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Statistical Korovkin-Type Theory For Matirx-Valued Functions Of Two Variables*

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Abstract

In this work, we investigate an approximation problem for matrix valued positive linear operators of two variables. Also using the A-statistical convergence which is stronger than Pringsheim convergence of double sequences, we prove a Korovkin-type approximation theorem for matrix valued positive linear operators of two variables. We also compute the rates of A-statistical convergence of this operators.

1 Introduction

One of the most important and basic results in approximation theory is the classical Bohman-Korovkin theorem (see for instance [10]). This theorem establishes the uniform convergence in the space $C[a,b]$ of all the continuous real functions defined on the interval $[a,b]$, for a sequence of positive linear operators (T_n) acting on $C[a,b]$, assuming the convergence only on the test functions $1, x, x^2$. Of course, this theory mainly gives the approximation a scaler-valued function by means of linear positive operators. However, in [16], Serra-Capizzano presented a new Korovkin-type result for matrix valued functions. Some other related topics may be found in the papers [13, 17, 18] and cited therein. In particular, the use of statistical convergence had a great impulse in recent years. Furthermore, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [1, 2, 3, 5, 8, 9]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. In [4], using the statistical convergence, Duman and Erkuş-Duman proved Korovkin-type approximation theorem for matrix valued positive linear operators. Our primary interest in the present paper is to obtain a general Korovkin-type approximation theorem for double sequences of matrix valued positive linear operators of two variables from $C(D, \mathbb{C}^{s\times t})$ to itself where D is a compact subset of $\mathbb{R}^2$.

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We begin with some definitions and notations which we will use in the sequel. As usual, a double sequence

$$x = (x_{mn}), \quad m, n \in \mathbb{N},$$

is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim\limits_{m,n} x = L$ (see [14]). In this case, we say that $x = (x_{mn})$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = (x_{mn})$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence needs not to be bounded.

Now let

$$A = [a_{prmn}], \; p, r, m, n \in \mathbb{N},$$

be a four-dimensional summability matrix. For a given double sequence $x = (x_{mn})$, the A-transform of x, denoted by $Ax := \{(Ax)_{pr}\}$, is given by

$$(Ax)_{pr} = \sum_{(m,n) \in \mathbb{N}^2} a_{prmn} x_{mn}, \quad p, r \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(p, r) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [7]). In 1926, Robinson [15] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, *RH*-regularity (see, [6, 15]).

Recall that a four dimensional matrix $A = [a_{prmn}]$ is said to be *RH*-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{prmn}]$ is *RH*-regular if and only if

(i) $P - \lim\limits_{p,r} a_{prmn} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $P - \lim\limits_{p,r} \sum\limits_{(m,n) \in \mathbb{N}^2} a_{prmn} = 1$,

(iii) $P - \lim\limits_{p,r} \sum\limits_{m \in \mathbb{N}} |a_{prmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim\limits_{p,r} \sum\limits_{n \in \mathbb{N}} |a_{prmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum\limits_{(m,n) \in \mathbb{N}^2} |a_{prmn}|$ is $P-$convergent,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{prmn}| < A$ holds for every $(p,r) \in \mathbb{N}^2$.

Now let $A = [a_{prmn}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then A-density of K is given by

$$\delta_A^{(2)}\{K\} := P - \lim_{p,r} \sum_{(m,n)\in K} a_{prmn}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real double sequence $x = (x_{mn})$ is said to be $A-$statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta_A^{(2)}\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\} = 0.$$

In this case, we write $st_A^{(2)} - \lim_{m,n} x_{mn} = L$. We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [11, 12]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

A P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold true.

2 A Korovkin-Type Approximation Theorem for Double Sequences

In this section, we give a Korovkin-type theorem for double sequences of matrix valued positive linear operators defined on $C(D, \mathbb{C}^{s\times t})$ using the concept of A-statistical convergence.

