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Generalized Dual Fibonacci Quaternions*

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Abstract

In this paper, we defined the generalized dual Fibonacci quaternions. Also, we investigated the relations between different generalized dual Fibonacci quaternions. Furthermore, we gave the Binet's formulas and Cassini identities for these quaternions.

1 Introduction

The quaternions are generalized numbers. They were first described by the Irish mathematician William Rowan Hamilton in 1843. Hamilton [1] introduced the set of quaternions which can be represented as

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i \text{ and } ki = -ik = j.$$

The split quaternions or co-quaternions are elements of a 4-dimensional associative algebra introduced by James Cockle in 1849, similar to the quaternions introduced by Hamilton. Cockle [2] introduced the set of split quaternions which can be represented as

$$H_S = \{q = q_0 + iq_1 + jq_2 + kq_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where

$$i^2 = -1,\ j^2 = k^2 = 1 \text{ and } ijk = 1.$$

Several authors worked on different quaternions and their generalizations [3, 5–18]. In 2013, Akyiğit et al. [17] defined split Fibonacci and split Lucas quaternions and obtained some identities for them. Complex split quaternions defined by Kula and Yayli [13] in 2007. In 1963, Horadam [3] firstly introduced the *nth* Fibonacci quaternion and generalized Fibonacci quaternions, which can be represented as

$$H_F = \{Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},\ n \geq 1\text{: } F_n \text{ is } n\text{-th Fibonacci number}\}$$

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where

$$i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k,\ jk = -kj = i \text{ and } ki = -ik = j.$$

In 1969, Iyer [5, 6] derived many relations for the Fibonacci quaternions. Also, in 1973, Swamy [8] considered generalized Fibonacci quaternions as follows

$$P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \text{ for } n \geq 1$$

where

$$\begin{cases} H_n = H_{n-1} + H_{n-2}, \\ H_1 = p, \\ H_2 = p + q, \end{cases} \quad \text{or} \quad H_n = (p-q)F_n + qF_{n+1}.$$

Here, H_n is the n-th generalized Fibonacci number defined in [4], see [8] for generalized Fibonacci quaternions. In 1977, Iakin [9, 10] introduced higher order quaternions and gave some identities for these quaternions. In 1993, Horadam [12] extended quaternions to the complex Fibonacci numbers defined by Harman [11]. In 2012, Halıcı [15] gave generating functions and Binet's formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [16] defined complex Fibonacci quaternions as follows

$$H_{FC} = \{R_n = C_n + e_1C_{n+1} + e_2C_{n+2} + e_3C_{n+3},\, n \geq 1 : C_n = F_n + iF_{n+1},\ i^2 = -1\}$$

where

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = e_1e_2e_3 = -1,\ e_1e_2 = -e_2e_1 = e_3,$$
$$e_2e_3 = -e_3e_2 = e_1 \text{ and } e_3e_1 = -e_1e_3 = e_2.$$

In 2009, Ata and Yaylı [17] defined dual quaternions with dual numbers $(a + \varepsilon b,\ a, b \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0)$ coefficient as follows

$$H(\mathbb{D}) = \{Q = A + Bi + Cj + Dk : A, B, C, D \in \mathbb{D} \text{ and } i^2 = j^2 = k^2 = -1 = ijk\}. \quad (1)$$

In 2014, Nurkan and Güven [18] defined dual Fibonacci quaternions as follows

$$\begin{aligned} H(\mathbb{D}) \quad = \quad & \Big\{\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + k\tilde{F}_{n+3},\, n \geq 1 : \tilde{F}_n = F_n + \epsilon F_{n+1}, \\ & \epsilon^2 = 0 \text{ and } \epsilon \neq 0\Big\} \end{aligned} \quad (2)$$

where

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i,$$
$$ki = -ik = j \text{ and } \tilde{Q}_n = Q_n + \varepsilon Q_{n+1}.$$

Essentially, these quaternions in equations (1) and (2) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. Majernik [20] defined dual quaternions as follows

$$H_{\mathbb{D}} = \{Q = a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = ijk = 0 \text{ and } ij = -ji = jk = -kj = ki = -ik = 0.$$

For more details on dual quaternions, see [19]. It is clear that $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets. In 2015, Yüce and Torunbalcı Aydın [21] defined dual Fibonacci quaternions as follows

$$\begin{aligned} H_{\mathbb{D}} \quad = \quad & \{Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},\ n \geq 1 : \\ & F_n \text{ is } n\text{-th Fibonacci number}\} \end{aligned} \tag{3}$$

where

$$i^2 = j^2 = k^2 = ijk = 0 \text{ and } ij = -ji = jk = -kj = ki = -ik = 0.$$

The Lucas sequence (L_n) and D_n^L which are the n-th term of the dual Lucas quaternion sequence (D_n^L) are defined by the following recurrence relations:

$$\begin{cases} L_{n+2} = L_{n+1} + L_n,\ \forall n \geq 0, \\ L_0 = 2,\ L_1 = 1, \end{cases}$$

$$D_n^L = L_n + iL_n + jL_{n+2} + kL_{n+3} \text{ and } i^2 = j^2 = k^2 = ijk = 0.$$

In this paper, we will define the generalized dual Fibonacci quaternions as follows

$$\begin{aligned} Q_{\mathbb{D}} \quad = \quad & \{\mathbb{D}_{\mathbf{n}} = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3},\ n \geq 1 : H_n \text{ is } n\text{-th generalized} \\ & \text{Fibonacci number}\}, \end{aligned}$$

where

$$i^2 = j^2 = k^2 = ijk = 0 \text{ and } ij = -ji = jk = -kj = ki = -ik = 0.$$

Also, we will give Binet's Formula and Cassini identities for the generalized dual Fibonacci quaternions.

