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Statistical Relative Uniform Convergence Of Double Sequences Of Positive Linear Operators*

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Abstract

In the present paper, we introduce the concept of A-statistical relative uniform convergence for double sequences of functions defined on a compact subset of the real two-dimensional space. Based upon this definition, we prove Korovkin-type approximation theorem. Finally, we compute the rate of convergence.

1 Introduction

Korovkin type approximation theorems are useful tools to check whether a given sequence (L_n) of positive linear operators on $C[a,b]$ of all continuous functions on the real interval $[a,b]$ is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1960 [17] for the functions 1, x and x^2 in the space $C[a,b]$ as well as for the functions 1, cos and sin in the space of all continuous 2π-periodic functions on the real line. Several mathematicians have worked on extending or generalizing the Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, summability theory and partial differential equations.

In recent years, Korovkin theory has been quite improved by some efficient tools in mathematics such as the concept of statistical convergence from summability theory, the fuzzy logic theory, the complex functions theory, the theory of q-calculus, and the theory of fractional analysis. E. H. Moore [18] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, E. W. Chittenden [5] gave the following definition of relatively uniform convergence is equivalent to the definition given by Moore:

A sequence (f_n) of functions, defined on any compact subset of the real space, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x)$,

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called a scale function $\sigma(x)$ such that for every $\varepsilon > 0$ there exists an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$|f_n(x) - f(x)| < \varepsilon |\sigma(x)|$$

holds uniformly in x. The sequence (f_n) is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniformly*.

It is observed that uniform convergence is the special case of relatively uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [4, 5, 6]).

Similarly, we can give the following definition for double sequences of functions:

A double sequence (f_{mn}) of functions, defined on any compact subset of the real two-dimensional space, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x,y)$, called a scale function $\sigma(x,y)$ such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $n, m > n_\varepsilon$ the inequality

$$|f_{mn}(x,y) - f(x,y)| < \varepsilon |\sigma(x,y)|$$

holds uniformly in (x,y). The double sequence (f_{mn}) is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniformly*.

EXAMPLE 1. For each $(m,n) \in \mathbb{N}^2$, define $f_{mn} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f_{mn}(x,y) = \frac{2nmxy}{1 + n^2m^2x^2y^2}.$$

This sequence does not converge uniformly, but converges to $f = 0$ uniformly relative to a scale function

$$\sigma(x,y) = \begin{cases} \frac{1}{xy} & \text{if } (x,y) \in (0,1] \times (0,1], \\ 1 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Let $A = (a_{jkmn})$ be a four-dimensional summability method. For a given double sequence $x = (x_{mn})$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{m,n=1,1}^{\infty,\infty} a_{jkmn} x_{mn}$$

provided the double series converges in the Pringsheim's sense for $(j,k) \in \mathbb{N}^2$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions ([15]). In 1926 Robison [20] presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, *RH*-regularity ([14], [20]).

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robinson- Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if

(i) $P - \lim_{j,k} a_{jkmn} = 0$ for each m and n,

(ii) $P - \lim_{j,k} \sum\limits_{m,n=1,1}^{\infty,\infty} a_{jkmn} = 1$,

(iii) $P - \lim_{j,k} \sum\limits_{m=1}^{\infty} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{j,k} \sum\limits_{n=1}^{\infty} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum\limits_{m,n=1,1}^{\infty,\infty} |a_{jkmn}|$ is P-convergent,

(vi) there exits finite positive integers A and B such that $\sum\limits_{m,n>B} |a_{jkmn}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Then A-*density of* K, denoted by $\delta^2_{(A)}(K)$, is given by:

$$\delta^2_{(A)}(K) := P - \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{mn})$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta^2_{(A)}(\{(m,n)\mathbb{N} : |x_{mn} - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st^2_{(A)} - \lim x = L$ ([19]).

Let f and f_{mn} belong to $C(D)$, which is the space of all continuous real valued functions on a compact subset D of the real two-dimensional space and $\|f\|_{C(D)}$ denotes the usual supremum norm of f in $C(D)$.

