

# Resonance States Completeness For A Model Of The Helmholtz Resonator With Line-Like Window\*

Igor Yurievich Popov<sup>†</sup>

16 July 2017

## Abstract

A model of the Helmholtz resonator with narrow slit is considered. The construction is based on the theory of self-adjoint extensions of symmetric operators. Resonance states are described. It is proved that the system of the resonance states is complete in  $L_2(\Omega)$ , where  $\Omega$  is the convex hull of the resonator with window.

## 1 Introduction

The Helmholtz resonator is an open resonator with small boundary window. The problem of resonances and resonance states for the Helmholtz resonator attracted great attention starting from famous Lord Rayleigh work [1]. But rigorous mathematical description of the problem was given only in the second half of 20-th century. Particularly, it became clear that resonances (not only for the Helmholtz resonator but for any bounded scatterer) are eigenvalues of some dissipative operator (see [2, 3, 4] and references in [5]). A few models and asymptotic approaches to the problem were developed (see, e.g., [6, 7, 8, 9] and references therein). One of the intriguing question in this field is: What is the minimal domain  $\Omega$  which gives one the completeness of the resonance states in  $L_2(\Omega)$ ? Our hypothesis is that it is the convex hull of the scatterer. It is not yet proved. There are only some examples of solved particular problems [10, 11]. There is an interesting relation between the scattering problem and functional model [12, 13, 14, 15, 16]. More precisely, the completeness is related to the factorization of the scattering matrix (correspondingly, characteristic function for the functional model). This model gives one an instrument for investigation of the analogous completeness problem for quantum graph with semi-infinite edges. A few results were obtained in this field [17, 18]. For quantum graph the problem reduces to the factorization of a scalar function.

The completeness of resonance states problem is not purely mathematical question. It corresponds to important physical property. The resonance states are related to the scattering problem. The completeness of resonance states in  $L_2(\Omega)$  for some domain  $\Omega$  means that any state (i.e., any vibration in acoustical or electro-magnetic cases) can be

---

\*Mathematics Subject Classifications: 47B38, 47E05.

<sup>†</sup>Department of Higher Mathematics, ITMO University, Kronverkskiy, 49, St. Petersburg 197101, Russia

excited by incoming plane waves. Conversely, the incompleteness means that there are states which can not be excited by incoming waves. This property is very important for physical applications.

## 2 Model Construction and Main Results

Consider convex bounded domain  $\Omega^{in}$  with smooth boundary  $\partial\Omega^{in}$ . Let  $L, L \in \partial\Omega^{in}$ , be smooth bounded curve without self-intersections. To switch on the interaction between  $\Omega^{in}$  and the external domain  $\Omega^{ex} = \mathbb{R}^3 \setminus \Omega^{in}$  through line-like window  $L$  we use so-called "restriction-extension" procedure (see, e.g., [8, 19, 20, 21, 22, 23, 24]). Let us start from the Neumann Laplacian in  $\Omega^{in}$ . Restrict this operator onto the set of smooth functions vanishing at  $L$ . The closure  $-\Delta_0$  of this restriction is a symmetric operator with infinite deficiency indices. To describe the operator extensions, we need the description of the adjoint operator domain. It is given by the following Lemma.

LEMMA 1 ([8]). The domain of the operator  $-\Delta_0^*$  is as follows

$$D(-\Delta_0^*) : u(x) = \int_L (\xi^{in}(s)h_{k_0}^{in}(x, s) + \xi^{ex}(s)h_{k_0}^{ex}(x, s))ds + u_0(x),$$

$$h_{k_1}^{in,ex}(x, s) = \begin{cases} G^{in,ex}(x, x(s), k_0), & x \in \Omega^{in,ex}, \\ 0 & x \in \Omega^{ex,in}, \end{cases}$$

$$x(s) \in L, u_0 \in W_2^2(\Omega^{in}) \oplus W_2^2(\Omega^{ex}), \xi^{in,ex} \in W_2^{-1}(L).$$

Here  $G^{in,ex}(x, y, k_0)$  is the Green function for  $\Omega^{in,ex}$  corresponding to some regular value  $k_0^2 < 0$  (i.e.  $\Re k_0 = 0$ ).