2 Generalized Dual Fibonacci Quaternions

The generalized Fibonacci sequence H_n is defined as

$$\text{either } H_n = H_{n-1} + H_{n-2},\ n \geq 2 \ \text{ or } \ H_n = (p-q)F_n + qF_{n+1},\ n \geq 0 \tag{4}$$

where $H_0 = q$, $H_1 = p$, $H_2 = p + q$ and $p, q \in \mathbb{Z}$. Here, H_n is the n-th generalized Fibonacci number that defined in [4]. We can define the generalized dual Fibonacci quaternions by using generalized Fibonacci numbers as follows

$$\begin{aligned} Q_{\mathbb{D}} \quad = \quad & \{\mathbb{D}_{\mathbf{n}} = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} : H_n \text{ is } n\text{-th generalized} \\ & \text{Fibonacci number}\} \end{aligned} \tag{5}$$

where

$$i^2 = j^2 = k^2 = ijk = 0 \text{ and } ij = -ji = jk = -kj = ki = -ik = 0.$$

Let $\mathbb{D}_{\mathbf{n}}^{\mathbf{1}}$ and $\mathbb{D}_{\mathbf{n}}^{\mathbf{2}}$ be the n-th generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}}^{\mathbf{1}})$ and $(\mathbb{D}_{\mathbf{n}}^{\mathbf{2}})$ such that

$$\mathbb{D}_{\mathbf{n}}^{\mathbf{1}} = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \tag{6}$$

and

$$\mathbb{D}_{\mathbf{n}}^{\mathbf{2}} = G_n + iG_{n+1} + jG_{n+2} + kG_{n+3}.$$

Then, the addition and subtraction of the generalized dual Fibonacci quaternions is defined by

$$\begin{aligned} \mathbb{D}_{\mathbf{n}}^{\mathbf{1}} \pm \mathbb{D}_{\mathbf{n}}^{\mathbf{2}} &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \pm (G_n + iG_{n+1} + jG_{n+2} + kG_{n+3}) \\ &= (H_n \pm G_n) + i(H_{n+1} \pm G_{n+1}) + j(H_{n+2} \pm G_{n+2}) + k(H_{n+3} \pm G_{n+3}). \end{aligned}$$

Multiplication of the generalized dual Fibonacci quaternions is defined by

$$\begin{aligned} \mathbb{D}_{\mathbf{n}}^{\mathbf{1}} \mathbb{D}_{\mathbf{n}}^{\mathbf{2}} &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(G_n + iG_{n+1} + jG_{n+2} + kG_{n+3}) \\ &= (H_n G_n) + i(H_n G_{n+1} + H_{n+1} G_n) + j(H_n G_{n+2} + H_{n+2} G_n) \\ &\quad + k(H_n G_{n+3} + H_{n+3} G_n) \\ &= (H_n G_n) + H_n(iG_{n+1} + jG_{n+2} + kG_{n+3}) \\ &\quad + (iH_{n+1} + jH_{n+2} + kH_{n+3})G_n. \end{aligned} \tag{7}$$

The scaler and the vector part of $\mathbb{D}_{\mathbf{n}}$ which is the n-th term of the generalized dual Fibonacci quaternion $(\mathbb{D}_{\mathbf{n}})$ are denoted by

$$S_{\mathbb{D}_{\mathbf{n}}} = H_n \text{ and } V_{\mathbb{D}_{\mathbf{n}}} = iH_{n+1} + jH_{n+2} + kH_{n+3}.$$

Thus, the generalized dual Fibonacci quaternion $\mathbb{D}_{\mathbf{n}}$ is given by $\mathbb{D}_{\mathbf{n}} = S_{\mathbb{D}_{\mathbf{n}}} + V_{\mathbb{D}_{\mathbf{n}}}$. Then, relation (7) is defined by

$$\mathbb{D}_{\mathbf{n}}^{\mathbf{1}} \mathbb{D}_{\mathbf{n}}^{\mathbf{2}} = S_{\mathbb{D}_{\mathbf{n}}^{\mathbf{1}}} S_{\mathbb{D}_{\mathbf{n}}^{\mathbf{2}}} + S_{\mathbb{D}_{\mathbf{n}}^{\mathbf{1}}} V_{\mathbb{D}_{\mathbf{n}}^{\mathbf{2}}} + S_{\mathbb{D}_{\mathbf{n}}^{\mathbf{2}}} V_{\mathbb{D}_{\mathbf{n}}^{\mathbf{1}}}.$$

