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Fixed Point Theorems For Weakly Contractive Mappings On S-Metric Spaces And a Homotopy Result*

Mohammad Mahdi Rezaee[†], Mahryam Shahraki[‡], Shaban Sedghi[§], Ishak Altun[¶]

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Abstract

The present paper contains some fixed point results for weakly contractive single valued mappings on S-metric spaces and a homotopy result, which is offered as an application of the obtained new results.

1 Introduction

Alber and Guerre-Delabriere in [1] defined the concept of weakly contractive mappings for single valued maps on Hilbert spaces and proved the existence of fixed points. Since then, many authors have obtained fixed point theorems for weakly contractive mappings; see for example, [2, 3, 7, 12, 17]. Rhoades [12] proved the following fixed point theorem.

THEOREM 1 ([12]). Let (X, d) be a complete metric space and let T be a φ-weak contraction on X; that is, for each $x, y \in X$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)). \tag{1}$$

Also if φ is a continuous and nondecreasing function, then T has a unique fixed point.

The above interesting result is one of the generalizations of the Banach contraction principle, because it contains contractions as special case ($\varphi(t) = (1 - k)t$). Also the weak contractions are related to maps of Boyd and Wong type ones [5] and Reich type ones [13]. Namely, if φ is a lower semi-continuous function from the right then

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†Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
‡Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
§Corresponding author. Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
¶King Saud University, College of Science, Riyadh, Saudi Arabia and Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

$\psi(t) = t - \varphi(t)$ is an upper semi-continuous function from the right, and, moreover, (1) turns into $d(Tx, Ty) \leq \psi(d(x, y))$. Therefore the φ-weak contraction with a function φ is of Boyd and Wong type [5]. And, if we define $k(t) = 1 - \varphi(t)/t$ for $t > 0$ and $k(0) = 0$, then (1) is replaced by $d(Tx, Ty) \leq k(d(x, y))d(x, y)$. Therefore the φ-weak contraction becomes a Reich type one.

THEOREM 2 ([17]). Let (X, d) be a complete metric space and let $T, S : X \longrightarrow X$ two mappings such that, for each $x, y \in X$,

$$d(Tx, Sy) \leq m(x, y) - \varphi(m(x, y)),$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$,

$$m(x, y) = max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}.$$

Then there exists a unique point $u \in X$ such that $u = Tu = Su$.

The aim of this paper is to prove the above results on an S-metric space. Before giving our main result we recall some of the basic concepts and results for S-metric spaces, and some similar spaces.

Let X be a nonempty set and $G : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following condition: for all $x, y, z, a \in X$,

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$,

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, x, a \in X$.

Then the function G is called a *generalized metric* or a G-metric on X and the pair (X, G) is called a G-metric space. More datails are available in [10].

Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \rightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D^*(x, y, z) \geq 0$,

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where p is a permutation function,

(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space. Examples of such a function are

(a) $D^*(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$.

(b) $D^*(x,y,z) = d(x,y) + d(y,z) + d(z,x)$ where d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$, then we define

$$D^*(x,y,z) = ||x+y-2z|| + ||x+z-2y|| + ||y+z-2x||.$$

(d) If $X = \mathbb{R}^+$, then we define

$$D^*(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

It is easy to see that every G-metric is a D^*-metric, but in general the converse is not hold.

EXAMPLE 1. If $X = \mathbb{R}$ then we define

$$D^*(x,y,z) = |x+y-2z| + |x+z-2y| + |y+z-2x|.$$

It is easy to see that $(\mathbb{R}, D^*)$ is a D^*-metric, but it is not G-metric. For, if one sets $x = 5, y = -5$ and $z = 0$ then $G(x,x,y) \le G(x,y,z)$ does not hold.

Now, we give the concept of S-metric spaces, which modifies the D-metric and G-metric spaces as follows:

DEFINITION 1 ([16]). Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0,\infty)$ that satisfies the following conditions, for each $x,y,z,a \in X$,

(1) $S(x,y,z) \ge 0$.

(2) $S(x,y,z) = 0$ if and only if $x = y = z$.

(3) $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $x,y,z,a \in X$.

The pair (X,S) is called an S-metric space.

Examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X. Then $S(x,y,z) = ||x-z|| + ||y-z||$ is an S-metric on X.

(2) Let X be a nonempty set, d is ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

It is easy to see that every D^*-metric is an S-metric, but in general the converse does not hold.

EXAMPLE 2. Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X. Then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X, but it not D^*-metric. For, it does not satisfy the symmetry condition.

LEMMA 1 ([16]). In an S-metric space, we have $S(x,x,y)=S(y,y,x)$.