DEFINITION 1 ([13]). (f_{mn}) is said to be statistically pointwise convergent to f on D if $st^2_{(A)} - \lim\limits_{m,n} f_{mn}(x,y) = f(x,y)$ for each $(x,y) \in D$, i.e., for every $\varepsilon > 0$ and for each $(x,y) \in D$,

$$\delta^2_{(A)}(\{(m,n) : |f_{mn}(x,y) - f(x,y)| \geq \varepsilon\}) = 0.$$

Then, it is denoted by $f_{mn} \to f$ $(stat)$ on D.

DEFINITION 2 ([13]). (f_{mn}) is said to be statistically uniform convergent to f on D if either

$$st^2_{(A)} - \lim_{m,n} \sup_{(x,y)\in D} |f_{mn}(x,y) - f(x,y)| = 0,$$

or

$$\delta^2_{(A)}\left(\left\{(m,n)\ : \|f_{mn} - f\|_{C(D)} \geq\ \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$. This limit is denoted by $f_{mn} \rightrightarrows f\ (stat)$ on D.

DEFINITION 3. (f_{mn}) is said to be statistically *relatively uniform* convergent to f on D if there exists a function $\sigma(x,y)$, $|\sigma(x,y)| > 0$, called a scale function $\sigma(x,y)$ such that for every $\varepsilon > 0$,

$$\delta^2_{(A)}\left(\left\{(m,n)\ :\ \sup_{(x,y)\in D}\left|\frac{f_{mn}(x,y) - f(x,y)}{\sigma(x,y)}\right| \geq\ \varepsilon\right\}\right) = 0.$$

This limit is denoted by $(st)^2_{(A)} - f_{mn} \rightrightarrows f\ (D;\sigma)$.

Using the above definitions, the next result follows immediately.

LEMMA 1. $f_{mn} \rightrightarrows f$ on D (in the ordinary sense) implies $f_{mn} \rightrightarrows f\ (stat)$ on D, which also implies $(st)^2_{(A)} - f_{mn} \rightrightarrows f\ (D;\sigma)$.

However, one can construct an example which guarantees that the converses of Lemma 1 are not always true. Such an example is presented as follows:

EXAMPLE 2. For each $(m,n) \in \mathbb{N}^2$, define $g_{mn} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g_{mn}(x,y) = \frac{2n^2m^2xy}{1 + n^3m^3x^2y^2}$$

Take $A = C(1,1)$, four dimensional Cesáro matrix. Then observe that

$$(st)^2_{(C(1,1))} - g_{mn} \rightrightarrows g = 0([0,1] \times [0,1];\sigma)$$

and

$$\sigma(x,y) = \begin{cases} \frac{1}{xy} & \text{if } (x,y) \in (0,1] \times (0,1], \\ 1 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

However, (g_{mn}) is not statistically (or ordinary) uniform convergent to the function $g = 0$ on the interval $[0,1] \times [0,1]$.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [12] and Steinhaus [21] independently in the same year 1951. Some Korovkin-type theorems in the setting of a statistical convergence were given by [2, 3, 7, 8, 9, 11, 16, 23]. In the present paper, using the Definition 3 we prove Korovkin-type approximation theorem for double sequences of functions defined on a compact subset of the real two-dimensional space.

2 A Korovkin-Type Approximation Theorem for Double Sequences

In this section we apply the notion of statistical uniform convergence of a double sequence of functions relative to a scale function to prove a Korovkin type approximation theorem.

Let L be a linear operator from $C(D)$ into itself. Then, as usual, we say that L is positive provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $(x, y) \in D$ by $L(f(u, v); x, y)$ or, briefly, $L(f; x, y)$.

First we recall the classical case of the Korovkin-type result for a double sequence introduced in [22].

THEOREM 1. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,

$$P - \lim_{m,n} \|L_{mn}(f) - f\|_{C(D)} = 0$$

if and only if

$$P - \lim_{m,n} \|L_{mn}(e_i) - e_i\|_{C(D)} = 0, \quad (i = 0, 1, 2, 3),$$

where $e_0(x, y) = 1$, $e_1(x, y) = x$, $e_2(x, y) = y$ and $e_3(x, y) = x^2 + y^2$.