The conjugate of generalized dual Fibonacci quaternion $\mathbb{D}_{\mathbf{n}}$ is denoted by $\overline{\mathbb{D}}_{\mathbf{n}}$ and it is

$$\overline{\mathbb{D}}_{\mathbf{n}} = H_n - iH_{n+1} - jH_{n+2} - kH_{n+3}. \tag{8}$$

The norm of $\mathbb{D}_{\mathbf{n}}$ is defined as

$$\|\mathbb{D}_{\mathbf{n}}\|^2 = \mathbb{D}_{\mathbf{n}} \overline{\mathbb{D}}_{\mathbf{n}} = (H_n)^2. \tag{9}$$

Then, we give the following theorem using statements (4), (5) and the generalized Fibonacci number in [4] as follows

$$\begin{aligned} H_n H_m + H_{n+1} H_{m+1} &= p^2 F_{n+m+1} + 2pq F_{n+m} + q^2 F_{n+m-1} \\ &= (2p - q) H_{n+m+1} - e F_{n+m+1} \end{aligned} \tag{10}$$

where $e = p^2 - pq - q^2$.

THEOREM 1. Let H_n and $\mathbb{D}_{\mathbf{n}}$ be the n-th terms of generalized Fibonacci sequence (H_n) and the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}})$, respectively. In this case, for $n \geq 1$ we can give the following relations

$$\mathbb{D}_{\mathbf{n}} + \mathbb{D}_{\mathbf{n+1}} = \mathbb{D}_{\mathbf{n+2}}, \tag{11}$$

$$(\mathbb{D}_{\mathbf{n}})^{\mathbf{2}} = 2H_n.\mathbb{D}_{\mathbf{n}} - H_n^2, \tag{12}$$

$$\mathbb{D}_{\mathbf{n}} - i\mathbb{D}_{\mathbf{n+1}} - j\mathbb{D}_{\mathbf{n+2}} - k\mathbb{D}_{\mathbf{n+3}} = H_n, \tag{13}$$

$$\mathbb{D}_{\mathbf{n}}\mathbb{D}_{\mathbf{m}} + \mathbb{D}_{\mathbf{n+1}}\mathbb{D}_{\mathbf{m+1}} = (2p-q)[2\mathbb{D}_{\mathbf{n+m+1}} - H_{n+m+1}] - e[2Q_{n+m+1} - F_{n+m+1}] \tag{14}$$

where Q_{n+m+1} is the dual Fibonacci quaternion.

PROOF. By

$$\mathbb{D}_{\mathbf{n}} = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \tag{15}$$

and

$$\mathbb{D}_{\mathbf{n+1}} = H_{n+1} + iH_{n+2} + jH_{n+3} + kH_{n+4}, \tag{16}$$

we see that

$$\begin{aligned}
\mathbb{D}_{\mathbf{n}} + \mathbb{D}_{\mathbf{n+1}} &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) + (H_{n+1} + iH_{n+2} + jH_{n+3} + kH_{n+4}) \\
&= H_n + H_{n+1} + i(H_{n+1} + H_{n+2}) + j(H_{n+2} + H_{n+3}) + k(H_{n+3} + H_{n+4}) \\
&= H_{n+2} + iH_{n+3} + jH_{n+4} + kH_{n+5} \\
&= \mathbb{D}_{\mathbf{n+2}}.
\end{aligned}$$

So (11) holds. We observe

$$\begin{aligned}
(\mathbb{D}_{\mathbf{n}})^{\mathbf{2}} &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
&= H_n^2 + 2i(H_nH_{n+1}) + 2j(H_nH_{n+2}) + k(H_nH_{n+3}) = 2H_n\mathbb{D}_{\mathbf{n}} - H_n^2.
\end{aligned}$$

So (12) holds. By (6) and conditions in the equation (5), we see that

$$\begin{aligned}
\mathbb{D}_{\mathbf{n}} - i\mathbb{D}_{\mathbf{n+1}} - j\mathbb{D}_{\mathbf{n+2}} - k\mathbb{D}_{\mathbf{n+3}} &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
&\quad -i(H_{n+1} + iH_{n+2} + jH_{n+3} + kH_{n+4}) \\
&\quad -j(H_{n+2} + iH_{n+3} + jH_{n+4} + kH_{n+5}) \\
&\quad -k(H_{n+3} + iH_{n+4} + jH_{n+5} + kH_{n+6}) \\
&= H_n.
\end{aligned}$$

So (13) holds. By (7) and (10), we see that

$$\begin{aligned}
\mathbb{D}_{\mathbf{n}}\mathbb{D}_{\mathbf{m}} &= H_nH_m + i(H_nH_{m+1} + H_{n+1}H_m) + j(H_nH_{m+2} - H_{n+2}H_m) \\
&\quad +k(H_nH_{m+3} + H_{n+3}H_m)
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\mathbb{D}_{\mathbf{n+1}}\mathbb{D}_{\mathbf{m+1}} &= H_{n+1}H_{m+1} + i(H_{n+1}H_{m+2} + H_{n+2}H_{m+1}) \\
&\quad +j(H_{n+1}H_{m+3} + H_{n+3}H_{m+1}) \\
&\quad +k(H_{n+1}H_{m+4} + H_{n+4}H_{m+1}).
\end{aligned} \tag{18}$$

