

# Generalization Of Semi Compatibility With Some Fixed Point Theorems Under Strict Contractive Condition\*

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## Abstract

The main purpose of this paper is to introduce a generalization of the concept of semi compatible mappings and some fixed point theorems are obtained by using the new notion under strict contractive condition. We also demonstrate that new notion is necessary for the existence of common fixed point.

## 1 Introduction and Preliminaries

In 1922 Banach [5], known as the father of fixed point theory introduced and studied the basic and very fruitful concept of contraction mappings. Probably till next five decades, all work involving fixed points used the Banach contraction principle. In 1968 Kannan [12] proved a fixed point theorems for a map satisfying contractive condition that did not require continuity at each point. It has been known from this paper that there exists maps that have a discontinuity in the domain but which have fixed point. During this time many fixed point theorems were proved for pair of maps by replacing  $x$  and  $y$  on right hand of inequality condition with continuous mappings  $S$  and  $T$ . However it was necessary to add some different kind of commutativity and continuity conditions. It was turning point in the study of fixed point when in 1982, the notion of weak commutativity was introduced by Sessa [23] as a generalization of commutativity and sharper tool to obtain common fixed points of mappings. This paper was strong foundation for many fixed point papers over the next two decades. This concept was generalized in regular timing by Jungck [10, 11] by introducing compatible and weak compatible mappings and examples to show that each of these generalizations of commutativity is proper extension of previous definitions. Possibly the first common fixed point theorem without continuity conditions was proved by Pant [14, 15] by introducing reciprocal continuous mappings. Recently, Pant et al. and Pant and Bisht [16, 17] generalized the notion of reciprocal continuity by introducing weak reciprocal continuity and conditionally reciprocal continuity respectively and obtained fixed point theorems. In this connection, the recent papers of Gopal et al. and Bisht et al.

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[7, 9] are also readable. In 2008, Al-Thagafi et al. [4] introduced the weaker form of weakly compatible maps by introducing new notion of occasionally weakly compatible (owc) mappings. Bisht et al. [6] have discussed that, under contractive conditions the existence of common fixed point and occasionally weak compatibility are equivalent conditions. Over the past few years, generalizations of compatible and commuting mappings have been widely used for obtaining fixed points. For this, one can read Patel et al. [18]. Recently, Alghamdi et al. [3] have shown that many recent results which employ different weaker non-commuting notions are not real generalization. Agarwal et al. [2] list a comparison of various non-commuting conditions in metric fixed point theory and their applications. In this connection one can read the paper of Rhoades [20] and Murthy [13]. These weaker non-commuting mappings can be reduced to different weaker forms of commuting mappings under fixed point setting. In this connection, one can follow the recent paper of Abbas et al. [1]. The generalization of compatible mappings called semi compatible mappings is introduced by Singh et al. [24] and it is proved by authors that the concept of semi compatible mappings is equivalent to the concept of compatible mappings under the conditions of mappings. This paper was genesis for many fixed point theorems over next decade. Recently, Saluja et al. [21, 22] generalized the notion of semi compatibility by introducing weak semi compatibility and conditional semi compatibility respectively and obtained some common fixed point theorems by using these notions. Motivated by the result of Saluja et al. [21], we introduce the more general form of semi compatible mappings named strong semi compatible mappings and proved some fixed point theorems under strict contractive condition. Also we have an example which shows independency of strong semi compatibility with noncompatibility of mappings.

Next, we discuss some relevant definitions and results.

DEFINITION 1 ([10]). Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called compatible if  $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$  where  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in X$ .

DEFINITION 2. Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called non-compatible if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in X$ , but  $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n)$  is non-zero or does not exist.

DEFINITION 3 ([24]). Two self maps  $f$  and  $g$  of metric space  $(X, d)$  are called semi compatible if  $\lim_{n \rightarrow \infty} fgx_n = gx$  holds when  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$  for some  $x \in X$ .

DEFINITION 4 ([19]). Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called  $R$ -weak commuting of type  $(A_g)$  if there exists some positive real number  $R$  such that  $d(gfx, ffx) \leq Rd(fx, gx)$  for all  $x \in X$ .

DEFINITION 5 ([19]). Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called  $R$ -weak commuting of type  $(A_f)$  if there exists some positive real number  $R$  such that  $d(fgx, ggx) \leq Rd(fx, gx)$  for all  $x \in X$ .