Now we recall the statistical case of the Korovkin-type result introduced in [11],

THEOREM 2. Let $A = (a_{jkmn})$ be a nonnegative *RH*-regular summability matrix method. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,

$$st^2_{(A)} - \lim_{m,n} \|L_{mn}(f) - f\|_{C(D)} = 0$$

if and only if

$$st^2_{(A)} - \lim_{m,n} \|L_{mn}(e_i) - e_i\|_{C(D)} = 0, \quad (i = 0, 1, 2, 3),$$

where $e_0(x, y) = 1$, $e_1(x, y) = x$, $e_2(x, y) = y$ and $e_3(x, y) = x^2 + y^2$.

Now we have the following main result.

THEOREM 3. Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix method. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,

$$(st)^2_{(A)} - L_{mn}(f) \rightrightarrows f \ (D; \sigma) \tag{1}$$

if and only if

$$(st)^2_{(A)} - L_{mn}(e_i) \rightrightarrows e_i \ (D; \sigma_i),\ i = 0, 1, 2, 3 \tag{2}$$

where $\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}$, $|\sigma_i(x,y)| > 0$ and $\sigma_i(x,y)$ is unbounded for $i = 0,1,2,3$.

PROOF. Since each $f_i \in C(D), (i = 0,1,2,3)$, the implication (1)→(2) is obvious. We prove the converse part. By the continuity of f on compact set D, we can write

$$|f(x,y)| \leq K$$

where $K := \|f\|_{C(D)}$. Also, since f is continuous on D, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ for all $(u,v) \in D$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, putting $\varphi(u,v) = (u-x)^2 + (v-y)^2$, we get

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2K}{\delta^2}\varphi(u,v).$$

This means that

$$-\varepsilon - \frac{2K}{\delta^2}\varphi(u,v) < f(u,v) - f(x,y) < \varepsilon + \frac{2K}{\delta^2}\varphi(u,v).$$

Since $L_{mn}(f;x,y)$ is monotone and linear, we obtain

$$\begin{aligned} L_{mn}(e_0;x,y)\left(-\varepsilon - \frac{2K}{\delta^2}\varphi(u,v)\right) &< L_{mn}(e_0;x,y)\left(f(u,v) - f(x,y)\right) \\ &< L_{mn}(e_0;x,y)\left(\varepsilon + \frac{2K}{\delta^2}\varphi(u,v)\right). \end{aligned}$$

Note that x,y are fixed and so $f(x,y)$ is a constant number. Therefore,

$$\begin{aligned} -\varepsilon L_{mn}(e_0;x,y) - \frac{2K}{\delta^2}L_{mn}(\varphi;x,y) &< L_{mn}(f;x,y) - f(x,y)\,L_{mn}(e_0;x,y) \\ &< \varepsilon L_{mn}(e_0;x,y) + \frac{2K}{\delta^2}L_{mn}(\varphi;x,y). \end{aligned} \tag{3}$$

Also

$$\begin{aligned} & L_{mn}(f;x,y) - f(x,y) \\ = \; & L_{mn}(f;x,y) - f(x,y)\,L_{mn}(e_0;x,y) + f(x,y)\,L_{mn}(e_0;x,y) - f(x,y) \\ = \; & [L_{mn}(f;x,y) - f(x,y)\,L_{mn}(e_0;x,y)] + f(x,y)\,[L_{mn}(e_0;x,y) - e_0(x,y)]. \end{aligned} \tag{4}$$

By (3) and (4), we get

$$\begin{aligned} & L_{mn}(f;x,y) - f(x,y) \\ < \; & \varepsilon L_{mn}(e_0;x,y) + \frac{2K}{\delta^2}\{[L_{mn}(e_3;x,y) - e_3(x,y)] \\ & -2x\,[L_{mn}(e_1;x,y) - e_1(x,y)] - 2y[L_{mn}(e_2;x,y) - e_2(x,y)] \\ & +(x^2+y^2)\,[L_{mn}(e_0;x,y) - e_0(x,y)]\} \\ & +f(x,y)\,[L_{mn}(e_0;x,y) - e_0(x,y)] \end{aligned}$$