So (14) holds. Finally, adding equations (17) and (18) side by side, we obtain

$$\begin{aligned}
\mathbb{D}_{\mathbf{n}}\mathbb{D}_{\mathbf{m}} + \mathbb{D}_{\mathbf{n+1}}\mathbb{D}_{\mathbf{m+1}} &= (H_nH_m + H_{n+1}H_{m+1}) \\
&\quad + i[H_nH_{m+1} + H_{n+1}H_m + H_{n+1}H_{m+2} + H_{n+2}H_{m+1}] \\
&\quad + j[H_nH_{m+2} + H_{n+2}H_m + H_{n+1}H_{m+3} + H_{n+3}H_{m+1}] \\
&\quad + k[H_nH_{m+3} + H_{n+3}H_m + H_{n+1}H_{m+4} + H_{n+4}H_{m+1}] \\
&= [p^2F_{n+m+1} + 2pqF_{n+m} + q^2F_{n+m+3}] \\
&\quad + 2i[p^2F_{n+m+2} + 2pqF_{n+m+1} + q^2F_{n+m}] \\
&\quad + 2j[p^2F_{n+m+3} + 2pqF_{n+m+2} + q^2F_{n+m+1}] \\
&\quad + 2k[p^2F_{n+m+4} + 2pqF_{n+m+3} + q^2F_{n+m+2}] \\
&= 2(2p-q)[H_{n+m+1} + iH_{n+m+2}] \\
&\quad + 2(2p-q)[jH_{n+m+3} + kH_{n+m+4}] \\
&\quad - e[2F_{n+m+1} + 2iF_{n+m+2} + 2jF_{n+m+3} + 2kF_{n+m+4}] \\
&= (2p-q)[2\mathbb{D}_{\mathbf{n+m+1}} - H_{n+m+1}] - e[2Q_{n+m+1} - F_{n+m+1}]
\end{aligned}$$

where Q_{n+m+1} is the dual Fibonacci quaternion.

THEOREM 2. Let $\mathbb{D}_{\mathbf{n}}$ and D_n^L be n-th terms of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}})$ and the dual Lucas quternion sequence (D_n^L), respectively. The following relations are satisfied

$$\mathbb{D}_{\mathbf{n+1}} + \mathbb{D}_{\mathbf{n-1}} = pD_n^L + qD_{n-1}^L \text{ and } \mathbb{D}_{\mathbf{n+2}} - \mathbb{D}_{\mathbf{n-2}} = pD_n^L + qD_{n-1}^L.$$

PROOF. From equations (15), (16) and identities $H_n = (p-q)F_n + qF_{n+1}$ and $H_{n+1} + H_{n-1} = pL_n + qL_{n-1}$ between the generalized Fibonacci number and the Lucas number, we see that

$$\begin{aligned}
&\mathbb{D}_{\mathbf{n+1}} + \mathbb{D}_{\mathbf{n-1}} \\
={}& (H_{n+1} + H_{n-1}) + i(H_{n+2} + H_n) + j(H_{n+3} + H_{n+1}) + k(H_{n+4} + H_{n+2}) \\
={}& (pL_n + qL_{n-1}) + i(pL_{n+1} + qL_n) + j(pL_{n+2} + qL_{n+1}) + k(pL_{n+3} + qL_{n+2}) \\
={}& p(L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}) + q(L_{n-1} + iL_n + jL_{n+1} + kL_{n+2}) \\
={}& pD_n^L + qD_{n-1}^L
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{D}_{\mathbf{n+2}} - \mathbb{D}_{\mathbf{n-2}} \\
={}& (H_{n+2} - H_{n-2}) + i(H_{n+3} - H_{n-1}) + j(H_{n+4} - H_n) + k(H_{n+5} - H_{n+1}) \\
={}& p(L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}) + q(L_{n-1} + iL_n + jL_{n+1} + kL_{n+2}) \\
={}& pD_n^L + qD_{n-1}^L.
\end{aligned}$$

THEOREM 3. Let $\mathbb{D}_{\mathbf{n}}$ be the n-th term of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}})$. Then, we can give the following relations between these quaternions

$$\mathbb{D}_{\mathbf{n}} + \overline{\mathbb{D}}_{\mathbf{n}} = 2H_n, \tag{19}$$

$$\mathbb{D}_{\mathbf{n}}\overline{\mathbb{D}}_{\mathbf{n}}+\mathbb{D}_{\mathbf{n-1}}\overline{\mathbb{D}}_{\mathbf{n-1}}=H_n^2+H_{n-1}^2=(2p-q)H_{2n-1}-eF_{2n-1}, \tag{20}$$

$$\mathbb{D}_{\mathbf{n}}\overline{\mathbb{D}}_{\mathbf{n}}+\mathbb{D}_{\mathbf{n+1}}\overline{\mathbb{D}}_{\mathbf{n+1}}=H_n^2+H_{n+1}^2=(2p-q)H_{2n+1}-eF_{2n+1}, \tag{21}$$

$$\mathbb{D}_{\mathbf{n+1}}\overline{\mathbb{D}}_{\mathbf{n+1}}-\mathbb{D}_{\mathbf{n-1}}\overline{\mathbb{D}}_{\mathbf{n-1}}=H_{n+1}^2-H_{n-1}^2=(2p-q)H_{2n}-eF_{2n}, \tag{22}$$

