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# Uniqueness And Value Distribution Of Differences Of Meromorphic Functions*

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**Abstract**

The purpose of the paper is to study the uniqueness problems of difference polynomials of meromorphic functions sharing a small function. The results of the paper improve and generalize the recent results due to Liu, et al. [11] and Liu, et al. [12].

## 1 Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function $f$, we denote by $T(r,f)$ the Nevanlinna characteristic of $f$ and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite logarithmic measure. We denote by $T(r)$ the maximum of $T(r,F)$ and $T(r,G)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite logarithmic measure.

A meromorphic function $a(z)$ is called a small function with respect to $f$, provided that $T(r,a)=S(r,f)$. The order of $f$ is defined by

$$\sigma(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Recently, the topics of difference equations and difference products in complex plane $\mathbb{C}$ has attracted many mathematicians. Many papers have focused on value distribution of differences and differences operators analogues of Nevanlinna theory ([2, 4, 9, 10]) and many people dealt with the uniqueness problems related to meromorphic functions and their shifts or their difference operators and obtained some interesting results ([11, 12, 13, 16]).

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In 2011, K. Liu, X. L. Liu and T. B. Cao studied the uniqueness of the difference monomials and obtained the following results.

THEOREM A ([11]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

THEOREM B ([11]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

We now explain the notation of weighted sharing as introduced in [8].

DEFINITION 1 ([8]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$ then $f$, $g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$ respectively.

In 2015, Y. Liu, J. P. Wang and F. H. Liu improved Theorems A, B and obtained the following results.

THEOREM C ([12]). Let $c \in \mathbb{C} \setminus \{0\}$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14)$, $k(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$ for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

THEOREM D ([12]). Let $c \in \mathbb{C} \setminus \{0\}$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$, for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

THEOREM E ([12]). Let $c \in \mathbb{C} \setminus \{0\}$ and let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 22)$ be a positive integer. If $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) \equiv t_2$, for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Now it is quite natural to ask the following question.

QUESTION 1. What can be said if the sharing value 1 in Theorems C, D and E is replaced by a nonzero polynomial ?

Now taking the possible answer of the above question into background we obtain the following results.

THEOREM 1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. If $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$, then one of the following two cases holds:

(1) $f(z) \equiv tg(z)$ for some constant $t$ such that $t^{n+1} = 1$,

(2) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant $c$ and $t$ is a constant such that $t^{n+1} = c^2$

THEOREM 2. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 1)$. Then conclusion of Theorem 1 holds.

THEOREM 3. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p(z)(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 0)$. Then conclusion of Theorem 1 holds.

We now make the following definitions and notations which are used in the paper.

DEFINITION 2 ([7]). Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid\leq p)$ the counting function of those $a$-points of $f$ (counted with multiplicities) whose multiplicities are not greater than $p$. By $\bar{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r, a; f \mid\geq p)$ and $\bar{N}(r, a; f \mid\geq p)$.

DEFINITION 3 ([8]). Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k$ times if $m > k$. Then

$$N_k(r, a; f) = \bar{N}(r, a; f) + \bar{N}(r, a; f \mid\geq 2) + ... + \bar{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \bar{N}(r, a; f)$.

## 2 Lemma

In this section we present the lemma which will be needed in the sequel.

Let $F$, $G$ be two non-constant meromorphic functions. Henceforth we shall denote by $H$ the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

LEMMA 1 ([14]). Let $f$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

LEMMA 2 ([2]). Let $f(z)$ be a meromorphic function of finite order $\sigma$, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have

$$m(r, \frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = O(r^{\sigma-1+\varepsilon}).$$

The following lemma is a slight modifications of the original version (Theorem 2.1 of [2])

LEMMA 3. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

LEMMA 4 ([3]). Let $f$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r, f), \quad N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$

$$\bar{N}(r, 0; f(z+c)) \leq \bar{N}(r, 0; f(z)) + S(r, f), \quad \bar{N}(r, \infty; f(z+c)) \leq \bar{N}(r, \infty; f) + S(r, f).$$

Taking $m = 0$ in Lemma 2.4 [11], we obtain the following lemma.