DEFINITION 6. Let  $X$  be a set, and  $f$  and  $g$  be self maps of  $X$ . A point  $x$  in  $X$  is called coincidence point of  $f$  and  $g$  iff  $fx = gx$ . If  $C(f, g)$  is a set of coincidence points, then it can be given by  $C(f, g) = \{x : fx = gx \text{ where } x \in X\}$ .

DEFINITION 7 ([4]). A pair  $(f, g)$  of self mappings defined on a nonempty set  $X$  is said to be occasionally weakly compatible mappings (in short owc) if there exists a point  $x$  in  $X$ , which is a coincidence point of  $f$  and  $g$  at which  $f$  and  $g$  commute.

DEFINITION 8. ([11]). A pair  $(f, g)$  of self mappings of nonempty set  $X$  is said to be weakly compatible if the mappings commute at their coincidence points, i.e.,  $fx = gx, (x \in X)$  implies  $fgx = gfx$ .

DEFINITION 9 ([21]). A pair  $(f, g)$  of self mappings of metric space  $(X, d)$  is called conditional semi compatible (in short csc) if the set of sequence  $\{x_n\}$  satisfying  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n$  is nonempty, then there exists at least a sequence  $\{y_n\}$  satisfying  $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = t$  such that  $\lim_{n \rightarrow \infty} fgy_n = gt$  and  $\lim_{n \rightarrow \infty} gfy_n = ft$ .

Notice that semi compatibility is independent from conditional semicompatibility. The following examples illustrate this fact.

EXAMPLE. Let  $X = [2, \infty)$  with the usual metric  $d$ ,

$$f(x) = \begin{cases} 2 & \text{if } 2 \leq x < 4, \\ 4 & \text{if } x \geq 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3 & \text{if } 2 \leq x < 4, \\ x & \text{if } x \geq 4. \end{cases}$$

Clearly,  $f, g : X \rightarrow X$ . We take sequence  $x_n = 4 + \varepsilon_n$  where  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . It follows that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 4, \quad \lim_{n \rightarrow \infty} fgx_n = 4 = g(4) \quad \text{and} \quad \lim_{n \rightarrow \infty} gfy_n = 4 = f(4).$$

So  $f$  and  $g$  are semi compatible but not conditional semi compatible.

EXAMPLE. Let  $X = [2, 8]$  with usual metric  $d$ ,

$$f(x) = \begin{cases} x & \text{if } 2 \leq x \leq 5, \\ x + 2 & \text{if } 5 < x \leq 7, \\ 2 & \text{if } 7 < x \leq 8, \end{cases}$$

and

$$g(x) = \begin{cases} (x + 2)/2 & \text{if } 2 \leq x \leq 5, \\ 2x - 3 & \text{if } 5 < x \leq 7, \\ 3 & \text{if } 7 < x \leq 8. \end{cases}$$

Clearly,  $f, g : X \rightarrow X$ . We consider a sequence  $x_n = 5 + \frac{1}{n}$ . Then

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 7,$$

$$\lim_{n \rightarrow \infty} fgx_n = 2 \neq g(7) \text{ and } \lim_{n \rightarrow \infty} gfx_n = 3 \neq f(7).$$

If the sequence  $x_n = 2 + \frac{1}{n}$  is considered, then

$$\begin{aligned} \lim_{n \rightarrow \infty} fx_n &= \lim_{n \rightarrow \infty} gx_n = 2, \\ \lim_{n \rightarrow \infty} fgx_n &= 2 = g(2) \text{ and } gfx_n = 2 = f(2). \end{aligned}$$

So the pair  $(f, g)$  is conditional semi compatible but not semi compatible.

The two examples show that conditional semi compatible and semi compatible mappings are independent notions. In the following example,  $f$  and  $g$  are conditional semi compatible but they are not necessarily occasionally weakly compatible.

EXAMPLE. Let  $X = [2, 8]$  with the usual metric  $d$ ,

$$f(x) = \begin{cases} x^2 & \text{if } 2 \leq x \leq 4, \\ x - 2 & \text{if } 4 < x \leq 8 \end{cases} \text{ and } g(x) = \begin{cases} 4 & \text{if } 2 \leq x \leq 4, \\ x/2 & \text{if } 4 < x \leq 8. \end{cases}$$

Clearly,  $f, g : X \rightarrow X$ . We take a sequence  $x_n = 2 + \frac{1}{n}$ . It follows that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 4 \text{ and } \lim_{n \rightarrow \infty} fgx_n = 16 \neq g(4).$$

In addition, if we take a sequence  $x_n = 4 + \frac{1}{n}$ . It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} fx_n &= \lim_{n \rightarrow \infty} gx_n = 2, \\ \lim_{n \rightarrow \infty} fgx_n &= 4 = g(2) \text{ and } \lim_{n \rightarrow \infty} gfx_n = 4 = f(2). \end{aligned}$$

Since  $f(2) = g(2)$  and  $fg(2) \neq gf(2)$ , we see that the pair  $(f, g)$  is conditional semi compatible but not owc.