$$\begin{aligned}(\mathbb{D}_{\mathbf{n}})^{\mathbf{2}}+(\mathbb{D}_{\mathbf{n-1}})^{\mathbf{2}} &= 2\mathbb{D}_{\mathbf{n}}H_n-H_n^2+2\mathbb{D}_{\mathbf{n-1}}H_{n-1}-H_{n-1}^2\\ &= (2p-q)[2\mathbb{D}_{\mathbf{2n-1}}-H_{2n-1}]-e[2Q_{2n-1}-F_{2n-1}] \end{aligned}\tag{23}$$

where Q_{2n-1} is the dual Fibonacci quaternion.

PROOF. By (8), we get

$$\begin{aligned}\mathbb{D}_{\mathbf{n}}+\overline{\mathbb{D}}_{\mathbf{n}} &= (H_n+iH_{n+1}+jH_{n+2}+kH_{n+3})\\ &\quad +(H_n-iH_{n+1}-jH_{n+2}-kH_{n+3})\\ &= 2H_n+i(H_{n+1}-H_{n+1})+j(H_{n+2}-H_{n+2})+k(H_{n+3}-H_{n+3})\\ &= 2H_n.\end{aligned}$$

Then (19) holds. By (8) and (9), we get

$$\mathbb{D}_{\mathbf{n}}\overline{\mathbb{D}}_{\mathbf{n}}+\mathbb{D}_{\mathbf{n-1}}\overline{\mathbb{D}}_{\mathbf{n-1}}=(H_n^2+H_{n-1}^2)=(2p-q)H_{2n-1}-eF_{2n-1}.$$

Then (20) holds. By (8) and (9), we get

$$\mathbb{D}_{\mathbf{n}}\overline{\mathbb{D}}_{\mathbf{n}}+\mathbb{D}_{\mathbf{n+1}}\overline{\mathbb{D}}_{\mathbf{n+1}}=(H_n^2+H_{n+1}^2)=(2p-q)H_{2n+1}-eF_{2n+1}.$$

Then (21) holds. By (8) and (9), we get

$$\mathbb{D}_{\mathbf{n+1}}\overline{\mathbb{D}}_{\mathbf{n+1}}-\mathbb{D}_{\mathbf{n-1}}\overline{\mathbb{D}}_{\mathbf{n-1}}=(H_{n+1}^2-H_{n-1}^2)=(2p-q)H_{2n}-eF_{2n}.$$

Then (22) holds. By (9), we get

$$\begin{aligned}\mathbb{D}_{\mathbf{n}}^{\mathbf{2}}+\mathbb{D}_{\mathbf{n-1}}^{\mathbf{2}} &= (2\mathbb{D}_{\mathbf{n}}H_n-H_n^2)+(2\mathbb{D}_{\mathbf{n-1}}H_{n-1}-H_{n-1}^2)\\ &= 2\mathbb{D}_{\mathbf{n}}H_n+2\mathbb{D}_{\mathbf{n-1}}H_{n-1}-(H_n^2+H_{n-1}^2)\\ &= (2p-q)[2\mathbb{D}_{\mathbf{2n-1}}-H_{2n-1}]-e[2Q_{2n-1}-F_{2n-1}]\end{aligned}$$

where Q_{2n-1} is the dual Fibonacci quaternion. Then (23) holds.

THEOREM 4. Let $\mathbb{D}_{\mathbf{n}}$ be the n-th term of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}})$. Then we have the following identities

$$\sum_{s=1}^{n}\mathbb{D}_{\mathbf{s}}=\mathbb{D}_{\mathbf{n+2}}-\mathbb{D}_{\mathbf{2}}, \tag{24}$$

$$\sum_{s=0}^{p}\mathbb{D}_{\mathbf{n+s}}+\mathbb{D}_{\mathbf{n+1}}=\mathbb{D}_{\mathbf{n+p+2}}, \tag{25}$$

$$\sum_{s=1}^{n} \mathbb{D}_{\mathbf{2s-1}} = \mathbb{D}_{\mathbf{2n}} - \mathbb{D}_{\mathbf{0}}, \tag{26}$$

$$\sum_{s=1}^{n} \mathbb{D}_{\mathbf{2s}} = \mathbb{D}_{\mathbf{2n+1}} - \mathbb{D}_{\mathbf{1}}. \tag{27}$$

PROOF. Since $\sum_{t=a}^{n} H_t = H_{n+2} - H_{a+1}$ [4], we get

$$\begin{aligned}
\sum_{s=1}^{n} \mathbb{D}_{\mathbf{s}} &= \sum_{s=1}^{n} H_s + i\sum_{s=1}^{n} H_{s+1} + j\sum_{s=1}^{n} H_{s+2} + k\sum_{s=1}^{n} H_{s+3} \\
&= (H_{n+2} - H_2) + i(H_{n+3} - H_3) + j(H_{n+4} - H_4) + k(H_{n+5} - H_5) \\
&= (H_{n+2} + iH_{n+3} + jH_{n+4} + kH_{n+5}) - (H_2 + iH_3 + jH_4 + kH_5) \\
&= \mathbb{D}_{\mathbf{n+2}} - \mathbb{D}_{\mathbf{2}}.
\end{aligned}$$