LEMMA 5. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed and let $\Phi(z) = f^n(z)f(z+c)$, where $n \in \mathbb{N}$. Then we have

$$(n-1)\, T(r, f) \quad \leq \quad T(r, \Phi) + S(r, f).$$

LEMMA 6. Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. Let $p(z)$ be a nonzero polynomial such that $\deg(p) < (n-1)/2$. Then

(1) if $\deg(p) \geq 1$, then $f^n(z)f(z+c)g^n(z)g(z+c) \not\equiv p^2(z)$;

(2) if $p(z) = c \in \mathbb{C} \setminus \{0\}$, then the relation $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$ always implies that $fg = t$, where $t$ is a constant such that $t^{n+1} = c^2$.

PROOF. Suppose

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z). \tag{2}$$

Let $h_1 = fg$. Then by (2), we have

$$h_1^n(z) \equiv \frac{p^2(z)}{h_1(z+c)}. \tag{3}$$

We now consider following two cases.

**Case 1.** Suppose $h_1$ is a transcendental meromorphic function. Now by Lemmas 1, 2 and 4, we get

$$\begin{aligned} n\,T(r,h_1) = T(r,h_1^n) + S(r,h_1) &= T(r,\frac{p^2}{h_1(z+c)}) + S(r,h_1) \\ &\le N(r,0;h_1(z+c)) + m(r,\frac{1}{h_1(z+c)}) + S(r,h_1) \\ &\le N(r,0;h_1(z)) + m(r,\frac{1}{h_1(z)}) + S(r,h_1) \\ &\le T(r,h_1) + S(r,h_1), \end{aligned}$$

which is a contradiction.

**Case 2.** Suppose $h_1$ is a rational function. Let

$$h_1 = \frac{h_2}{h_3}, \tag{4}$$

where $h_2$ and $h_3$ are two nonzero relatively prime polynomials. By (4), we have

$$T(r,h_1) = \max\{\deg(h_2), \deg(h_3)\}\log r + O(1). \tag{5}$$

Now by (3)–(5), we have

$$\begin{aligned} & n\ \max\{\deg(h_2), \deg(h_3)\}\ \log r \\ = {} & T(r,h_1^n) + O(1) \\ \le {} & T(r,h_1(z+c)) + 2\,T(r,p) + O(1) \\ = {} & \max\{\deg(h_2), \deg(h_3)\}\ \log r + 2\ \deg(p)\ \log r + O(1). \end{aligned} \tag{6}$$

We see that $\max\{\deg(h_2), \deg(h_3)\} \ge 1$. Now by (6), we deduce that $(n-1)/2 \le \deg(p)$, which contradicts our assumption that $\deg(p) < (n-1)/2$. Hence $h_1$ must be a nonzero constant. Let

$$h_1 = t \in \mathbb{C} \setminus \{0\}. \tag{7}$$

Now when $\deg(p) \ge 1$, by (3) and (7), we arrive at a contradiction. Therefore in this case we have $f^n(z)f(z+c)g^ng(z+c) \not\equiv p^2(z)$. Suppose $p(z) = c \in \mathbb{C} \setminus \{0\}$. So by (3) we see that $h_1^{n+1} \equiv c^2$. By (7) we get $t^{n+1} \equiv c^2$. This completes the proof.

LEMMA 7 ([8]). Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds:

(i) $T(r,f) \le N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f) + S(r,g)$,

(ii) $fg \equiv 1$,

(iii) $f \equiv g$.

LEMMA 8 ([1]). Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then

$$\begin{aligned} T(r,F) \quad &\le \quad N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\,\bar{N}(r,0;F) \\ &\quad + \frac{1}{2}\,\bar{N}(r,\infty;F) + S(r,F) + S(r,G). \end{aligned}$$

LEMMA 9 ([1]). Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then

$$\begin{aligned} T(r,F) \quad &\le \quad N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\,\bar{N}(r,0;F) \\ &\quad + \bar{N}(r,0;G) + 2\,\bar{N}(r,\infty;F) + \bar{N}(r,\infty;G) + S(r,F) + S(r,G). \end{aligned}$$

LEMMA 10 ([15]). Let $H$ be defined as in (1). If $H \equiv 0$ and

$$\limsup_{r\longrightarrow\infty} \frac{\bar{N}(r,0;F) + \bar{N}(r,0;G) + \bar{N}(r,\infty;F) + \bar{N}(r,\infty;G)}{T(r)} < 1, \quad r \in I,$$

where $I$ is a set of infinite linear measure, then $F \equiv G$ or $F.G \equiv 1$.

