mathcal{C}$ for all $\lambda \in \mathbb{R}^+$, and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

THEOREM 1 (Krasnosel'skii's compression-expansion theorem [1]). Let E be a Banach space, $\mathcal{C} \subset E$ a cone and assume that $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a completely continuous map such that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:

(i) $\|Tu\| \leq \|u\|$ for $\|u\| = r$ and $\|Tu\| \geq \|u\|$ for $\|u\| = R$,

(ii) $\|Tu\| \geq \|u\|$ for $\|u\| = r$ and $\|Tu\| \leq \|u\|$ for $\|u\| = R$.

Then T has a fixed point u with $r \leq \|u\| \leq R$.

2 Existence of Positive Solution

To prove existence of a positive solution for the Problem (1), we give assumptions on the nonlinear term which allows us to apply the Krasnosel'skii's compression-expansion fixed point theorem. Let E be the Banach space defined by

$$C_0(\overline{\Omega}, \mathbb{R}) = \{u \in C(\overline{\Omega}, \mathbb{R}) : u = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $\|u\|_0 = \sup_{x\in\overline{\Omega}} |u(x)|$. Let K a subset of $\overline{\Omega}$; for a function $h : \overline{\Omega} \to \mathbb{R}$, by $h_{|_K}$ we mean the function $h_{|_K}(x) = h(x)$ if $x \in K$ and $h_{|_K}(x) = 0$ if $x \in \overline{\Omega} \setminus K$. We shall assume that the following *global weak Harnack inequality* holds:

(H) There exist a compact set $K \subset \Omega$ and a number $\eta > 0$ such that $u(x) \geq \eta\|u\|_0$ for all $x \in K$ and every nonnegative superharmonic function $u \in C^1(\overline{\Omega}, \mathbb{R})$ with $u = 0$ on $\partial\Omega$.

THEOREM 2. Assume that

(S$_1$) $f : \overline{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function and there exists $\sigma > 0$ with $\sigma \neq 1$ such that

$$f(x,u) \leq a(x) + b(x)u^{\sigma} \text{ for } (x,u) \in \overline{\Omega} \times \mathbb{R}^+,$$

where $a, b : \overline{\Omega} \longrightarrow \mathbb{R}^+$ are continuous and positive functions.

(S$_2$) there exists $R > 0$ such that

$$\min_{x \in K, u \in [R\eta, R]} f(x,u) > R \|(-\Delta)^{-1} 1_{|K}\|_0^{-1}.$$

(S$_3$)

$$M_2 (\frac{1}{\sigma M_2})^{\frac{\sigma}{\sigma-1}} - (\frac{1}{\sigma M_2})^{\frac{1}{\sigma-1}} + M_1 < 0 \text{ when } \sigma > 1,$$

where $M_1 = \|(-\Delta^{-1})a\|_0$ and $M_2 = \|(-\Delta^{-1})b\|_0$.

Then the Problem (1) has at least one positive solution.

PROOF. Let $F : C_0\left(\overline{\Omega}, \mathbb{R}\right) \longrightarrow C\left(\overline{\Omega}, \mathbb{R}\right)$ be the Nemytskii operator defined by $Fu(x) = f(x, u(x))$ and $T : \mathcal{C} \longrightarrow C_0(\overline{\Omega}, \mathbb{R})$ the operator given by $Tu = (-\Delta)^{-1}Fu$, where $\mathcal{C}$ is the cone defined by

$$\mathcal{C} = \{u \in C_0(\overline{\Omega}, \mathbb{R}) : u \geq 0 \text{ on } \overline{\Omega} \text{ and } \min_{x \in K} u(x) \geq \eta \|u\|_0\}.$$

One can see that u is a solution of Problem (1) if and only if

$$u = (-\Delta)^{-1} Fu,$$

that is, a solution of the Problem (1) is a fixed point of the operator T. In order to show that the operator T has a fixed point, we shall prove that the hypotheses of Theorem 1 are satisfied. It is clear that the operator T satisfies

$$\begin{cases} -\Delta(Tu) = f\left(x,u\right) & \text{in } \Omega, \\ Tu = 0 & \text{on } \partial\Omega. \end{cases}$$