DEFINITION 10. Two self maps  $f$  and  $g$  of metric space  $(X, d)$  are said to be strong semi compatible iff  $f$  and  $g$  are conditional semi compatible and owc as well.

EXAMPLE. Let  $X = [2, 8]$ ,  $d$  be the usual metric on  $X$ ,

$$f(x) = \begin{cases} x^2 & \text{if } 2 \leq x < 5, \\ (x-1)/2 & \text{if } 5 \leq x \leq 8 \end{cases} \text{ and } g(x) = \begin{cases} 4 & \text{if } 2 \leq x < 5, \\ x-3 & \text{if } 5 \leq x \leq 8. \end{cases}$$

Clearly,  $f, g : X \rightarrow X$ . We take a sequence  $x_n = 2 + \frac{1}{n}$ . Then we obtain

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 4 \text{ and } \lim_{n \rightarrow \infty} fgx_n = 16 \neq g(4).$$

Next, we consider a sequence  $y_n = 5 + \frac{1}{n}$ . Then

$$\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = 2,$$

$$\lim_{n \rightarrow \infty} fgy_n = 4 = g(2) \quad \text{and} \quad \lim_{n \rightarrow \infty} gfy_n = 4 = f(2).$$

Here 5 is the coincidence point of  $f$  and  $g$  and they commute at their coincidence point. It shows that  $f$  and  $g$  are strong semi compatible mappings.

Here we demonstrate that the notion "strong semicompatibility" and noncompatibility are independent concepts. The following examples illustrate this fact.

EXAMPLE. Let  $X$  be a real set with the usual metric  $d$ , and  $f, g : X \rightarrow X$  where

$$fx = 1 + x \quad \text{and} \quad gx = 1 - x \quad \text{for all } x.$$

We take a sequence  $x_n = 1/n$ . It follows that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(fgx_n, gfy_n) = 2.$$

So the pair of maps  $(f, g)$  is non-compatible, but not strong semi compatible.

EXAMPLE. Let  $X = [2, 8]$  with the usual metric  $d$ ,

$$f(x) = \begin{cases} 2x + 1 & \text{if } 2 \leq x < 5, \\ x - 3 & \text{if } 5 \leq x \leq 8 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 3 & \text{if } 2 \leq x < 5, \\ 2 & \text{if } 5 \leq x \leq 8. \end{cases}$$

Clearly,  $f, g : X \rightarrow X$ . In the present example, pair  $(f, g)$  is strong semi compatible but not non-compatible. To see this, we consider a sequence  $x_n = 2 + \epsilon_n$  where  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} fx_n &= \lim_{n \rightarrow \infty} gx_n = 5, \\ \lim_{n \rightarrow \infty} fgx_n &= 2 = g(5) \quad \text{and} \quad \lim_{n \rightarrow \infty} gfy_n = 2 = f(5). \end{aligned}$$

Also 5 is a coincidence point of pair of maps  $(f, g)$  and they commute at their coincidence point.

## 2 Main Results

THEOREM 1. Let  $f$  and  $g$  be non-compatible strong semi compatible self mappings of a usual metric space  $(X, d)$  such that

- (a)  $f(x) \subseteq g(x)$ ;
- (b)  $d(fx, fy) < d(gx, gy)$ , whenever  $gx \neq gy$ ;
- (c) either  $f$  and  $g$  are  $R$ -weak commuting of type  $A_f$  or  $A_g$ .

Then  $f$  and  $g$  have common fixed point in  $X$ .