Let s, t be two fixed natural numbers and D a compact subset of $\mathbb{R}^2$. By $C(D, \mathbb{C}^{s\times t})$ we denote the space of all continuous functions F acting on D and having values in the space $\mathbb{C}^{s\times t}$ of the complex $s \times t$ matrices such that

$$F(x,y) := [f_{jk}(x,y)]_{s\times t}\,,\ ((x,y) \in D,\ 1 \leq j \leq s,\ 1 \leq k \leq t), \tag{1}$$

where the symbol $[b_{jk}]_{s\times t}$ denotes the $s \times t$ matrix. Here, by the continuity of F we mean that all scalar valued functions f_{jk} are continuous on D. Then, the norm $\|\cdot\|_{s\times t}$ on the space $C(D, \mathbb{C}^{s\times t})$ is defined by

$$\|F\|_{s\times t} := \max_{1\leq j\leq s,\ 1\leq k\leq t} \|f_{jk}\| := \max_{1\leq j\leq s,\ 1\leq k\leq t} \left(\sup_{(x,y)\in D} |f_{jk}(x,y)| \right).$$

Throughout this paper we use the following test functions

$$E_{0jk}(u,v) = E_{jk},\ E_{1jk}(u,v) = uE_{jk}, \tag{2}$$

$$E_{2jk}(u,v) = vE_{jk} \text{ and } E_{3jk}(u,v) = \left(u^2+v^2\right)E_{jk}, \tag{3}$$

for $(u,v) \in D$, $1 \le j \le s$, $1 \le k \le t$, where E_{jk} denotes the matrix of the canonical basis of $\mathbb{C}^{s\times t}$ being 1 in the position (j,k) and zero otherwise.

Let $\Phi : C(D,\mathbb{C}^{s\times t}) \to C(D,\mathbb{C}^{s\times t})$ be an operator, and let us assume that

(i) $\Phi(aF+bG) = a\Phi(F)+b\Phi(G)$ for any $\alpha,\beta \in \mathbb{C}$ and $F,G \in C(D,\mathbb{C}^{s\times t})$,

(ii) $\Phi(F) \le K\Phi(|F|)$ for any function $F \in C(D,\mathbb{C}^{s\times t})$ and for a fixed positive constant K.

Under the above-mentioned assumptions, the operator Φ is said to be a matrix linear positive operator, or briefly, *mLPO* (see, for details, [16]). The inequality appearing in *(ii)* is understood to be componentwise, i.e., holding for any component $(j,k) \in \{1,2,...,s\} \times \{1,2,...,t\}$.

Now we have the following main result.

THEOREM 1. Let $A = [a_{prmn}]$ be a nonnegative *RH*-regular summability matrix and let (Φ_{mn}) be a double sequence of *mPLOs* acting from $C(D,\mathbb{C}^{s\times t})$ into itself. Then for all $(j,k) \in \{1,2,...,s\} \times \{1,2,...,t\}$ and for each $i = 0,1,2,3$,

$$st_A^2 - \lim_{m,n} \|\Phi_{mn}(E_{ijk}) - E_{ijk}\|_{s\times t} = 0, \tag{4}$$

where E_{ijk} is given by (2) and (3), if and only if for every $F \in C(D,\mathbb{C}^{s\times t})$ as in (1),

$$st_A^2 - \lim_{m,n} \|\Phi_{mn}(F) - F\|_{s\times t} = 0. \tag{5}$$

PROOF. Since each $E_{ijk} \in C(D,\mathbb{C}^{s\times t})$, $(i = 0,1,2,3)$, the implication (5)$\Rightarrow$(4) is obvious. Suppose now that (4) holds. Let $F \in C(D,\mathbb{C}^{s\times t})$ and $(x,y) \in D$ be fixed. Fist, we calculate the expression $|\Phi_{nm}(F;x,y) - F(x,y)|$, the symbol $|B|$ denotes the matrix having entries equal to the absolute value of the entries of the matrix B. We can show that $F(x,y) := [f_{jk}(x,y)]_{s\times t}$, $1 \le j \le s$, $1 \le k \le t$, can be written as follows:

$$F(x,y) := \sum_{j=1}^{s}\sum_{k=1}^{t} f_{jk}(x,y)E_{jk} = \sum_{j=1}^{s}\sum_{k=1}^{t} f_{jk}(x,y)E_{0jk}(u,v).$$

Hence, we obtain

$$\Phi_{mn}(F(x,y);x,y) = \sum_{j=1}^{s}\sum_{k=1}^{t} f_{jk}(x,y)\Phi_{mn}\left(E_{0jk}(u,v);x,y\right). \tag{6}$$

Also, the continuity of f_{jk} on D, for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u,v) \in D$ satisfying

$$\sqrt{(u-x)^2+(v-y)^2} \le \delta.$$

We have

$$|f_{jk}(u,v) - f_{jk}(x,y)| \le \varepsilon \text{ for } 1 \le j \le s, 1 \le k \le t. \tag{7}$$