Because Tu is superharmonic, then by the global weak Harnack inequality (H) we have $T(\mathcal{C}) \subset \mathcal{C}$. Indeed if $v \in T(\mathcal{C})$, there exists $u \in \mathcal{C}$ such that $v = Tu$. We have

$$\min_{x \in K} v(x) = \min_{x \in K} Tu(x) \geq \eta \left\|Tu\right\|_0 \geq \eta \|v\|_0.$$

Moreover, because f is continuous, then by Ascoli-Arzéla compactness criterion, it is easy to see that the operator T is compact.
Now, let r be a positive number which will be selected later and consider the two sets

$$S_1 = \left\{u \in C_0(\overline{\Omega}, \mathbb{R}) : \|u\|_0 < r\right\} \text{ and } S_2 = \left\{u \in C_0(\overline{\Omega}, \mathbb{R}) : \|u\|_0 < R\right\}.$$

Let $u \in \mathcal{C} \cap \partial S_1$, then using assumption (S_1) and monotonicity of the operator $(-\Delta)^{-1}$, we have

$$\begin{aligned}
\|Tu\|_0 &= \|(-\Delta)^{-1} Fu\|_0 \leq \|(-\Delta)^{-1}(a(.) + b(.)u^\sigma)\|_0 \\
&\leq \|(-\Delta)^{-1} a\|_0 + \|(-\Delta)^{-1} b(.)u^\sigma\|_0 \\
&\leq \|(-\Delta)^{-1} a\|_0 + \|\|u\|_0^\sigma (-\Delta)^{-1} b\|_0 \\
&\leq \|(-\Delta)^{-1} a\|_0 + r^\sigma \|(-\Delta)^{-1} b\|_0 \leq M_1 + r^\sigma M_2.
\end{aligned}$$

Note that when $\sigma < 1$, we have $r - M_1 - r^\sigma M_2 \longrightarrow +\infty$ as $r \longrightarrow +\infty$. Hence, if r is chosen sufficiently large ($r > R$), we get $M_1 + r^\sigma M_2 < r$. On the other hand, by using (S_3), we deduce that for $\sigma > 1$, there exists a suitable $r > 0$ which satisfies $M_1 + r^\sigma M_2 < r$. Then, for any $\sigma > 0$, $\sigma \neq 1$, we have $\|Tu\|_0 \leq r = \|u\|_0$.

Now, if $u \in \mathcal{C} \cap \partial S_2$, by using assumption ($\mathrm{S}_2$) and by monotonicity of the operator $(-\Delta)^{-1}$, we get

$$\begin{aligned}
\|Tu\|_0 &= \|(-\Delta)^{-1} Fu\|_0 \geq \|(-\Delta)^{-1} (Fu)_{|_K}\|_0 \\
&\geq \min_{x \in K, y \in [R\eta,\ R]} f(x,y) \sup_{x \in K} |(-\Delta)^{-1} 1_{|K})| \\
&\geq R \|(-\Delta)^{-1} 1_{|K})\|_0^{-1} \|(-\Delta)^{-1} 1_{|K})\|_0 \\
&= R = \|u\|_0.
\end{aligned}$$

Hence, $\|Tu\|_0 \geq \|u\|_0$ for $\|u\|_0 = R$. Then, using Krasnosel'skii's fixed point Theorem 1, the existence result holds.

3 General Existence Results

Now, we will use Schauder's fixed point theorem to prove existence of solutions for the Problem (1).

THEOREM 3. Assume that the nonlinear term f satisfies

$$|f(x,u)| \leq H(x, |u|) \text{ for } x \in \overline{\Omega} \text{ and } u \in \mathbb{R},$$

where $H : \overline{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies

(A_1) H is continuous, nondecreasing with respect to its second variable,

(A_2) there exists $M_0 > 0$ such that

$$H(x, M_0) \leq \frac{M_0}{\|(-\Delta)^{-1} 1_{\overline{\Omega}}\|_0} \text{ for all } x \in \Omega.$$

Then the Problem (1) has at least one solution.