Then (24) holds. We can write

$$\begin{aligned}
\sum_{s=0}^{p} \mathbb{D}_{\mathbf{n+s}} + \mathbb{D}_{\mathbf{n+1}} &= (H_{n+p+2} - H_{n+1} + H_{n+1}) + i(H_{n+p+3} - H_{n+2} + H_{n+2}) \\
&\quad + j(H_{n+p+4} - H_{n+3} + H_{n+3}) + k(H_{n+p+5} - H_{n+4} + H_{n+4}) \\
&= H_{n+p+2} + iH_{n+p+3} + jH_{n+p+4} + kH_{n+p+5} \\
&= \mathbb{D}_{\mathbf{n+p+2}}.
\end{aligned}$$

Then (25) holds. By

$$\sum_{i=1}^{n} H_{2i-1} = H_{2n} - q \text{ and } \sum_{i=1}^{n} H_{2i} = H_{2n+1} - p \quad [4],$$

we get

$$\begin{aligned}
\sum_{s=1}^{n} \mathbb{D}_{\mathbf{2s-1}} &= (H_{2n} - q) + i(H_{2n+1} - p) + j(H_{2n+2} - q - p) + k(H_{2n+3} - 2p - q) \\
&= [H_{2n} + iH_{2n+1} + jH_{2n+2} + kH_{2n+3}] - [q + ip + j(p+q) + k(2p+q)] \\
&= \mathbb{D}_{\mathbf{2n}} - [H_0 + iH_1 + jH_2 + kH_3] \\
&= \mathbb{D}_{\mathbf{2n}} - \mathbb{D}_{\mathbf{0}}.
\end{aligned}$$

Then (26) holds. By $\sum\limits_{i=1}^{n} H_{2i} = H_{2n+1} - p$ [4], we get

$$\begin{aligned}
\sum_{s=1}^{n} \mathbb{D}_{\mathbf{2s}} &= (H_{2n+1} - p) + i(H_{2n+2} - q - p) + j(H_{2n+3} - 2p - q) + k(H_{2n+4} - 3p - q) \\
&= [H_{2n+1} + iH_{2n+2} + jH_{2n+3} + kH_{2n+4}] \\
&\quad - [p + i(p+q) + j(2p+q) + k(3p+2q)] \\
&= \mathbb{D}_{\mathbf{2n+1}} - [H_1 + iH_2 + jH_3 + kH_4] \\
&= \mathbb{D}_{\mathbf{2n+1}} - \mathbb{D}_{\mathbf{1}}.
\end{aligned}$$

Then (27) holds.

THEOREM 5. Let $\mathbb{D}_\mathbf{n}$ and Q_n be the n-th terms of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_\mathbf{n})$ and the dual Fibonacci quaternion sequence (Q_n), respectively. Then, we have

$$Q_n\overline{\mathbb{D}}_\mathbf{n} - \overline{Q}_n\mathbb{D}_\mathbf{n} = 2[H_nQ_n - F_n\mathbb{D}_\mathbf{n}], \tag{28}$$

$$Q_n\overline{\mathbb{D}}_\mathbf{n} + \overline{Q}_n\mathbb{D}_\mathbf{n} = 2F_nH_n, \tag{29}$$

$$Q_n\mathbb{D}_\mathbf{n} - \overline{Q}_n\overline{\mathbb{D}}_\mathbf{n} = 2[F_n\mathbb{D}_\mathbf{n} + H_nQ_n - 2F_nH_n]. \tag{30}$$

PROOF. By (6) and (3), we get

$$\begin{aligned}
Q_n\overline{\mathbb{D}}_\mathbf{n} - \overline{Q}_n\mathbb{D}_\mathbf{n} \quad = \quad & (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})(H_n - iH_{n+1} - jH_{n+2} - kH_{n+3}) \\
& -(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3})(H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
= \quad & (F_nH_n - F_nH_n) + 2i(-F_nH_{n+1} + F_{n+1}H_n) \\
& +2j(-F_nH_{n+2} + F_{n+2}H_n) + 2k(-F_nH_{n+3} + F_{n+3}H_n) \\
= \quad & -2F_n[H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}] \\
& +2H_n[F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}] \\
= \quad & 2[H_nQ_n - F_n\mathbb{D}_\mathbf{n}].
\end{aligned}$$

Then (28) holds. By (6) and (3), we get

$$\begin{aligned}
Q_n\overline{\mathbb{D}}_\mathbf{n} + \overline{Q}_n\mathbb{D}_\mathbf{n} \quad = \quad & (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})(H_n - iH_{n+1} - jH_{n+2} - kH_{n+3}) \\
& +(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3})(H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
= \quad & (F_nH_n + F_nH_n) + i(-F_nH_{n+1} + F_{n+1}H_n + F_nH_{n+1} - F_{n+1}H_n) \\
& +j(-F_nH_{n+2} + F_{n+2}H_n + F_nH_{n+2} - F_{n+2}H_n) \\
& +k(-F_nH_{n+3} + F_{n+3}H_n + F_nH_{n+3} - F_{n+3}H_n) \\
= \quad & 2F_nH_n.
\end{aligned}$$