DEFINITION 2 ([16]). Let (X,S) be an S-metric space. For $r>0$ and $x\in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S\ [x,r]$ with center x and radius r respectively as

$$B_s(x,r)=\{y\in X: S(y,y,x)<r\}$$

and

$$B_s[x,r]=\{y\in X: S(y,y,x)\leq r\}.$$

EXAMPLE 3 ([16]). Let $X=\mathbb{R}$. Denote $S(x,y,z)=|y+z-2x|+|y-z|$ for all $x,y,z\in X$. Then

$$\begin{aligned} B_s(1,2) &= \{y\in X: S(y,y,1)<2\} \\ &= \{y\in\mathbb{R}: |y-1|<2\} \\ &= (0,2). \end{aligned}$$

DEFINITION 3 ([16]). Let (X,S) be an S-metric space and $A\subset X$.

(1) If for every $x\in A$ there exists an $r>0$ such that $B_S(x,r)\subset A$, then the subset A is called open subset of X.

(2) Subset A of X is said to be S-bounded if there exists an $r>0$ such that $S(x,x,y)<r$ for all $x,y\in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x)\to 0$ as $n\to\infty$; that is, for each $\varepsilon>0$ there exists an $n_0\in\mathbb{N}$ such that

$$\forall n\geq n_0 \Longrightarrow S(x_n,x_n,x)<\varepsilon,$$

and we denote this by $\lim\limits_{n\longrightarrow\infty} x_n=x$.

(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for each $\varepsilon>0$, there exists an $n_0\in\mathbb{N}$ such that $S(x_n,x_n,x_m)<\varepsilon$ for each $n,m\geq n_0$.

(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A\subset X$ with $x\in A$ if and only if there exists an $r>0$ such that $B_S(x,r)\subset A$. Then τ is a topology on X(induced by the S-metric S).

2 Main Results

THEOREM 3. Let (X, S) be a complete S-metric space and $T : X \longrightarrow X$ a function such that for, all $x, y, z \in X$,

$$S(Tx, Ty, Tz) \leq M(x, y, z) - \varphi(M(x, y, z)), \tag{2}$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$ and

$$M(x, y, z) = \max\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz)\}. \tag{3}$$

Then there exists a unique point $u \in X$ such that $Tu = u$.

PROOF. First we show that $M(x, y, z) = 0$ if and only if $x = y = z$ is a common fixed point of T. Indeed, if $x = y = z = Tx = Ty = Tz$, then $M(x, y, z) = 0$. Let $M(x, y, z) = 0$. Then $S(x, y, z) = 0$, which implies that $x = y = z$. The condition $S(x, x, Tx) = 0$ implies that $x = Tx$. Therefore $x = y = z = Tx = Ty = Tz$. Let $x_0 \in X$ be an arbitrary point and choose a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n > 0$. By (2), (3) and the property of φ, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)) \\ &= max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\ &\quad -\varphi(max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned} \tag{4}$$

If there exists an n such that $S(x_n, x_n, x_{n+1}) = 0$, then x_n is a fixed point of T. Therefore we shall assume that $S(x_n, x_n, x_{n+1}) \neq 0$ for every n. Now, if $S(x_n, x_n, x_{n+1}) > S(x_{n-1}, x_{n-1}, x_n)$, for some n, then

$$S(x_n, x_n, x_{n+1}) \leq S(x_n, x_n, x_{n+1}) - \varphi(S(x_n, x_n, x_{n+1})),$$

a contradiction. Thus, for all n,

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)) \\ &\leq M(x_{n-1}, x_{n-1}, x_n) \\ &\leq S(x_{n-1}, x_{n-1}, x_n), \end{aligned}$$

and $\{S(x_n, x_n, x_{n+1})\}$ is monotone nonincreasing and bounded below. So there exists a number $r \geq 0$ such that

$$\lim_{n \longrightarrow \infty} S(x_n, x_n, x_{n+1}) = \lim_{n \longrightarrow \infty} M(x_{n-1}, x_{n-1}, x_n) = r.$$

From the lower semi continuity of φ we have

$$\varphi(r) \leq \lim_{n \longrightarrow \infty} inf \varphi(M(x_{n-1}, x_{n-1}, x_n)).$$

We claim that $r = 0$. In fact, taking upper limits as $n \longrightarrow \infty$ on either side of the following inequality:

$$S(x_n, x_n, x_{n+1}) \leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)),$$

we have

$$r \leq r - \lim_{n \longrightarrow \infty} inf \varphi(M(x_{n-1}, x_{n-1}, x_n)) \leq r - \varphi(r);$$

i.e. $\varphi(r) \leq 0$. Thus $\varphi(r) = 0$, from the definition of φ, and $r = 0$. Hence

$$\lim_{n \longrightarrow \infty} S(x_n, x_n, x_{n+1}) = 0. \tag{5}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose this is not true. Then there is an $\varepsilon > 0$ such that, for an integer k, there exist integers $m(k) > n(k) > k$ such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{6}$$