$$\begin{aligned}
&= \varepsilon\left[L_{mn}(e_0;x,y)-e_0\left(x,y\right)\right]+\varepsilon+\frac{2K}{\delta^2}\left\{\left[L_{mn}(e_3;x,y)-e_3\left(x,y\right)\right]\right.\\
&\quad -2x\left[L_{mn}(e_1;x,y)-e_1\left(x,y\right)\right]+-2y\left[L_{mn}(e_2;x,y)-e_2\left(x,y\right)\right]+\\
&\quad \left.\left(x^2+y^2\right)\left[L_{mn}(e_0;x,y)-e_0\left(x,y\right)\right]\right\}+f\left(x,y\right)\left[L_{mn}(e_0;x,y)-e_0\left(x,y\right)\right]
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|L_{mn}\left(f;x,y\right)-f(x,y)\right|\\
\leq\ & \varepsilon+\left(\varepsilon+K+\frac{2K\left\|e_3\right\|_{C(D)}}{\delta^2}\right)\left|L_{mn}(e_0;x,y)-e_0(x,y)\right|\\
&+\frac{4K\left\|e_1\right\|_{C(D)}}{\delta^2}\left|L_{mn}(e_1;x,y)-e_1(x,y)\right|+\frac{4K\left\|e_2\right\|_{C(D)}}{\delta^2}\left|L_{mn}(e_2;x,y)-e_2(x,y)\right|\\
&+\frac{2K}{\delta^2}\left|L_{mn}(e_3;x,y)-e_3(x,y)\right|\\
\leq\ & \varepsilon+M\left\{\left|L_{mn}(e_0;x,y)-e_0(x,y)\right|+\left|L_{mn}(e_1;x,y)-e_1(x,y)\right|\right.\\
&\left.+\left|L_{mn}(e_2;x,y)-e_2(x,y)\right|+\left|L_{mn}(e_3;x,y)-e_3(x,y)\right|\right\}
\end{aligned}$$

where

$$M=\varepsilon+K+\frac{2K}{\delta^2}\left(\left\|e_3\right\|_{C(D)}+2\left\|e_2\right\|_{C(D)}+2\left\|e_1\right\|_{C(D)}+1\right).$$

We get

$$\begin{aligned}
&\sup_{(x,y)\in D}\left|\frac{L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\\
\leq\ & \sup_{(x,y)\in D}\frac{\varepsilon}{\left|\sigma(x,y)\right|}+M\left\{\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_0;x,y)-e_0\left(x,y\right)}{\sigma_0(x,y)}\right|\right.\\
&+\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_1;x,y)-e_1\left(x,y\right)}{\sigma_1(x,y)}\right|+\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_2;x,y)-e_2\left(x,y\right)}{\sigma_2(x,y)}\right|\\
&\left.+\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_3;x,y)-e_3\left(x,y\right)}{\sigma_3(x,y)}\right|\right\} \qquad (5)
\end{aligned}$$

where $\sigma(x)=\max\left\{\left|\sigma_i\left(x\right)\right|;i=0,1,2,3\right\}$. Now, for a given $r>0$, choose $\varepsilon>0$ such that $\sup_{(x,y)\in D}\frac{\varepsilon}{|\sigma(x,y)|}<r$. Then,

$$R:=\left\{(m,n):\sup_{(x,y)\in D}\left|\frac{L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\geq r\right\}$$

and

$$R_i:=\left\{(m,n):\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_i;x,y)-e_i(x,y)}{\sigma_i(x,y)}\right|\geq\frac{r-\sup_{(x,y)\in D}\frac{\varepsilon}{|\sigma(x,y)|}}{3M}\right\},\ i=0,1,2,3$$

It follows from (5) that $R \subset \bigcup_{i=0}^{3} R_i$ and so

$$\sum_{(m,n)\in R} a_{jkmn} \leq \sum_{(m,n)\in R_1} a_{jkmn} + \sum_{(m,n)\in R_2} a_{jkmn} + \sum_{(m,n)\in R_3} a_{jkmn}.$$

Then using the hypothesis (2), we get

$$(st)^2_{(A)} - L_{mn}(f) \rightrightarrows f \ (D;\sigma),$$

where $\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}$. This completes the proof of the theorem.