PROOF. Noncompatibility of  $f$  and  $g$  implies that there exists some sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = t$  and  $\lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in X$  but

$\lim_{n \rightarrow \infty} d(fgx_n, gfx_n)$  is either non zero or does not exist. Since  $f$  and  $g$  are strong semi compatible maps,  $\lim_{n \rightarrow \infty} fx_n = t$  and  $\lim_{n \rightarrow \infty} gx_n = t$ , there exists a sequence  $\{y_n\}$  in  $X$  satisfying  $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = u$  for some  $u \in X$  such that

$$\lim_{n \rightarrow \infty} fgy_n = gu \text{ and } \lim_{n \rightarrow \infty} gfy_n = fu.$$

When  $f$  and  $g$  are  $R$ -weak commuting of type  $A_f$ , this yields

$$d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n) \text{ as } R > 0.$$

Limiting  $n \rightarrow \infty$  yields  $\lim_{n \rightarrow \infty} ggy_n = gu$ . Let  $fu \neq gu$ , then by (b) we have  $d(fgy_n, fu) < d(ggy_n, gu)$ . On limiting  $n \rightarrow \infty$  we have  $d(gu, fu) < d(gu, gu)$  which is a contradiction and hence  $fu = gu$ . Since pair  $(f, g)$  is strong semi compatible, therefore  $fgu = gfu$  for some  $u$  in  $X$ , where  $u \in C(f, g)$  the set of coincidence points. It yields further  $fgu = gfu = ffu = ggu$ . Now again by (b) when supposed  $ffu \neq fu$ ,  $d(fu, ffu) < d(gu, gfu)$ . This gives  $d(fu, ffu) < d(fu, ffu)$ , which is a contradiction and hence  $ffu = fu$ . This concludes  $ffu = gfu = fu$  or  $fu$  is common fixed point of  $f$  and  $g$ .

When  $f$  and  $g$  are  $R$ -weak commuting of type  $A_g$ , this yields

$$d(gfy_n, ffy_n) \leq Rd(fy_n, gy_n) \text{ as } R > 0.$$

Limiting  $n \rightarrow \infty$  yields  $\lim_{n \rightarrow \infty} ffy_n = fu$ . Let  $fu \neq u$  then by (b) we have  $d(ffy_n, fy_n) < d(gfy_n, gy_n)$ . On limiting  $n \rightarrow \infty$  gives  $d(fu, u) < d(fu, u)$ , which is a contradiction and hence  $fu = u$ . Since  $f(X) \subseteq g(X)$ , there exists some point  $v$  in  $X$  such that  $fu = gv$ . Now by (b) when assuming

$$fv \neq gv, d(fv, ffy_n) < d(gv, gfy_n).$$

Limiting  $n \rightarrow \infty$  yields  $d(fv, fu) < d(fu, fu)$ , which is a contradiction and so  $fv = gv$ . Since  $f$  and  $g$  are strong semi compatible mappings, this yields  $fgv = gfv$  for some  $v$  in  $X$  such that  $v \in C(f, g)$  the set of coincidence points. It yields further

$$fgv = gfv = ffv = ggv.$$

If  $fgv \neq fv$ . Then by (b),  $d(fgv, fv) < d(ggv, gv)$ . It gives further  $d(fgv, fv) < d(fgv, fv)$ , which is a contradiction and hence  $fgv = fv$  or  $fgv = gv$ .

EXAMPLE. Let  $x, y \in X$  ( $x \neq y$ ) where  $X = [1, 10]$  and  $d$  be the usual metric on  $X$ . Define  $f, g : X \rightarrow X$  as follows:

$$fx = \begin{cases} x & \text{if } 1 \leq x < 5, \\ 9 & \text{if } 5 \leq x \leq 10 \end{cases} \text{ and } gx = \begin{cases} 3x - 2 & \text{if } 1 \leq x < 5, \\ 2x - 1 & \text{if } 5 \leq x \leq 10. \end{cases}$$

If sequence  $x_n = 5 + \epsilon_n$  is taken where  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , then we have  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 9$  but  $\lim_{n \rightarrow \infty} gfx_n \neq f(9)$ . If the sequence  $x_n = 1 + \epsilon_n$  is taken where  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 1, \quad \lim_{n \rightarrow \infty} gfx_n = 1 = g(1)$$

and

$$\lim_{n \rightarrow \infty} g f x_n = 1 = f(1).$$

We observe that  $f(5) = g(5)$ ,  $f g(5) \neq g f(5)$ ,  $f(1) = g(1)$  and  $f g(1) = g f(1)$ . So  $f$  and  $g$  are strong semi compatible mappings. It follows that maps  $f$  and  $g$  satisfy all conditions as  $f(X) \subseteq g(X)$ ,  $R$ -weak commuting type of  $A_g$  and noncompatibility. Finally, we see that

$$d(fx, fy) = |x - y| \text{ and } d(gx, gy) = 3|x - y| \text{ for } x \in [1, 5]$$

and that

$$d(fx, fy) = 0 \text{ and } d(gx, gy) = 2|x - y| \text{ for } x \in [5, 10].$$

So  $f$  and  $g$  satisfy condition (b). Therefore  $f$  and  $g$  satisfy all conditions of theorem and have a common fixed point at  $x = 1$ .