For each integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (6) and such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{7}$$

Then

$$\begin{aligned}
\varepsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leq S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\
&= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}).
\end{aligned}$$

And, by (5) and (7) it follows that

$$\lim_{k \longrightarrow \infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{8}$$

Since $m(k)$ is the least positive integer exceeding $n(k)$ satisfying (6) and (8), we have

$$\varepsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}),$$

and, by (2) we get

$$\begin{aligned}
\varepsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}) \\
&\leq S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\
&= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \\
&\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + M(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \\
&+ max\{S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}), S(x_{m(k)}, x_{m(k)}, x_{m(k)+1})\}.
\end{aligned}$$

Letting $k \longrightarrow \infty$ and using (5) and (8), we have

$$\varepsilon \leq \lim_{k \longrightarrow \infty} M(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq \varepsilon,$$

and so

$$\lim_{k\longrightarrow\infty} M(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon.$$

Since φ is lower semicontinuous, we have

$$\varphi(\varepsilon) \leq \lim_{n\longrightarrow\infty} inf M(x_{n(k)}, x_{n(k)}, x_{m(k)}).$$

From (2) we get

$$S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \leq M(x_{n(k)}, x_{n(k)}, x_{m(k)}) - \varphi(M(x_{n(k)}, x_{n(k)}, x_{m(k)})),$$

and, taking upper limit as $k \longrightarrow \infty$, we have

$$\begin{aligned} \varepsilon &\leq \varepsilon - \lim_{n\longrightarrow\infty} inf M(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &\leq \varepsilon - \varphi(\varepsilon), \end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence.

It follows from the completeness of X that there exists a $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$.

We shall now prove that $u = Tu$. Indeed, suppose that $u \neq Tu$. Then

$$\begin{aligned} S(x_n, x_n, Tu) &\leq M(x_{n-1}, x_{n-1}, u) - \varphi(M(x_{n-1}, x_{n-1}, u)) \\ &\leq max\{S(x_{n-1}, x_{n-1}, u), S(x_{n-1}, x_{n-1}, x_n), S(u, u, Tu)\} \\ &\quad -\varphi(M(x_{n-1}, x_{n-1}, u)), \end{aligned}$$

and, taking the upper limit as $n \longrightarrow \infty$, we have

$$S(u, u, Tu) \leq S(u, u, Tu) - \varphi(S(u, u, Tu)),$$

which is a contradiction. Thus $u = Tu$. To prove uniqueness, suppose that $u \neq v$ and $Tv = v$. Then (2) implies that

$$\begin{aligned} S(u, u, v) &= S(Tu, Tu, Tv) \\ &\leq M(u, u, v) - \varphi(M(u, u, v)) \\ &\leq max\{S(u, u, v), S(u, u, Tu), S(v, v, Tv)\} \\ &- \varphi(M(u, u, v)), \\ &\leq S(u, u, v) - \varphi(S(u, u, v)), \end{aligned}$$

which is a contradiction. Hence $u = v$.

COROLLARY 1. Let (X, S) be a complete S-metric space and $T : X \longrightarrow X$ a function such that, for all $x, y, z \in X$,

$$S(Tx, Ty, Tz) \leq S(x, y, z) - \varphi(S(x, y, z)),$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. Then there exists a unique point $u \in X$ such that $Tu = u$.

COROLLARY 2. Let (X, S) be a complete S-metric space and $T : X \longrightarrow X$ a function such that for, all $x, y, z \in X$,

$$S(Tx, Ty, Tz) \le LS(x, y, z),$$

where $L \in (0, 1/2)$. Then there exists a unique point $u \in X$ such that $Tu = u$.

COROLLARY 3. Let (X, S) be a complete S-metric space, $x_0 \in X$ and $r > 0$. Suppose that $T : B_s(x_0, r) \longrightarrow X$ a function such that, for all $x, y, z \in B_s(x_0, r)$,

$$S(Tx, Ty, Tz) \le LS(x, y, z),$$

with

$$S(Tx_0, Tx_0, x_0) < (1 - 2L)r,$$

where $L \in (0, 1/2)$. Then there exists a unique point $u \in B_s(x_0, r)$ such that $Tu = u$.