Then (29) holds. By (6) and (3), we get

$$\begin{aligned}
Q_n\mathbb{D}_\mathbf{n} - \overline{Q}_n\overline{\mathbb{D}}_\mathbf{n} \quad = \quad & (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})(H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
& -(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3})(H_n - iH_{n+1} - jH_{n+2} - kH_{n+3}) \\
= \quad & (F_nH_n - F_nH_n) + i(2F_nH_{n+1} + 2F_{n+1}H_n) \\
& +j(2F_nH_{n+2} + 2F_{n+2}H_n) + k(2F_nH_{n+3} + 2F_{n+3}H_n) \\
= \quad & F_n\mathbb{D}_\mathbf{n} + F_n(\mathbb{D}_\mathbf{n} - 2H_n) + 2H_n(Q_n - F_n) \\
= \quad & 2[F_n\mathbb{D}_\mathbf{n} + H_nQ_n - 2F_nH_n].
\end{aligned}$$

Then (30) holds.

THEOREM 6. **(Binet's Formulas)**. Let $\mathbb{D}_\mathbf{n}$ be the n-th term of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_\mathbf{n})$. For $n \geq 1$, the Binet's formulas for these quaternions are as follows:

$$\mathbb{D}_\mathbf{n} = \frac{1}{\alpha - \beta}\left(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n\right) \tag{31}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2},$$

$$\begin{aligned} \hat{\alpha} &= (p - q\beta) + i[p(1-\beta) + q] + j[p(2-\beta) + q(1-\beta)] \\ &\quad + k[p(3-2\beta) + q(2-\beta)] \end{aligned}$$

and

$$\begin{aligned} \hat{\beta} &= (p - q\alpha) + i[p(1-\alpha) + q] + j[p(2-\alpha) + q(1-\alpha)] \\ &\quad + k[(p(3-2\alpha) + q(2-\alpha)]. \end{aligned}$$

PROOF. The characteristic equation of recurrence relation $\mathbb{D}_\mathbf{n+2} = \mathbb{D}_\mathbf{n+1} + \mathbb{D}_\mathbf{n}$ is $t^2 - t - 1 = 0$. The roots of this equation are

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$. The Binet's formulas for Fibonacci sequence, generalized Fibonacci sequence and dual Fibonacci quaternion sequence respectively, are as follows

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right), \quad H_n = \frac{1}{2\sqrt{5}}\left(l\alpha^n - m\beta^n\right)$$

and

$$Q_n = \frac{1}{\sqrt{5}}\left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right),$$

cf. [3, 4, 21]. Using recurrence relation and initial values

$$\mathbb{D}_\mathbf{0} = (q, p, p+q, 2p+q) \text{ and } \mathbb{D}_\mathbf{1} = (p, p+q, 2p+q, 3p+2q),$$

the Binet's formula for $\mathbb{D}_\mathbf{n}$ is

$$\mathbb{D}_\mathbf{n} = A\alpha^n + B\beta^n = \frac{1}{\sqrt{5}}\left[\hat{\alpha}\alpha^n - \hat{\beta}\,\beta^n\right]$$

where

$$A = \frac{\mathbb{D}_\mathbf{1} - \mathbb{D}_\mathbf{0}\beta}{\alpha - \beta}, \quad B = \frac{\alpha\mathbb{D}_\mathbf{0} - \mathbb{D}_\mathbf{1}}{\alpha - \beta},$$

$$\hat{\alpha} = (p - q\beta) + i[p(1-\beta) + q] + j[p(2-\beta) + q(1-\beta)] + k[(3-2\beta) + q(2-\beta)]$$

and

$$\hat{\beta} = (p - q\alpha) + i[p(1-\alpha) + q] + j[p(2-\alpha) + q(1-\alpha)] + k[p(3-2\alpha) + q(2-\alpha)].$$

THEOREM 7. **(Cassini-like Identity)**. Let $\mathbb{D}_{\mathbf{n}}$ be the n-th term of the generalized dual Fibonacci quaternion sequence $(\mathbb{D}_{\mathbf{n}})$. For $n \geq 1$, the Cassini-like identity for $\mathbb{D}_{\mathbf{n}}$ is as follows

$$\mathbb{D}_{\mathbf{n-1}}\mathbb{D}_{\mathbf{n+1}} - (\mathbb{D}_{\mathbf{2}})^{\mathbf{2}} = (-1)^n e(1 + i + 3j + 4k). \tag{32}$$