Self-adjoint extensions are restrictions of the adjoint operator. We are not interested in the whole set of such extensions. We consider only one (the most natural) self-adjoint extension given by the following Lemma.

LEMMA 2. The restriction of  $-\Delta_0^*$  with the following domain gives one a self-adjoint operator:

$$D(-\Delta) = \{u : u \in D(-\Delta_0^*), \xi^{in} = -\xi^{ex}, u_0^{in}|_L = u_0^{ex}|_L\}. \quad (1)$$

To obtain the scattering matrix, we construct the solution of the scattering problem.

LEMMA 3. The solution of the scattering problem has the form

$$\psi(x, \nu, k) = \begin{cases} \int_L G^{in}(x, x(s), k)\alpha^{in}(s)ds, & x \in \Omega^{in}, \\ \psi^{ex}(x, \nu, k) + \int_L G^{ex}(x, x(s), k)\alpha^{ex}(s)ds, & x \in \Omega^{ex}. \end{cases} \quad (2)$$

Here  $\psi^{ex}(x, \nu, k)$  is the solution of the problem of scattering of plane wave with wave vector  $k\nu$  by  $\Omega^{ex}$  without line-like window,  $\alpha^{in,ex}, \alpha^{in,ex} \in W_2^{-1}(L)$ ,  $\alpha^{in}(s) = -\alpha^{ex}(s)$ , is a solution of the equation:

$$\int_L Q(s, t)\alpha^{in}(t)dt = \psi^{ex}(x(s), \nu, k), \quad (3)$$

where

$$Q(s, t) = G^{in}(x(s), x(t), k) - G^{in}(x(s), x(t), k_0) + G^{ex}(x(s), x(t), k) - G^{ex}(x(s), x(t), k_0).$$

PROOF. Taking the solution of the scattering problem in general form (2), one should satisfy condition (1) of self-adjointness. To this purpose, one rewrites the expression in the following way

$$\begin{aligned} & \int_L G^{in,ex}(x, x(s), k) \alpha^{in,ex}(s) ds \\ = & \int_L G^{in,ex}(x, x(s), k_0) \alpha^{in,ex}(s) ds + \int_L (G^{in,ex}(x, x(s), k) \\ & - G^{in,ex}(x, x(s), k_0)) \alpha^{in,ex}(s) ds. \end{aligned}$$

It gives, immediately,  $\xi^{in,ex}(s) = \alpha^{in,ex}(s)$ . Then, condition (1) gives one equation (3) which finishes the proof.

LEMMA 4. Resonances are roots of the equation:

$$-1 = \frac{ik}{2\pi} \int_{\Sigma} \int_L ds \int_L dt [Q]^{-1}(s, t) \overline{\psi^{ex}(x(t), \nu, \bar{k})} \psi^{ex}(x(s), \nu, k) d\nu. \quad (4)$$

Here  $[Q]^{-1}(s, t)$  is the kernel of the inverse operator for integral operator  $Q$ .

PROOF. Consider the asymptotic expansion for  $\psi(x, \nu, k)$  as  $|x| \rightarrow \infty, x = |x|\omega$  which has the form

$$\psi(x, \nu, k) = \exp(ik(x, \nu)) + \frac{\exp(ik|x|)}{4\pi|x|} f(\omega, \nu, k) + o\left(\frac{1}{|x|}\right),$$

where  $f(\omega, \nu, k)$  is the scattering amplitude.