Now let $D = [0,1] \times [0,1] \subset \mathbb{R}^2$ and $A = C(1,1) := (c_{jkmn})$, the double Cesaro matrix, defined by

$$c_{jkmn} = \begin{cases} \frac{1}{jk} & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the double Bernstein polynomials

$$B_{mn}(f;x,y) = \sum_{s=0}^{m}\sum_{t=0}^{n} f\left(\frac{s}{m},\frac{t}{n}\right) x^s y^t (1-x)^{m-s}(1-y)^{n-t}$$

on $C(D)$. Using these polynomials, we introduce the following positive linear operators on $C(D)$:

$$P_{mn}(f;x,y) = (1+g_{mn}(x,y))B_{mn}(f;x,y), \ \ (x,y) \in D \text{ and } f \in C(D), \qquad (6)$$

where $g_{mn}(x,y)$ is given in Example 2. Then, we observe that

$$\begin{aligned} P_{mn}(e_0;x,y) &= (1+g_{mn}(x,y))e_0(x,y), \\ P_{mn}(e_1;x,y) &= (1+g_{mn}(x,y))e_1(x,y), \\ P_{mn}(e_2;x,y) &= (1+g_{mn}(x,y))e_2(x,y), \\ P_{mn}(e_3;x,y) &= (1+g_{mn}(x,y))\left[e_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n}\right]. \end{aligned}$$

Since $(st)^2_{(A)} - g_{mn} \rightrightarrows g = 0 \ (D;\sigma)$,

$$\sigma(x,y) = \begin{cases} \frac{1}{xy} & \text{if } (x,y) \in (0,1] \times (0,1], \\ 1 & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

we conclude that

$$(st)^2_{(A)} - P_{mn}(e_i) \rightrightarrows e_i \ (D;\sigma) \text{ for each } i = 0,1,2.$$

So, by Theorem 3, we immediately see that

$$(st)^2_{(A)} - P_{mn}(f) \rightrightarrows f \ \ (D;\sigma) \text{ for all } f \in C(D).$$

However, since (g_{mn}) is not statistically uniform convergent to the function $g = 0$ on the compact set D, we can say that Theorem 2 does not work for our operators defined by (6). Furthermore, since (g_{mn}) is not uniformly convergent (in the ordinary sense) to the function $g = 0$ on D, the classical Korovkin theorem does not work either. Therefore, this application clearly shows that our Theorem 3 is a non-trivial generalization of the classical and the statistical cases of the Korovkin results introduced in [22] and [11], respectively.

3 Rates of Statistical Relative Uniform Convergence in Theorem 3

In this section we study the rates of statistical relative uniform convergence of a sequence of positive linear operators defined on $C(D)$ with the help of modulus of continuity. We now present the following definition.

DEFINITION 4. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and let (α_{mn}) be a positive non-increasing double sequence. A double sequence (f_{mn}) is said to converge *statistically relatively uniform to the scale function* $\sigma(x, y)$, $|\sigma(x, y)| > 0$, to f on D with the rate of $o(\alpha_{mn})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k} \frac{1}{\alpha_{jk}} \sum_{(m,n)\in K(\varepsilon)} a_{jkmn} = 0,$$

where

$$K(\varepsilon) = \left\{ (m, n) : \sup_{(x,y)\in D} \left| \frac{f_{mn}(x, y) - f(x, y)}{\sigma(x, y)} \right| \geq \varepsilon \right\}.$$

In this case, it is denoted by

$$(st)^2_{(A)} - (f_{mn} - f) = o(\alpha_{mn})\ (D; \sigma).$$

DEFINITION 5. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and let (α_{mn}) be a positive non-increasing double sequence. A double sequence (f_{mn}) is said to converge *statistically relatively uniform to the scale function* $\sigma(x, y)$, $|\sigma(x, y)| > 0$, to f on D with the rate of $o_{mn}(\alpha_{mn})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n)\in M(\varepsilon)} a_{jkmn} = 0,$$

where

$$M(\varepsilon) = \left\{ (m, n) : \sup_{(x,y)\in D} \left| \frac{f_{mn}(x, y) - f(x, y)}{\sigma(x, y)} \right| \geq \varepsilon\alpha_{mn} \right\}.$$

In this case, it is denoted by

$$(st)^2_{(A)} - (f_{mn} - f) = o_{mn}(\alpha_{mn})\ (D; \sigma).$$

Then we first need the following lemma to get the rates of convergence in Theorem 3 by using Definition 4.