PROOF. By (15) and (16), we get

$$\begin{aligned}
& \mathbb{D}_{\mathbf{n-1}}\mathbb{D}_{\mathbf{n+1}} - (\mathbb{D}_{\mathbf{2}})^{\mathbf{2}} \\
= \;& (H_{n-1} + iH_n + jH_{n+1} + kH_{n+2})(H_{n+1} + iH_{n+2} + jH_{n+3} + kH_{n+4}) \\
& -[H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}]^2 \\
= \;& [H_{n-1}H_{n+1} - H_n^2] + i[H_{n-1}H_{n+2} + H_nH_{n+2} - 2H_nH_{n+1}] \\
& +j[H_{n-1}H_{n+3} - 2H_nH_{n+2} + H_{n+1}^2) + k[H_{n-1}H_{n+4} + H_{n+1}H_{n+2} - 2H_nH_{n+3}] \\
= \;& (-1)^n e(1 + i + 3j + 4k)
\end{aligned}$$

where we use identity of the Fibonacci number

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n}$$

and identities of the generalized Fibonacci numbers as follows

$$H_{n+1}H_{n-1} - H_n^2 = (-1)^n(p^2 - pq - q^2) = (-1)^n e, \tag{33}$$

$$H_{n+2}H_{n-1} - H_nH_{n+1} = (-1)^n(p^2 - pq - q^2) = (-1)^n e, \tag{34}$$

$$H_{n+3}H_{n-1} - H_{n+1}H_{n+1} - 2H_nH_{n+2} = 3(-1)^n(p^2 - pq - q^2) = 3(-1)^n e, \tag{35}$$

$$H_{n+4}H_{n-1} - H_{n+2}H_{n+1} - 2H_nH_{n+3} = 4(-1)^n(p^2 - pq - q^2) = 4(-1)^n e, \tag{36}$$

$$e = p^2 - pq - q^2.$$

So (32) holds.

SPECIAL CASE. From the equations (33)–(36) for $p = 1$ and $q = 0$, we obtain all results in [21] as a special case.

We will give an example in which we check in a particular case the Cassini identity for the generalized dual Fibonacci quaternions.

EXAMPLE 1. Let $\mathbb{D}_{\mathbf{1}}$, $\mathbb{D}_{\mathbf{2}}$,$\mathbb{D}_{\mathbf{3}}$ and $\mathbb{D}_{\mathbf{4}}$ be the generalized dual Fibonacci quaternions such that

$$\begin{cases}
\mathbb{D}_{\mathbf{1}} = p + i(p+q) + j(2p+q) + k(3p+2q), \\
\mathbb{D}_{\mathbf{2}} = (p+q) + i(2p+q) + j(3p+2q) + k(5p+3q), \\
\mathbb{D}_{\mathbf{3}} = (2p+q) + i(3p+2q) + j(5p+3q) + k(8p+5q), \\
\mathbb{D}_{\mathbf{4}} = (3p+2q) + i(5p+3q) + j(8p+5q) + k(13p+8q).
\end{cases}$$

In this case,

$$\begin{aligned}
\mathbb{D}_{\mathbf{1}}\mathbb{D}_{\mathbf{3}}-(\mathbb{D}_{\mathbf{2}})^{\mathbf{2}} &= [p+i(p+q)+j(2p+q)+k(3p+2q)] \\
&\quad \times[(3p+2q)+i(3p+2q)+j(5p+3q)+k(8p+5q)] \\
&\quad -[[(p+q)+i(2p+q)+j(3p+2q)+k(5p+3q)^2]^2 \\
&= (p^2-pq-q^2)+i(p^2-pq-q^2)+j(3p^2-3pq-3q^2) \\
&\quad +k(4p^2-4pq-4q^2) \\
&= (p^2-pq-q^2))(1+i+3j+4k) \\
&= (-1)^2e(1+i+3j+4k)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{D}_{\mathbf{2}}\mathbb{D}_{\mathbf{4}}-(\mathbb{D}_{\mathbf{3}})^{\mathbf{2}} &= [(p+q)+i(2p+q)+j(3p+2q)+k(5p+3q)] \\
&\quad [(3p+2q)+i(5p+3q)+j(8p+5q)+k(13p+8q)] \\
&\quad -[(2p+q)+i(3p+2q)+j(5p+3q)+k(8p+5q)^2]^2 \\
&= (-p^2+pq+q^2)+i(-p^2+pq+q^2) \\
&\quad +j(-3p^2+3pq+3q^2)+k(-4p^2+4pq+4q^2) \\
&= (-1)^3(p^2-pq-q^2))(1+i+3j+4k) \\
&= (-1)^3e(1+i+3j+4k).
\end{aligned}$$

3 Conclusion

The generalized dual Fibonacci quaternions are given by

$$\mathbb{D}_{\mathbf{n}} = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \tag{37}$$

where H_n is the n-th generalized Fibonacci number and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = 0 \text{ and } ij = -ji = jk = -kj = ki = -ik = 0.$$

Furthermore, from the generalized dual Fibonacci quaternions for $p = 1$ and $q = 0$, we obtain results of the dual Fibonacci quaternions given by Yüce and Torunbalcı Aydın [21] as a special case. Then, this study fills the gap in the literature by providing the generalization of dual quaternion as in the generalized Fibonacci quaternion [8]. Also, Binet's formula is obtained. In this way, n-th generalized dual Fibonacci quaternion is obtained practically.

There have been several studies on curve theory and magnetism by using the isomorphism between dual quaternion space and Galilean space G^4. Due to this study, application areas of Fibonacci and dual Fibonacci sequence leads up to generalization of quaternions by using the properties of dual Fibonacci quaternions and the generalized dual Fibonacci quaternions.

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