The kernel of the  $S$ -matrix is related to the scattering amplitude in the conventional way:

$$S(\omega, \nu, k) = \delta(\omega - \nu) + ik(2\pi)^{-1} f(\omega, \nu, k).$$

As a result, one obtains

$$S(\omega, \nu, k) = S^{ex}(\omega, \nu, k) + ik(2\pi)^{-1} \int_L \alpha^{ex}(s) \psi^{ex}(x(s), \omega, k) ds. \quad (5)$$

The resonances are the zeros of the operator-valued function  $S(k)$ :

$$\int_{\Sigma} S(\omega, \nu, k) e(\nu) d\nu = 0.$$

Here  $\Sigma$  is the unit sphere. In view of the above expression (5) for  $S$ , the equation gets the form:

$$\int_{\Sigma} S^{ex}(\omega, \nu, k) e(\nu) d\nu = -ik(2\pi)^{-1} \int_{\Sigma} \int_L \alpha^{ex}(s) \psi^{ex}(x(s), \omega, k) ds e(\nu) d\nu. \quad (6)$$

Consider (6) in a neighborhood of the real axis. It is known [2] that if the scatterer ( $\Omega^{in}$ ) is convex or star-like, then there are no zeros of the operator-valued function  $S^{ex}(k)$  in some strip near the real axis. Hence, in this domain the operator  $S^{ex}$  has a bounded inverse  $(S^{ex})^{-1}$ . Then, the equation takes the form:

$$-e(\omega) = (S^{ex})^{-1} ik(2\pi)^{-1} \int_{\Sigma} \int_L \alpha^{ex}(s) \psi^{ex}(x(s), \omega, k) ds e(\nu) d\nu. \quad (7)$$

Taking into account the known relation

$$(S^{ex})^{-1} \psi^{ex}(x, \omega, k) = \overline{\psi^{ex}(x, \omega, \bar{k})},$$

and (3), one transforms (7) into the following homogeneous integral equation:

$$-e(\omega) = \frac{ik}{2\pi} \int_{\Sigma} \left( \int_L ds \int_L dt [Q]^{-1}(s, t) \overline{\psi^{ex}(x(t), \omega, \bar{k})} \psi^{ex}(x(s), \nu, k) \right) e(\nu) d\nu.$$

The condition for existence of non-trivial solution gives one the needed equation (4) for resonances. QED.

Let us consider a neighborhood of eigenvalue  $\lambda_n^{in} = (k_n^{in})^2, k_n^{in} \geq 0$ . It contains the resonance  $\lambda_n = k_n^2$ . Let us select the corresponding term in (3):

$$\begin{aligned} & - \int_L \frac{\psi_n(x(s)) \overline{\psi_n(x(t))}}{((k_n^{in})^2 - k^2)} \alpha^{in}(t) dt \\ = & \int_L \frac{\psi_n(x(s)) \overline{\psi_n(x(t))}}{((k_n^{in})^2 - k_0^2)} \alpha^{in}(t) dt \\ & + (k^2 - k_0^2) \int_L \sum_{m, m \neq n} \frac{\psi_m(x(s)) \overline{\psi_m(x(t))}}{((k_m^{in})^2 - k^2)((k_m^{in})^2 - k_0^2)} \alpha^{in}(t) dt \\ & + \int_L (k^2 - k_0^2) \int_{\mathbb{R}^3} \frac{\psi^{ex}(x(s), |\kappa|, \nu) \overline{\psi^{ex}(x(t), |\kappa|, \nu)}}{(\kappa^2 - k^2)(\kappa^2 - k_0^2)} d^3 \kappa \alpha^{in}(t) dt + \alpha^{in}(s). \end{aligned}$$

The expression for resonance state in coordinate representation is as follows

$$\psi_n(x, k_n) = \begin{cases} \int_L G^{in}(x, x(s), k_n) \alpha^{in}(s) ds, & x \in \Omega^{in}, \\ \int_L G^{ex}(x, x(s), k_n) \alpha^{ex}(s) ds, & x \in \Omega^{ex}, \end{cases}$$

where  $k_n$  is  $n$ -th root of (4) (resonance),  $\alpha^{in, ex}$  is a solution of (3).