REMARK. It is well known that for existence of common fixed point under strict contraction condition, the Cauchy sequence should be considered. But here, this theorem is proved without taking completeness and even no Cauchy sequences are considered.

COROLLARY 1. Let  $f$  and  $g$  be strong semi compatible mappings of usual metric space  $(X, d)$  satisfying all conditions of Theorem 1 except condition (b) and instead of (b)  $f$  and  $g$  satisfying

$$d(fx, fy) \leq kd(gx, gy), \quad 0 \leq k < 1.$$

Then  $f$  and  $g$  have a common fixed point in  $X$ .

THEOREM 2. Let  $f$  and  $g$  be non-compatible strong semi compatible self mappings of a usual metric space  $(X, d)$  such that

(a)  $f(x) \subseteq g(x)$ ,

(b)

$$d(fx, fy) < \max \left\{ d(gx, gy), \frac{d(fx, gx) + d(fy, gy)}{2}, \frac{d(fx, gy) + d(fy, gx)}{2} \right\}$$

where the right hand side is positive,

(c) either  $f$  and  $g$  are  $R$ -weak commuting of type  $A_f$  or  $A_g$ .

Then  $f$  and  $g$  have common fixed point in  $X$ .

PROOF. Noncompatibility of  $f$  and  $g$  implies that there exists some sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} f x_n = t$  and  $\lim_{n \rightarrow \infty} g x_n = t$  for some  $t \in X$  but  $\lim_{n \rightarrow \infty} d(f g x_n, g f x_n)$  is either non zero or does not exist. Since  $f$  and  $g$  are strong

semi compatible mappings and  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  then there is a sequence  $\{y_n\}$  in  $X$  satisfying  $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = u$  for some  $u \in X$  such that  $\lim_{n \rightarrow \infty} fgy_n = gu$  and  $\lim_{n \rightarrow \infty} gfy_n = fu$ .

When  $f$  and  $g$  are  $R$ -weak commuting of type  $A_f$ , this yields  $d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n)$  as  $R > 0$ . Now limiting  $n \rightarrow \infty$ , which yields  $\lim_{n \rightarrow \infty} ggy_n = gu$ . Then we assert that  $fu = gu$ . If the assertion is not true, then by (b),

$$d(fgy_n, fu) < \max \left\{ d(ggy_n, gu), \frac{d(fgy_n, ggy_n) + d(fu, gu)}{2}, \frac{d(fgy_n, gu) + d(fu, ggy_n)}{2} \right\}.$$

As  $n \rightarrow \infty$ , we obtain

$$d(gu, fu) < \frac{1}{2}d(fu, gu),$$

which is a contradiction. So  $fu = gu$ . Since pair  $(f, g)$  is strong semi compatible, we see that  $fgu = gfu$  for some  $u \in X$  satisfying  $u \in C(f, g)$ , the set of coincidence points. It yields further  $fgu = gfu = ffu = ggu$ . If  $ffu \neq fu$ , by (b), we see that

$$d(ffu, fu) < \max \left\{ d(gfu, gu), \frac{d(ffu, gfu) + d(fu, gu)}{2}, \frac{d(ffu, gu) + d(fu, gfu)}{2} \right\}.$$

Then we further see that  $d(ffu, fu) < d(ffu, fu)$ . It is a contradiction. So  $ffu = fu$ . This implies that  $ffu = gfu = fu$  and  $fu$  is a common fixed point of  $f$  and  $g$ .

When  $f$  and  $g$  are  $R$ -weak commuting of type  $A_g$ , we obtain that  $d(gfy_n, ffy_n) \leq Rd(fy_n, gy_n)$  as  $R > 0$ . Then

$$\lim_{n \rightarrow \infty} ffy_n = fu.$$

Next, we assert that  $fu = u$ . If the assertion is not true, by (b), we see that

$$d(ffy_n, fy_n) < \max \left\{ d(gfy_n, gy_n), \frac{d(ffy_n, gfy_n) + d(fy_n, gy_n)}{2}, \frac{d(ffy_n, gy_n) + d(fy_n, gfy_n)}{2} \right\}.$$