LEMMA 2. Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix method. Let (f_{mn}) and (g_{mn}) be function sequences belonging to $C(D)$. Assume that $(st)^2_{(A)} - (f_{mn} - f) = o(\alpha_{mn})$ $(D; \sigma_0)$ and $(st)^2_{(A)} - (g_{mn} - g) = o(\beta_{mn})(D; \sigma_1)$, $|\sigma_i(x, y)| > 0$, $i = 0, 1$. Let $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$. Then the following statements hold:

(i) $(st)^2_{(A)} - (f_{mn} + g_{mn}) - (f + g) = o(\gamma_{mn})$ $(D; \max\{|\sigma_i(x, y)|; i = 0, 1\})$

(ii) $(st)^2_{(A)} - (f_{mn} - f)(g_{mn} - g) = o(\gamma_{mn})$ $(D; \sigma_0(x, y)\, \sigma_1(x, y))$,

(iii) $(st)^2_{(A)} - (\lambda(f_{mn} - f)) = o(\alpha_{mn})$ $(D; \sigma_0(x, y))$ for any real number λ,

(iv) $(st)^2_{(A)} - \sqrt{|f_{mn} - f|} = o(\alpha_{mn})$ $(D; \sqrt{|\sigma_0(x, y)|})$.

PROOF. (i) Assume that $(st)^2_{(A)} - (f_{mn} - f) = o(\alpha_{mn})$ $(D; \sigma_0)$ and that $(st)^2_{(A)} - (g_{mn} - g) = o(\beta_{mn})(D; \sigma_1)$. Also, for every $\varepsilon > 0$ define the following sets:

$$\begin{aligned}
\Psi &: = \left\{ (m, n) : \sup_{(x,y) \in D} \left| \frac{(f_{mn} + g_{mn})(x, y) - (f + g)(x, y)}{\sigma(x, y)} \right| \geq \varepsilon \right\}, \\
\Psi_1 &: = \left\{ (m, n) : \sup_{(x,y) \in D} \left| \frac{f_{mn}(x, y) - f(x, y)}{\sigma_0(x, y)} \right| \geq \frac{\varepsilon}{2} \right\}, \\
\Psi_2 &: = \left\{ (m, n) : \sup_{(x,y) \in D} \left| \frac{g_{mn}(x, y) - g(x, y)}{\sigma_1(x, y)} \right| \geq \frac{\varepsilon}{2} \right\},
\end{aligned}$$

where $\sigma(x, y) = \max\{|\sigma_i(x, y)|; i = 0, 1\}$. Then observe that

$$\Psi \subset \Psi_1 \cup \Psi_2. \tag{7}$$

Therefore, since $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$, we conclude that, for all $(j, k) \in \mathbb{N}^2$,

$$\frac{1}{\gamma_{jk}} \sum_{(m,n) \in \Psi} a_{jkmn} \leq \frac{1}{\alpha_{jk}} \sum_{(m,n) \in \Psi_1} a_{jkmn} + \frac{1}{\beta_{jk}} \sum_{(m,n) \in \Psi_2} a_{jkmn}. \tag{8}$$

Now by taking limit as $j, k \to \infty$ in (8) and using the hypotheses, we conclude that

$$P - \lim_{j,k} \frac{1}{\gamma_{jk}} \sum_{(m,n) \in \Psi} a_{jkmn} = 0,$$

which completes the proof of (i). Since the proofs of (ii)–(iv) are similar, we omit them.

On the other hand, we recall that the modulus of continuity of a function $f \in C(D)$ is defined by

$$w(f,\delta) = \sup\left\{|f(u,v) - f(x,y)| : (u,v),(x,y) \in D, \sqrt{(u-x)^2 + (v-y)^2} \leq \delta\right\}.$$

where $\delta > 0$. Then we have the following result.