Taking into account that in a neighborhood of  $(k_n^{in})^2$  one has

$$(k_n^{in})^2 - k^2 = 2k_n^{in}(k_n^{in} - k) + k_n^{in} o(1 - k/k_n^{in}),$$

it is simple to show that the roots of the equation can be estimated as

$$c_1 n^{-2/3} < |\Re \lambda_n - \lambda_n^{in}| < c_2 n^{-2/3}$$

and

$$c_3 n^{-1} < |\Im \lambda_n| < c_4 n^{-1},$$

where  $c_1, c_2, c_3, c_4$  do not depend on  $n$ . Let us denote the resonance state corresponding to the eigenfunction  $\psi_n$  by  $\phi_n$ . To prove the completeness of the resonance states in  $L_2(\Omega^{in})$ , we will use the following theorem.

**THEOREM 1** ([25]). If a set  $\{\psi_n\}$  is complete orthogonal and normalized system of functions, a system of  $\{\phi_n\}$  is  $\omega$ -linearly independent and

$$\sum_n \|\psi_n - \phi_n\|^2 < \infty,$$

then the set  $\{\phi_n\}$  forms a basis.

To prove the first condition (linear independence) for the resonance states, we use the following theorem.

**DEFINITION** ([25]). Operator  $T$  is decomposable in respect to  $X = M \dot{+} N$  if

$$PD(T) \in D(T), \quad TM \in M, \quad TN \in N,$$

where  $P$  is  $N$ -parallel projector onto  $M$ .

**THEOREM 2** ([25]). Let the spectrum  $\sigma(T)$  of closed operator  $T, T : X \rightarrow X$ , contain a bounded part  $\sigma'$  separated from the rest of the spectrum  $\sigma'', \sigma'' = \sigma \setminus \sigma'$ , i.e. there exists a simple closed curve  $\Gamma$  such that  $\sigma' \in \text{Int}\Gamma, \sigma'' \cap \text{Int}\Gamma = \emptyset$ . Then there exist subspaces  $M', M'', X = M' \dot{+} M''$ , such that  $T$  is decomposable in respect to  $M', M''$  and  $\sigma(T_{M'}) - \sigma', \sigma(T_{M''}) - \sigma''$  and  $T_{M'}$  is bounded operator.

Thus, the linear independence for the resonance states takes place due to the fact that these states are eigenstates for a dissipative operator [2]. It should be mentioned that an eigenstate of the initial operator can preserve for the perturbed operator (with line-like window). It takes place if  $L$  is a part of a nodal line for this eigenfunction. In this case, we add such eigenfunction to the set of the resonance states. To come to the result, we need the asymptotics of  $\lambda_n^{in}$ . It is given by the well-known Weyl asymptotics for eigenvalues of the Laplace operator in bounded domain in  $\mathbb{R}^d$ :  $\lambda_n \sim n^{2/d}$ . In our case  $d = 3$ . Taking into account this asymptotics and properties of the curve  $L$  (boundedness), we come to our main theorem:

**THEOREM 4.** Let  $\Omega^{in}$  be convex bounded domain with smooth boundary  $\partial\Omega^{in}$ ,  $L$  be smooth bounded curve,  $L \in \partial\Omega^{in}$ . Then, the system of resonance states is complete in  $L_2(\Omega^{in})$ .

**Acknowledgement.** This work was partially financially supported by the Government of the Russian Federation (grant 074-U01), by grant 16-11-10330 of Russian Science Foundation.