## 3 Proofs of the Theorems

PROOF OF THEOREM 1. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Then $F$ and $G$ share $(1,2)$ except for the zeros of $p(z)$. Now by Lemma 7, we see that one of the following three cases holds.

**Case 1.** Suppose

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G).$$

Now by applying Lemmas 1 and 4, we have

$$\begin{aligned} & T(r,F) \\ \le \quad & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,f) + S(r,g) \\ = \quad & N_2(r,0;f^n f(z+c)) + N_2(r,0;g^n g(z+c)) \end{aligned}$$

$$\begin{aligned}
&+N_2(r,\infty;f^n f(z+c)) + N_2(r,\infty;g^n g(z+c)) + S(r,f) + S(r,g)\\
\leq\ & N_2(r,0;f^n) + N_2(r,0;f(z+c)) + N_2(r,0;g^n) + N_2(r,0;g(z+c))\\
&+N_2(r,\infty;f^n) + N_2(r,\infty;f(z+c)) + N_2(r,\infty;g^n) + N_2(r,\infty;g(z+c))\\
&+S(r,f) + S(r,g)\\
\leq\ & 2\,N(r,0;f) + N(r,0;f(z+c)) + 2\,N(r,0;g) + N(r,0;g(z+c)) + 2\,N(r,\infty;f)\\
&+N(r,\infty;f(z+c)) + 2\,N(r,\infty;g) + N(r,\infty;g(z+c)) + S(r,f) + S(r,g)\\
\leq\ & 4T(r,f) + N(r,0;f) + N(r,\infty;f) + 4T(r,g) + N(r,0;g) + N(r,\infty;g)\\
&+S(r,f) + S(r,g)\\
\leq\ & 6\,T(r,f) + 6\,T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

By Lemma 5, we have

$$(n-1)\,T(r,f) \leq 6\,T(r,f) + 6\,T(r,g) + S(r,f) + S(r,g) \leq 12\,T_1(r) + S_1(r), \tag{8}$$

where $T_1(r)$ is the maximum of $T(r,f)$ and $T(r,g)$ and $S_1(r)$ denotes any quantity satisfying $S_1(r) = o(T_1(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite logarithmic measure. Similarly we have

$$(n-1)\,T(r,g) \leq 12\,T_1(r) + S_1(r). \tag{9}$$

Combining (8) and (9) we get $(n-1)\,T_1(r) \leq 12\,T_1(r) + S_1(r)$, which contradicts with $n \geq 14$.

**Case 2.** $F \equiv G$. Then we have

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c). \tag{10}$$

Let $h = \frac{f}{g}$. Then by (10), we have

$$h^n(z) \equiv \frac{1}{h(z+c)}. \tag{11}$$

Now by Lemmas 1, 2 and 4, we get

$$\begin{aligned}
n\,T(r,h) = T(r,h^n) + S(r,h) \ &= \ T(r,\frac{1}{h(z+c)}) + S(r,h)\\
&\leq \ N(r,0;h(z+c)) + m(r,\frac{1}{h(z+c)}) + S(r,h)\\
&\leq \ N(r,0;h(z)) + m(r,\frac{1}{h(z)}) + S(r,h)\\
&\leq \ T(r,h) + S(r,h).
\end{aligned}$$

Since $n \geq 2$, we see that $h$ is a constant. By (11), we have $h^{n+1} = 1$. Thus $f(z) = tg(z)$ and $t^{n+1} = 1$.

**Case 3.** $F.G \equiv 1$. Then we have $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$. Hence Theorem 1 follows by Lemma 6. This completes the proof.

PROOF OF THEOREM 2. Let $F(z)=\frac{f^n(z)f(z+c)}{p(z)}$ and $G(z)=\frac{g^n(z)g(z+c)}{p(z)}$. Then $F$ and $G$ share $(1,1)$ except for the zeros of $p(z)$. We now consider the following two cases.

