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A Quadratic Tail Of Zeta*

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Abstract

Quadratic trigamma functions and reciprocal binomial coefficients sums are investigated in this paper. Closed form representations and integral expressions are developed for the infinite series.

1 Introduction and Preliminaries

In a recent paper Furdui [2] evaluates, in closed form, a quadratic series with trigamma function to obtain the result

$$\sum_{n\geq 1} \frac{\left(\psi'(n+1)\right)^2}{n} = 5\zeta(2)\zeta(3) - 9\zeta(5). \tag{1}$$

A generalization of (1) may be expressed for

$$S(k) \equiv \sum_{n=1}^{\infty} \frac{\left(\psi'(n+1)\right)^2}{n^p \binom{n+k}{k}}, \tag{2}$$

for $k \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ and for $p \in \{0, 1\}$. In this paper we shall represent (2) in terms of rational coefficients of special functions. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers. The digamma function is defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\, dt.$$

The digamma (or Psi) function $\psi(z), z \in \mathbb{R}$ can be expressed in terms of harmonic numbers such that $\psi(n+1) = H_n - \gamma$, here γ is the Euler-Mascheroni constant. A generalized harmonic number $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m}, \quad (m, n \in \mathbb{N}) \qquad \text{and} \qquad H_0^{(m)} := 0 \quad (m \in \mathbb{N})$$

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and

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} \qquad (n \in \mathbb{N}_0).$$

The generalized harmonic numbers, $H_z^{(\alpha+1)}$, may be expressed in terms of polygamma functions

$$H_n^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}(n+1),\ n \neq \{-1,-2,-3,...\},$$

where $\zeta(z)$ is the zeta function.

While there are many results for sums of harmonic numbers, see for example [5], [6] and references therein, there are fewer results for sums of the type (2).

The following lemma will be useful in the development in the proofs of the main theorems.

LEMMA 1. For $p = 0$ and $k = 1$ we have, from (2),

$$X(1) \quad : \quad = \sum_{n\geq 1}\frac{\left(\psi'(n+1)\right)^2}{n+1} = \zeta(2)\zeta(3) + \zeta(5) - \frac{5}{2}\zeta(4) \tag{3}$$

$$= \quad -\int_0^1\int_0^1 \frac{(xy+\log(1-xy))\log x\ \log y}{xy(1-x)(1-y)}dxdy. \tag{4}$$

PROOF. To prove (3) we consider

$$X(1) = \sum_{n\geq 1}\frac{\left(\psi'(n+1)\right)^2}{n+1} = \sum_{n\geq 1}\frac{\left(\zeta(2) - H_n^{(2)}\right)^2}{n+1} = \sum_{n\geq 1}\frac{1}{n+1}\left(\sum_{m\geq 1}\frac{1}{(m+n)^2}\right)^2$$

and changing the index of summation we can write

$$\sum_{n\geq 2}\frac{\left(\psi'(n)\right)^2}{n} = \sum_{n\geq 1}\frac{\left(\psi'(n)\right)^2}{n} - \left(\psi'(1)\right)^2.$$

Since $\psi'(1) = \zeta(2)$ and from, $p \in \mathbb{N}$,

$$\psi^{(p)}(n+1) = \psi^{(p)}(n) + \frac{(-1)^p\, p!}{n^{p+1}}$$

we see that

$$\begin{aligned} X(1) &= \sum_{n\geq 1}\frac{\left(\psi'(n+1) + \frac{1}{n^2}\right)^2}{n} - \zeta^2(2) \\ &= \sum_{n\geq 1}\left(\frac{\left(\psi'(n+1)\right)^2}{n} + \frac{2\psi'(n+1)}{n^3} + \frac{1}{n^5}\right) - \zeta^2(2) \\ &= X(0) + 2\sum_{n\geq 1}\frac{\psi'(n+1)}{n^3} + \zeta(5) - \zeta^2(2). \end{aligned}$$

By (1), $X(0) = 5\zeta(2)\zeta(3) - 9\zeta(5)$, from [4]

$$\sum_{n\geq 1} \frac{\psi'(n+1)}{n^3} = -2\zeta(2)\zeta(3) + \frac{9}{2}\zeta(5)$$

and since $\zeta^2(2) = \frac{5}{2}\zeta(4)$, we see that (3) follows. For (4) we use the definition, for $p \in \mathbb{N}_0$,

$$H_n^{(p+1)} = \frac{(-1)^p}{p!} \int_0^1 \frac{\ln^p x}{1-x}(1-x^n)\,dx \tag{5}$$

such that

$$\psi'(n+1) = \zeta(2) - H_n^{(2)} = -\int_0^1 \frac{x^n \log x}{1-x} dx.$$

Then

$$\begin{aligned} X(1) &= \sum_{n\geq 1} \frac{\left(\psi'(n+1)\right)^2}{n+1} = \int_0^1 \int_0^1 \frac{\log x \ \log y}{(1-x)(1-y)} \sum_{n=1}^{\infty} \frac{(xy)^n}{n+1} dx \\ &= -\int_0^1 \int_0^1 \frac{\log x \ \log y \ \left(xy + \log(1-xy)\right)}{xy(1-x)(1-y)} dxdy. \end{aligned}$$