PROOF. Let

$$\mathcal{D} = \overline{B}(0, M_0) = \left\{ u \in C_0 \left(\overline{\Omega}, \mathbb{R} \right) : \|u\|_0 \leq M_0 \right\}$$

and

$$T:\mathcal{D}\longrightarrow C_0\left(\overline{\Omega},\mathbb{R}\right),\quad Tu=(-\Delta)^{-1}Fu.$$

T is compact and we have for every $v\in T(\mathcal{D})$ there exists $u\in\mathcal{D}$ such that $v=Tu$. By the monotonicity of $(-\Delta)^{-1}$ and because H is nondecreasing with respect to its second argument we have,

$$\begin{aligned}\|v\|_0 &= \|Tu\|_0=\|(-\Delta)^{-1}Fu\|_0\le\|(-\Delta)^{-1}H(.,|u|)\|_0\\ &\le \|(-\Delta)^{-1}H(.,M_0)\|_0\le\left\|(-\Delta)^{-1}\left(\frac{M_0}{\|(-\Delta)^{-1}1_{\overline{\Omega}}\|_0}\right)\right\|_0\\ &\le \frac{M_0}{\|(-\Delta)^{-1}1_{\overline{\Omega}}\|_0}\|(-\Delta)^{-1}1_{\overline{\Omega}}\|_0=M_0,\end{aligned}$$

meaning that $v\in\mathcal{D}$, i.e., $T(\mathcal{D})\subset\mathcal{D}$. By Schauder's fixed point theorem, we deduce that the operator T has a fixed point in $\mathcal{D}$. The proof is complete.

THEOREM 4. Assume now that

(H_1) There exist a nondecreasing function $\Phi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and a function $\psi\in C\left(\overline{\Omega},\mathbb{R}^+\right)$, such that

$$|f(x,u)|\le\psi(x)\Phi(|u|)\text{ for }(x,u)\in\overline{\Omega}\times\mathbb{R},$$

(H_2) there exists $R_0>0$ such that

$$\Phi(R_0)\le\frac{R_0}{\|(-\Delta)^{-1}\psi\|_0}.$$

Then the Problem (1) has at least one solution.

PROOF. Let's consider the compact operator

$$T:\mathcal{D}=\overline{B}(0,R_0)=\left\{u\in C_0\left(\overline{\Omega},\mathbb{R}\right):\|u\|_0\le R_0\right\}\longrightarrow C_0\left(\overline{\Omega},\mathbb{R}\right)$$

defined by

$$Tu=(-\Delta)^{-1}Fu\text{ for }u\in\mathcal{D}.$$

We have $T(\mathcal{D})\subset\mathcal{D}$. Indeed, $v\in T(\mathcal{D})$ implies that there exists $u\in C_0\left(\overline{\Omega},\mathbb{R}\right)$ with $\|u\|\le R_0$ and $v=Tu$. Using assumptions (H_1), (H_2) and the monotonicity of the operator $(-\Delta)^{-1}$ we get

$$\begin{aligned}\|v\|_0 &= \|Tu\|_0=\|(-\Delta)^{-1}Fu\|_0\le\left\|(-\Delta)^{-1}\psi(.)\Phi(|u|)\right\|_0\\ &\le \left\|(-\Delta)^{-1}\psi(.)\Phi(\|u\|_0)\right\|_0\le\left\|(-\Delta)^{-1}\psi(.)\Phi(R_0)\right\|_0\\ &\le \Phi(R_0)\|(-\Delta)^{-1}\psi\|_0\le R_0,\end{aligned}$$

that is $T(\mathcal{D})\subset\mathcal{D}$. Hence, by Schauder's fixed point theorem, T has a fixed point, which is a solution of the Problem (1).

REMARK 1. If in Theorems 3 and 4, we assume that there exists $x_0\in\overline{\Omega}$ such that $f(x_0,0)\ne 0$, then the solutions obtained are nontrivial.

Acknowledgment. The authors are thankful to the anonymous referee for his/her careful reading of the original manuscript, which led to substantial improvements.

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