## References

- [1] Lord Rayleigh. The theory of Helmholtz resonator, *Proc. Roy. Soc. London A.*, 92(1916), 265–275.
- [2] P. D. Lax and R. S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.
- [3] P. D. Lax and R. S. Phillips, *Scattering Theory for Automorphic Functions*, Princeton Univ. Press, Princeton, N. J., and Univ. of Tokyo Press, Tokyo 1976.
- [4] V. M. Adamyan and D. Z. Arov, On a class of scattering operators and characteristic operator-functions of contractions. *Dokl. Akad. Nauk SSSR.*, 160(1965), 9–12.
- [5] I. Y. Popov, P. A. Kurasov, S. N. Naboko, A. A. Kiselev, A. E. Ryzhkov, A. M. Yafyasov, G. P. Miroshnichenko, Y. E. Karpeshina, V. I. Kruglov, T. F. Pankratova and A. I. Popov, A distinguished mathematical physicist Boris S. Pavlov. *Nanosystems: Phys. Chem. Math.*, 7(2016), 782–788.
- [6] P. D. Hislop and A. Martinez, Scattering resonances of Helmholtz resonator, *Indiana Univ. Math. J.*, 40(1991), 767–788.
- [7] R. R. Gadyl'shin, Existence and asymptotics of poles with small imaginary part for the Helmholtz resonator, *Russian Mathematical Surveys*, 52(1997), 1–72.
- [8] I. Y. Popov, Extension theory and localization of resonances for domains of trap type, *Mathematics of the USSR-Sbornik*, 71(1992), 209–234.
- [9] I. Y. Popov, The resonator with narrow slit and the model based on the operator extensions theory, *J. Math. Phys.*, 33(1992), 3794–3801.
- [10] A. A. Shushkov, Structure of resonances for symmetric scatterers, *Theor. Math. Phys.*, 64(1985), 944–949.
- [11] A. M. Vorobiev and I. Y. Popov, Model of quantum dot and resonant states for the Helmholtz resonator, *J. Phys.: Conf. Series*, 643(2015), 012097.
- [12] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kerchy, *Harmonic Analysis of Operators on Hilbert Space*, 2nd edition. Springer, Berlin 2010.
- [13] S. V. Khrushchev, N. K. Nikol'skii, B. S. Pavlov, *Unconditional Bases of Exponentials and of Reproducing Kernels, Complex Analysis and Spectral Theory (Leningrad, 1979/1980)*. *Lecture Notes in Math.*, vol. 864 (1981), Springer-Verlag, Berlin, New York, 214–335.
- [14] N. K. Nikol'skii, *Treatise on the Shift Operator: Spectral Function Theory. V.* 273. Springer Science and Business Media, Berlin, 2012.
- [15] V. V. Peller, *Hankel Operators and Their Applications*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

- [16] S. N. Naboko, Functional model of perturbation theory and its applications to scattering theory, *Boundary value problems of mathematical physics*, 10. *Trudy Mat. Inst. Steklov.*, 147(1980), 86–114.
- [17] I. Y. Popov and A. I. Popov, Line with attached segment as a model of Helmholtz resonator: Resonant states completeness, *Journal of King Saud University - Science*, 29(2017), 133–136.
- [18] D. A. Gerasimov and I. Y. Popov, Completeness of resonance states for quantum graph with two semi-infinite edges, *Complex Var. Elliptic Equ.*, DOI: 10.1080/17476933.2017.128951
- [19] I. Y. Popov, Justification of the model of cracks of zero width for the Dirichlet problem. *Siberian Mathematical J.*, 30(1989), 428–432.
- [20] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Second edition. With an appendix by Pavel Exner. AMS Chelsea Publishing, Providence, RI, 2005.
- [21] J. Brasche, P. Exner, Y. A. Kuperin and P. Seba. Schrödinger operator with sigular interactions, *J. Math. Anal. Appl.*, 184(1994), 112–139.
- [22] J. Behrndt, M. Langer and V. Lotoreichik, Boundary triples for Schrodinger operators with singular interactions on hypersurfaces, *Nanosystems: Physics, Chemistry, Mathematics*, 7(2016), 290–302.
- [23] I. Y. Popov and S. L. Popova, Zero-width slit model and resonances in mesoscopic systems, *Europhysics Letters*, 24(1993), 373–377.
- [24] I. Y. Popov and S. L. Popova, The extension theory and resonances for a quantum waveguide, *Physics Lett. A.*, 173(1993), 484– 488.
- [25] T. Kato. *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York 1966.