It is of interest to note that since

$$\int_0^1 \int_0^1 \frac{\log x \ \log y}{(1-x)(1-y)} dxdy = \zeta^2(2) = \frac{5}{2}\zeta(4),$$

then from (4) we infer the highly oscillatory integral

$$-\int_0^1 \int_0^1 \frac{\log x \ \log y \ \log(1-xy)}{xy(1-x)(1-y)} dxdy = \zeta(2)\zeta(3) + \zeta(5).$$

The case

$$X(0) = \sum_{n\geq 1} \frac{\left(\psi'(n+1)\right)^2}{n} = 5\zeta(2)\zeta(3) - 9\zeta(5)$$

is proved by Furdui [2], moreover

$$-\int_0^1 \int_0^1 \frac{\log x \ \log y \ \log(1-xy)}{(1-x)(1-y)} dxdy = \sum_{n\geq 1} \frac{\left(\psi'(n+1)\right)^2}{n}.$$

LEMMA 2. Let $r \in \mathbb{N}$. The following equality holds:

$$\begin{aligned} Z(r) &= \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n(n+r)} \\ &= \frac{1}{r}\left(\zeta(3) + H_{r-1}\zeta(2) - \sum_{j=1}^{r-1} \frac{H_j}{j^2}\right), \end{aligned} \tag{6}$$

and, for $r=0$,

$$Z(0)=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\frac{7}{4}\zeta(4).$$

PROOF. The proof of (6) is given in the paper in [4].

LEMMA 3. For $r\in\mathbb{N}$, we have the identity

$$Y(r)=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2(n+r)}=\frac{1}{r}\left(\frac{7}{4}\zeta(4)-Z(r)\right). \tag{7}$$

For $r=0$,

$$Y(0)=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5).$$

PROOF. We have

$$\begin{aligned}Y(r) &= \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2(n+r)}=\frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}-\frac{H_n^{(2)}}{n(n+r)}\right)\\ &= \frac{1}{r}\left(\frac{7}{4}\zeta(4)-Z(r)\right).\end{aligned}$$

LEMMA 4. For $r\in\mathbb{N}$. the following equality holds:

$$\begin{aligned}X(r) &= \sum_{n\geq1}\frac{\left(\psi'(n+1)\right)^2}{n+r}=X(1)+\frac{5(r-1)}{2r}\zeta(4)-H_{r-1}^{(2)}\zeta(3)+H_{r-1}^{(3)}\zeta(2)\\ &\quad -2\sum_{j=1}^{r-1}\left(\frac{H_j\zeta(2)}{j^2}+\frac{H_j}{2j^4}-\frac{Z(j)}{j}\right),\end{aligned} \tag{8}$$

where $X(1)$ and $Z(j)$ are given by (3) and (6), respectively.

PROOF. Consider

$$X(r)=\sum_{n\geq1}\frac{\left(\psi'(n+1)\right)^2}{n+r}=\sum_{n\geq1}\frac{\left(\zeta(2)-H_n^{(2)}\right)^2}{n+r}=\sum_{n\geq1}\frac{1}{n+r}\left(\sum_{m\geq1}\frac{1}{(m+n)^2}\right)^2$$

and changing the index of summation n

$$\begin{aligned} X(r) &= \sum_{n\geq 2}\frac{\left(\psi'(n)\right)^2}{n+r-1}=\sum_{n\geq 1}\frac{\left(\zeta(2)-H_n^{(2)}+\frac{1}{n^2}\right)^2}{n+r-1}-\frac{\zeta^2(2)}{r} \\ &= \sum_{n\geq 1}\frac{\left(\zeta(2)-H_n^{(2)}\right)^2}{n+r-1}+2\zeta(2)\sum_{n\geq 1}\frac{1}{n^2(n+r-1)} \\ &\quad -2\sum_{n\geq 1}\frac{H_n^{(2)}}{n^2(n+r-1)}+\sum_{n\geq 1}\frac{1}{n^4(n+r-1)}-\frac{5}{2r}\zeta(4) \\ &= X(r-1)+\frac{5\zeta(4)}{2r(r-1)}-\frac{\zeta(3)}{(r-1)^2}-\frac{2H_{r-1}\zeta(2)}{(r-1)^2} \\ &\quad +\frac{\zeta(2)}{(r-1)^3}-\frac{H_{r-1}}{(r-1)^4}+\frac{2Z(r-1)}{r-1}. \end{aligned}$$