THEOREM 4. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Assume that the following conditions hold:

(a) $(st)^2_{(A)} - (L_{mn}(e_0) - e_0) = o(\alpha_{mn})\ (D;\sigma_0)$,

(b) $(st)^2_{(A)} - w(f,\delta_{mn}) = o(\beta_{mn})(D;\sigma_1)$ where $\delta_{mn} := \sqrt{\|L_{mn}(\varphi)\|_{C(D)}}$ with

$$\varphi(u,v) = \varphi_{xy}(u,v) = (u-x)^2 + (v-y)^2.$$

Then we have that, for all $f \in C(D)$,

$$(st)^2_{(A)} - (L_{mn}(f) - f) = o(\gamma_{mn})(D; |\sigma_0(x,y)\sigma_1(x,y)|)$$

and

$$\sigma(x,y) = \max\{|\sigma_0(x,y)|,\ |\sigma_1(x,y)|,\ |\sigma_0(x,y)\sigma_1(x,y)|\},$$

where $\gamma_{mn} = \max\{\alpha_{mn},\beta_{mn}\}$, $|\sigma_i(x,y)| > 0$ and $\sigma_i(x,y)$ is unbounded for $i = 0,1$.

PROOF. Let $f \in C(D)$ and $(x,y) \in D$. Then it is well-known that

$$\begin{aligned} |L_{mn}(f;x,y) - f(x,y)| \ &\leq\ w(f,\delta_{mn})\,|L_{mn}(e_0;x,y) - e_0(x,y)| \\ &\quad +2w(f,\delta_{mn}) + M\,|L_{mn}(e_0;x,y) - e_0(x,y)| \end{aligned}$$

where $M = \|f\|_{C(D)}$ and (see, for instance, [1, 10]). This yields that

$$\begin{aligned} &\sup_{(x,y)\in D}\left|\frac{L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)}\right| \\ \leq\ &\sup_{(x,y)\in D}\frac{w(f,\delta_{mn})}{|\sigma_1(x,y)|}\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)}\right| + 2\sup_{(x,y)\in D}\frac{w(f,\delta_{mn})}{|\sigma_1(x,y)|} \\ &+M\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)}\right| \end{aligned}$$

Now given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} K\ &:\ = \left\{(m,n) : \sup_{(x,y)\in D}\left|\frac{L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)}\right| \geq \varepsilon\right\} \\ K_1\ &:\ = \left\{(m,n) : \sup_{(x,y)\in D}\frac{w(f,\delta_{mn})}{|\sigma_1(x,y)|}\sup_{(x,y)\in D}\left|\frac{L_{mn}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)}\right| \geq \frac{\varepsilon}{3}\right\} \end{aligned}$$

$$K_2 \quad : \quad = \left\{ (m,n) : \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{|\sigma_1(x,y)|} \geq \frac{\varepsilon}{6} \right\}$$
$$K_3 \quad : \quad = \left\{ (m,n) : \sup_{(x,y)\in D} \left| \frac{L_{mn}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)} \right| \geq \frac{\varepsilon}{3M} \right\}.$$

Then, it is easily see that $K \subset K_1 \cup K_2 \cup K_3.$ Also, defining

$$K_4 \quad : \quad = \left\{ (m,n) : \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{|\sigma_1(x,y)|} \geq \sqrt{\frac{\varepsilon}{3}} \right\}$$
$$K_5 \quad : \quad = \left\{ (m,n) : \sup_{(x,y)\in D} \left| \frac{L_{mn}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)} \right| \geq \sqrt{\frac{\varepsilon}{3}} \right\},$$

we have $K_1 \subset K_4 \cup K_5$, which yields $K \subseteq \bigcup_{i=2}^{5} K_i$. Therefore, since $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$, we conclude that, for all $(j,k) \in \mathbb{N}^2$,

$$\begin{aligned} \frac{1}{\gamma_{jk}} \sum_{(m,n)\in K} a_{jkmn} \quad &\leq \quad \frac{1}{\beta_{jk}} \sum_{(m,n)\in K_2} a_{jkmn} + \frac{1}{\alpha_{jk}} \sum_{(m,n)\in K_3} a_{jkmn} \\ &+ \frac{1}{\beta_{jk}} \sum_{(m,n)\in K_4} a_{jkmn} + \frac{1}{\alpha_{jk}} \sum_{(m,n)\in K_5} a_{jkmn}. \end{aligned} \tag{9}$$

Letting $j,k \to \infty$ on both sides of (9), we get

$$P - \lim_{j,k} \frac{1}{\gamma_{jk}} \sum_{(m,n)\in K} a_{jkmn} = 0.$$

Therefore, the proof is completed.

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