As  $n \rightarrow \infty$ , we obtain  $d(fu, u) < d(fu, u)$ . It is a contradiction. So  $fu = u$ . Since  $f(X) \subseteq g(X)$ , there exists a point  $v \in X$  such that  $fu = gv$ . By (b) with assuming  $fv \neq gv$ ,

$$d(fv, ffy_n) < \max \left\{ d(gv, gfy_n), \frac{d(fv, gv) + d(ffy_n, gfy_n)}{2}, \frac{d(fv, gfy_n) + d(ffy_n, gv)}{2} \right\}.$$

As  $n \rightarrow \infty$ , we obtain

$$d(fv, gv) < \frac{1}{2}d(fv, gv),$$



which is a contradiction. So  $fv = gv$ . Since pair  $(f, g)$  is strong semi compatible, we see that  $fgv = gfv$  for some  $v \in X$  satisfying  $v \in C(f, g)$ , the set of coincidence points. It yields further  $fgv = gfv = ffv = ggv$ . Again by (b) with letting  $fgv \neq gv$ ,

$$d(fgv, fv) < \max \left\{ d(ggv, gv), \frac{d(fgv, ggv) + d(fv, gv)}{2}, \frac{d(fgv, gv) + d(fv, ggv)}{2} \right\}$$

On simplifying, this yields  $d(fgv, gv) < d(fgv, gv)$ . It is a contradiction. So  $fgv = gv$ . The conclusion raises that  $fgv = ggv = gv$ . Therefore  $gv$  is a common fixed point of  $f$  and  $g$ .

EXAMPLE. Let  $x, y \in X$  ( $x \neq y$ ) where  $X = [1, 6]$  and  $d$  be the usual metric on  $X$ . Define  $f, g : X \rightarrow X$  as follows:

$$fx = \begin{cases} 2x + 1 & \text{if } x \in [1, 2), \\ x/2 & \text{if } x \in [2, 3), \\ 4 & \text{if } x \in [3, 6], \end{cases} \quad \text{and } gx = \begin{cases} 3x & \text{if } x \in [1, 2), \\ 2x - 3 & \text{if } x \in [2, 3), \\ x & \text{if } x \in [3, 6]. \end{cases}$$

Then  $f$  and  $g$  satisfy all the conditions of Theorem 2 and have common fixed point at  $x = 4$ . It can be verified in this example that 2 is coincidence point of  $f$  and  $g$  and they commute at their coincidence point. Furthermore,  $f$  and  $g$  are non-compatible. Also  $f$  and  $g$  are conditional semi-compatible. To see this, let us consider a sequence  $x_n = 2 + \epsilon_n$ , where  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\lim f x_n = \lim g x_n = 1$  and  $\lim f g x_n = g(1)$ ,  $\lim g f x_n = f(1)$ .

REMARK. The result of Theorem 2 will remains same if one replace to  $R$ -weak commutativity of type  $A_f$  or  $A_g$  by compatibility, the  $f$ -compatibility or  $g$ -compatibility of mappings  $f$  and  $g$ .

REMARK. Strong semi compatibility is necessary condition for existence of common fixed points of given mappings  $f$  and  $g$ . Let  $f$  and  $g$  are self mappings of metric space  $(X, d)$ . Let  $v$  be the fixed point of  $f$  and  $g$ . Therefore  $fv = gv = v$  also  $fgv = gfv$ . If we choose the sequence  $x_n = v$ , then  $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = v$ . Also,

$$\lim_{n \rightarrow \infty} f g x_n = f g v = v = g v, \quad \lim_{n \rightarrow \infty} g f x_n = g f v = v = f v.$$

Therefore  $f$  and  $g$  are strong semi compatible mappings. This shows necessary condition for existence of common fixed point of maps  $f$  and  $g$ . Whereas the strong semi compatible mappings is not a sufficient condition for existence of common fixed points. We take following example to ensure it

$$fx = \begin{cases} x + 2 & \text{if } x \in [2, 4), \\ 6 & \text{if } x \in [4, 6], \end{cases} \quad \text{and } gx = \begin{cases} 3x - 2 & \text{if } x \in [2, 4), \\ x + 2 & \text{if } x \in [4, 6]. \end{cases}$$

If the sequence  $x_n = 2 + \epsilon_n$  is taken, where  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = 4, \quad \lim_{n \rightarrow \infty} f g x_n = 6 = g(4),$$

and

$$\lim_{n \rightarrow \infty} g f x_n = 6 = f(4).$$

Also  $f(2) = g(2)$  and  $f g(2) = g f(2)$ . This concludes that maps  $f$  and  $g$  are strong semi compatible but they do not have any common fixed points.

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