**Case 1.** $H\not\equiv 0$. By Lemma 3, we have

$$\begin{aligned}\bar{N}(r,0;F)=\bar{N}(r,0;f^nf(z+c)) &\leq \bar{N}(r,0;f^n)+\bar{N}(r,0;f(z+c))\\ &\leq \bar{N}(r,0;f)+\bar{N}(r,0;f(z+c))\\ &\leq N(r,0;f)+N(r,0;f(z+c))\leq 2T(r,f)+S(r,f).\end{aligned}$$

Similarly we have $\bar{N}(r,\infty;F)\leq 2T(r,f)+S(r,f)$. Now by applying Lemmas 1, 4 and 8 we have

$$\begin{aligned}&T(r,F)\\ \leq\ & N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\,\bar{N}(r,0;F)\\ &+\frac{1}{2}\,\bar{N}(r,\infty;F)+S(r,f)+S(r,g)\\ \leq\ & 6\,T(r,f)+6\,T(r,g)+T(r,f)+T(r,f)+S(r,f)+S(r,g)\\ \leq\ & 8\,T(r,f)+6\,T(r,g)+S(r,f)+S(r,g)\end{aligned}$$

By Lemma 5, we have

$$(n-1)\,T(r,f)\leq 8\,T(r,f)+6\,T(r,g)+S(r,f)+S(r,g)\leq 14\,T_1(r)+S_1(r). \qquad (12)$$

Similarly, we have

$$(n-1)\,T(r,g)\leq 14\,T_1(r)+S_1(r). \qquad (13)$$

Combining (12) and (13) we get $(n-1)\,T_1(r)\leq 14T_1(r)+S_1(r)$, which contradicts with $n\geq 16$.

**Case 2.** $H\equiv 0$. In view of Lemmas 4 and 5, we get

$$\begin{aligned}&\bar{N}(r,0;F)+\bar{N}(r,0;G)+\bar{N}(r,\infty;F)+\bar{N}(r,\infty;G)\\ \leq\ & 4\,T(r,f)+4\,T(r,g)+S(r,f)+S(r,g)\\ \leq\ & \frac{4}{n-1}\,T(r,F)+\frac{4}{n-1}\,T(r,G)+S(r,F)+S(r,G)\leq\frac{8}{n-1}\,T(r)+S(r).\end{aligned}$$

Since $n>12$, we have

$$\limsup_{r\longrightarrow\infty}\frac{\bar{N}(r,0;F)+\bar{N}(r,0;G)+\bar{N}(r,\infty;F)+\bar{N}(r,\infty;G)}{T(r)}<1$$

and so by Lemma 10, we have either $F\equiv G$ or $F.G\equiv 1$. Hence Theorem 2 follows by the proof of Theorem 1. This completes the proof.

PROOF OF THEOREM 3. Let $F(z)=\frac{f^n(z)f(z+c)}{p(z)}$ and $G(z)=\frac{g^n(z)g(z+c)}{p(z)}$. Then $F$ and $G$ share $(1,0)$ except for the zeros of $p(z)$. We now consider the following two cases.

**Case 1.** $H \not\equiv 0$. By Lemma 3, we have

$$\bar{N}(r,0;F) \leq 2T(r,f) + S(r,f), \quad \bar{N}(r,\infty;F) \leq 2T(r,f) + S(r,f),$$

$$\bar{N}(r,0;G) \leq 2T(r,g) + S(r,g) \quad \text{and} \quad \bar{N}(r,\infty;G) \leq 2T(r,g) + S(r,g).$$

Now by Lemmas 1, 4 and 9, we have

$$\begin{aligned} & T(r,F) \\ \leq\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\ \bar{N}(r,0;F) + \bar{N}(r,0;G) \\ & +2\ \bar{N}(r,\infty;F) + \bar{N}(r,\infty;G) + S(r,f) + S(r,g) \\ \leq\ & 6\ T(r,f) + 6\ T(r,g) + 4T(r,f) + 2T(r,g) + 4T(r,f) + 2T(r,g) + S(r,f) + S(r,g) \\ \leq\ & 14\ T(r,f) + 10\ T(r,g) + S(r,f) + S(r,g) \end{aligned}$$

By Lemma 5, we have

$$(n-1)\ T(r,f) \leq 14\ T(r,f) + 10\ T(r,g) + S(r,f) + S(r,g) \leq 24\ T_1(r) + S_1(r). \quad (14)$$

Similarly we have

$$(n-1)\ T(r,g) \leq 24\ T_1(r) + S_1(r). \quad (15)$$

Combining (14) and (15) we get $(n-1)\ T_1(r) \leq 24T_1(r) + S_1(r)$, which contradicts with $n \geq 26$.

**Case 2.** $H \equiv 0$. Hence Theorem 3 follows from the proof of Theorems 1 and 2. This completes the proof.

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