This resulting recurrence relation, for $r\geq 2$,

$$\begin{aligned} X(r)-X(r-1) &= \frac{5\zeta(4)}{2r(r-1)}-\frac{\zeta(3)}{(r-1)^2}-\frac{2H_{r-1}\zeta(2)}{(r-1)^2} \\ &\quad +\frac{\zeta(2)}{(r-1)^3}-\frac{H_{r-1}}{(r-1)^4}+\frac{2Z(r-1)}{r-1} \end{aligned}$$

can be solved by the successive reduction of $X(r)$, $X(r-1)$, ..., $X(3)$, $X(2)$ such that we obtain (8).

The next few theorems relate the main results of this investigation, namely the closed form and integral representation of (2).

2 Closed form and Integral Identities

We now prove the following Theorems.

THEOREM 5. Let $k\in\mathbb{N}$, then from (2) with $p=0$ we have

$$S(k)=\sum_{n=1}^{\infty}\frac{\left(\psi'(n+1)\right)^2}{\binom{n+k}{k}}=\sum_{r=1}^{k}(-1)^{1+r}\; r\binom{k}{r}X(r), \tag{9}$$

where $X(r)$ is given by (8).

PROOF. Consider the expansion,

$$\begin{aligned}
S(k) &= \sum_{n=1}^{\infty} \frac{\left(\psi'(n+1)\right)^2}{\binom{n+k}{k}} = \sum_{n\geq 1} \frac{\left(\zeta(2) - H_n^{(2)}\right)^2}{\binom{n+k}{k}} \\
&= \sum_{n\geq 1} \frac{1}{\binom{n+k}{k}} \left(\sum_{m\geq 1} \frac{1}{(m+n)^2} \right)^2 \\
&= \sum_{n=1}^{\infty} \frac{k!\ \left(\psi'(n+1)\right)^2}{(n+1)_k} = \sum_{n=1}^{\infty} k!\ \left(\psi'(n+1)\right)^2\ \sum_{r=1}^{k} \frac{\Lambda_r}{n+r},
\end{aligned}$$

where

$$\Lambda_r = \lim_{n\to -r} \frac{n+r}{\prod\limits_{r=1}^{k}(n+r)} = \frac{(-1)^{r+1}\ r}{k!} \binom{k}{r}.$$

Hence

$$S(k) = \sum_{r=1}^{k} (-1)^{r+1}\ r \binom{k}{r} \sum_{n=1}^{\infty} \frac{\left(\psi'(n+1)\right)^2}{n+r} = \sum_{r=1}^{k} (-1)^{r+1}\ r \binom{k}{r} X(r).$$

We call the expression

$$\sum_{n\geq 1} \frac{1}{\binom{n+k}{k}} \left(\sum_{m\geq 1} \frac{1}{(m+n)^2} \right)^2$$

the quadratic tail of zeta. In fact this quadratic tail is associated with the Mordell-Tornheim sums, see for example, [1] and [3].

The other case of $p = 1$, from (2), can be evaluated in a similar fashion. We list the results in the next theorem.

THEOREM 6. Under the assumptions of Theorem 5, we have for $p = 1$,

$$\begin{aligned}
T(k) &= \sum_{n=1}^{\infty} \frac{\left(\psi'(n+1)\right)^2}{n\binom{n+k}{k}} = X(0) + \sum_{r=1}^{k} (-1)^r \binom{k}{r} X(r) \\
&= 5\zeta(2)\zeta(3) - 9\zeta(5) + \sum_{r=1}^{k} (-1)^r \binom{k}{r} X(r). \qquad (10)
\end{aligned}$$

PROOF. The proof follows directly from Theorem 5, and using the same technique.

It is possible to represent the harmonic number sums (6), (8), (9), and (10) in terms of an integral, this is developed in the next theorem.

THEOREM 7. Let $r \in \mathbb{N}$. Then we have:

$$\int_0^1 \frac{x\ \log x}{(1-x)}\ \Phi(x,1,r+1)\ dx = -2\zeta(3) + \frac{1}{r}\zeta(2) + \sum_{j=1}^{r-1} \frac{H_j}{j^2}, \tag{11}$$

where $\Phi(x,1,r+1)$ is the classical Hurwitz-Lerch transcendent. From (8) we identify the representation

$$\int_0^1 \int_0^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\ \Phi(xy,1,r+1)\ dxdy = X(r), \tag{12}$$

where $X(r)$ is given by (8).