PROOF. There exists an r_0 with $0 \le r_0 < r$ with $S(Tx_0, Tx_0, x_0) \le (1-2L)r_0$. We will show that $T : \overline{B_s(x_0, r_0)} \longrightarrow \overline{B_s(x_0, r_0)}$. To see this, note that, if $x \in \overline{B_s(x_0, r_0)}$, then

$$\begin{aligned} S(Tx, Tx, x_0) &\le S(Tx, Tx, Tx_0) + S(Tx, Tx, Tx_0) + S(x_0, x_0, Tx_0) \\ &= 2S(Tx, Tx, Tx_0) + S(Tx_0, Tx_0, x_0) \\ &\le 2LS(x, x, x_0) + (1 - 2L)r_0 \\ &\le r_0. \end{aligned}$$

Now apply Corollary 2 to deduce that T has a unique fixed point in $\overline{B_s(x_0, r_0)} \subseteq B_s(x_0, r)$. Again it is easy to see that T has only one fixed point in $B_s(x_0, r)$.

THEOREM 4. Let (X, S) be a complete S-metric space and U an open subset of X. Suppose that $H : \overline{U} \times [0, 1] \longrightarrow X$ and

(1) $x \ne H(x, \lambda)$ for every $x \in \partial U$ and $t \in [0, 1]$ (here ∂U denotes the boundary of U in X).

(2) For all $x, y, z \in \overline{U}$ and $\lambda \in [0, 1]$, $L \in (0, 1/2)$, such that

$$S(H(x, \lambda), H(y, \lambda), H(z, \lambda)) \le LS(x, y, z).$$

(3) There exists an $M \ge 0$, such that

$$S(H(x, \lambda), H(x, \lambda), H(x, \mu)) \le M|\lambda - \mu|,$$

for every $x \in \overline{U}$ and $\lambda, \mu \in [0, 1]$.

If $H(., 0)$ has a fixed point in U, then $H(., 1)$ has a fixed point in U.

PROOF. Consider the set

$$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\}.$$

Since $H(.,0)$ has a fixed point in U, then A is nonempty; that is, $0 \in A$. We will show that A is both open and closed in $[0,1]$, and hence by connectedness, we have that $A = [0,1]$. As a result, $H(.,1)$ has a fixed point in U. We first show that A is closed in $[0,1]$. To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \longrightarrow \lambda \in [0,1]$ as $n \longrightarrow \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1,2,3,\ldots$, there exist $x_n \in U$ with $x_n = H(x_n,\lambda)$. Also, for $n,m \in \{1,2,3,\ldots\}$ we have

$$\begin{aligned} S(x_n,x_n,x_m) &\le S(H(x_n,\lambda_n),H(x_n,\lambda_n),H(x_m,\lambda_m)) \\ &\le 2S(H(x_n,\lambda_n),H(x_n,\lambda_n),H(x_n,\lambda_m)) \\ &\quad +S(H(x_n,\lambda_m),H(x_n,\lambda_m),H(x_m,\lambda_m)) \\ &\le 2M|\lambda_n-\lambda_m| + LS(x_n,x_n,x_m); \end{aligned}$$

that is,

$$S(x_n,x_n,x_m) \le (\frac{2M}{1-L})|\lambda_n-\lambda_m|.$$

Since $\{\lambda_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we have that $\{x_n\}$ is also a Cauchy sequence, and, since X is complete there exists an $x \in \overline{U}$ such that $\{x_n\}$ is convergent to x. In addition, $x = H(x,\lambda)$, since

$$\begin{aligned} S(x_n,x_n,H(x,\lambda)) &\le S(H(x_n,\lambda_n),H(x_n,\lambda_n),H(x,\lambda)) \\ &\le 2S(H(x_n,\lambda_n),H(x_n,\lambda_n),H(x_n,\lambda)) \\ &\quad +S(H(x_n,\lambda),H(x_n,\lambda),H(x,\lambda)) \\ &\le 2M|\lambda_n-\lambda| + LS(x_n,x_n,x). \end{aligned}$$

Thus $\lambda \in A$ and A is closed in $[0,1]$. Next we show that A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists an $x_0 \in U$ with $x_0 = H(x_0,\lambda_0)$. Since U is open, then there exists an $r > 0$ such that $B_s(x_0,r) \subseteq U$.

Now fix an $\varepsilon > 0$, with

$$\varepsilon < \frac{(1-2L)r}{M}.$$

Let $\lambda \in (\lambda_0-\varepsilon,\lambda_0+\varepsilon)$, then

$$\begin{aligned} S(H(x_0,\lambda),H(x_0,\lambda),x_0) &= S(H(x_0,\lambda),H(x_0,\lambda),H(x_0,\lambda_0)) \\ &\le M|\lambda-\lambda_0| \\ &< (1-2L)r. \end{aligned}$$

Now apply Corollary 2 to deduce that $H(.,\lambda)$ has a fixed point in $B_s(x_0,r) \subseteq U$. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ and therefore A is open in $[0,1]$.

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