PROOF. From (5), we can therefore write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n(n+r)} &= -\int_0^1 \frac{\log x}{1-x} \sum_{n=1}^{\infty} \frac{1-x^n}{n(n+r)} dx \\ &= -\frac{1}{r} \int_0^1 \frac{\log x}{1-x}\ \left(x\Phi(x,1,r+1) + H_r + \ln(1-x)\right)\ dx \\ &= \frac{1}{r}\left(\zeta(3) + H_{r-1}\zeta(2) - \sum_{j=1}^{r-1} \frac{H_j}{j^2}\right).\end{aligned}$$

We can evaluate

$$-\frac{1}{r}\int_0^1 \frac{\log x}{1-x}\ \left(H_r + \ln(1-x)\right)\ dx = \frac{1}{r}\left(H_r\zeta(2) - \zeta(3)\right),$$

in which case

$$-\frac{1}{r}\int_0^1 \frac{x \log x}{1-x}\ \Phi(x,1,r+1)\ dx = \frac{1}{r}\left(2\zeta(3) - \frac{1}{r}\zeta(2) - \sum_{j=1}^{r-1}\frac{H_j}{j^2}\right),$$

hence (11) follows. It is also possible to recover the integral identity (11) from (7).

Similar integral representation can be obtained for $S(k)$ and $T(k)$. The results are recorded in the next theorem.

THEOREM 8. Let the conditions of Theorem 5 hold. Then we have for $k \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{1+k}\int_0^1 \int_0^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\ {}_2F_1\left[\left.\begin{matrix} 1,2 \\ 2+k \end{matrix}\right| xy\right] dxdy \\ =\ & S(k) = \sum_{n=1}^{\infty} \frac{\left(\psi'(n+1)\right)^2}{\binom{n+k}{k}}. \end{aligned} \tag{13}$$

Also for $T(k)$:

$$\begin{aligned}
& \frac{1}{1+k}\int_0^1\int_0^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\ {}_2F_1\left[\begin{array}{c}1,1\\ 2+k\end{array}\middle|\,xy\right]dxdy \\
= & \ T(k)=\sum_{n=1}^{\infty}\frac{\left(\psi'(n+1)\right)^2}{n\binom{n+k}{k}}, \qquad (14)
\end{aligned}$$

where

$${}_2F_1\left[\begin{array}{c}\cdot,\cdot\\ \cdot\end{array}\middle|\,z\right] \text{ is the classical Gauss hypergeometric function.}$$

PROOF. The proof follows the same pattern as that employed in Theorem 5.

EXAMPLES. Some illustrative examples follow. From (13), for $k=5$,

$$\begin{aligned}
& \frac{1}{6}\int_0^1\int_0^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\ {}_2F_1\left[\begin{array}{c}1,2\\ 7\end{array}\middle|\,xy\right]dxdy \\
= & \ \frac{2075}{576}\zeta(2)-\frac{895}{82944}-\frac{125}{48}\zeta(3)-\frac{15}{6}\zeta(4) \\
= & \ S(5)=\sum_{n=1}^{\infty}\frac{\left(\psi'(n+1)\right)^2}{\binom{n+5}{5}}.
\end{aligned}$$

From (14), for $k=3$,

$$\begin{aligned}
& \frac{1}{4}\int_0^1\int_0^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\ {}_2F_1\left[\begin{array}{c}1,1\\ 5\end{array}\middle|\,xy\right]dxdy \\
= & \ 4\zeta(2)\zeta(3)-10\zeta(5)-\frac{55}{12}\zeta(4)+\frac{7}{4}\zeta(3)-\frac{15}{8}\zeta(2)-\frac{45}{32} \\
= & \ T(3)=\sum_{n=1}^{\infty}\frac{\left(\psi'(n+1)\right)^2}{n\binom{n+3}{3}}.
\end{aligned}$$

From (12), for $r=3$, we have

$$\begin{aligned}
& \lim_{(\varepsilon,\delta)\to(0,0)}\int_{\varepsilon}^1\int_{\delta}^1 \frac{xy\ \log x\ \log y}{(1-x)(1-y)}\left(\frac{1}{x^3y^3}+\frac{1}{2x^2y^2}+\frac{\log(1-xy)}{x^4y^4}\right)dxdy \\
= & \ \frac{51}{32}+\frac{3}{8}\zeta(2)-\frac{5}{4}\zeta(3)-\zeta(2)\zeta(3)-\zeta(5).
\end{aligned}$$

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