Also we get for all $(x,y), (u,v) \in D$ satisfying $\sqrt{(u-x)^2 + (v-y)^2} > \delta$ that

$$|f_{jk}(u,v) - f_{jk}(x,y)| \le \frac{2M_{jk}}{\delta^2}\varphi(u,v) \text{ for } 1 \le j \le s, 1 \le k \le t, \tag{8}$$

where $M := \sup_{(x,y)\in D} |f_{jk}(x,y)|$ and $\varphi(u,v) = (u-x)^2 + (v-y)^2$. Combining (7) and (8) we have for $(u,v) \in D$ that

$$|f_{jk}(u,v) - f_{jk}(x,y)| \le \varepsilon + \frac{2M_{jk}}{\delta^2}\varphi(u,v). \tag{9}$$

Then observe that, by (9),

$$|F(u,v) - F(x,y)| \le \varepsilon E + \frac{2M}{\delta^2}\varphi(u,v)E, \tag{10}$$

where E is the $s \times t$ matrix such that all entires 1, and

$$M := \max_{1\le j\le s,\ 1\le k\le t} M_{jk} = \|F\|_{s\times t}.$$

Since Φ_{mn} is a $mPLO$, from (6) and (10), we get

$$\begin{aligned} & |\Phi_{mn}(F(u,v);x,y) - F(x,y)| \\ \le\ & K\Phi_{mn}(|F(u,v) - F(x,y)|;x,y) + |\Phi_{mn}(F(x,y);x,y) - F(x,y)| \\ \le\ & (\varepsilon K + M)\sum_{j=1}^{s}\sum_{k=1}^{t} |\Phi_{mn}(E_{0jk}(u,v);x,y) - E_{0jk}(x,y)| \\ & + \frac{2KM}{\delta^2}\Phi_{mn}(\varphi(u,v)E;x,y) + \varepsilon KE \end{aligned} \tag{11}$$

for a fixed positive constant K. Now we compute the expression $\Phi_{mn}(\varphi(u,v)E;x,y)$ on the left-hand side of (11). By a simple calculation, we get

$$\begin{aligned} \Phi_{mn}(\varphi(u,v)E;x,y) &= \Phi_{mn}\Big(\sum_{j=1}^{s}\sum_{k=1}^{t}\varphi(u,v)E_{jk};x,y\Big) \\ &= \sum_{j=1}^{s}\sum_{k=1}^{t}\{\Phi_{mn}(E_{3jk};x,y) - 2x\Phi_{mn}(E_{1jk};x,y) \\ &\quad -2y\Phi_{mn}(E_{2jk};x,y) + (x^2+y^2)\Phi_{mn}(E_{0jk};x,y)\}. \end{aligned}$$

Then,

$$
\begin{aligned}
& \Phi_{mn}(\left((u-x)^2+(v-y)^2\right)E;x,y) \\
\leq\ & \sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{3jk};x,y\right)-E_{3jk}(x,y)\right| \\
& +2A\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{1jk};x,y\right)-E_{1jk}(x,y)\right| \\
& +2B\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{2jk};x,y\right)-E_{2jk}(x,y)\right| \\
& +\left(A^2+B^2\right)\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}(x,y),\right| \qquad (12)
\end{aligned}
$$

where $A := \max|x|$ and $B := \max|y|$. Combining (11) and (12), we obtain

$$
\begin{aligned}
& \left|\Phi_{mn}(F(u,v);x,y)-F\left(x,y\right)\right| \\
\leq\ & (\varepsilon K+M)\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right)-E_{0jk}(x,y)\right| \\
& +\varepsilon KE+\frac{4AKM}{\delta^2}\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{1jk};x,y\right)-E_{1jk}(x,y)\right| \\
& +\frac{4BKM}{\delta^2}\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{2jk};x,y\right)-E_{2jk}(x,y)\right| \\
& +\frac{2KM}{\delta^2}\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{3jk};x,y\right)-E_{3jk}(x,y)\right| \\
& +\frac{2\left(A^2+B^2\right)KM}{\delta^2}\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right)-E_{0jk}(x,y)\right| \\
\leq\ & \varepsilon KE+C\sum_{j=1}^{s}\sum_{k=1}^{t}\sum_{i=0}^{3}\left|\Phi_{mn}\left(E_{ijk};x,y\right)-E_{ijk}(x,y)\right|,
\end{aligned}
$$

where

$$
C := \max\left\{\varepsilon K+M+\frac{2\left(A^2+B^2\right)KM}{\delta^2},\frac{4AKM}{\delta^2},\frac{4BKM}{\delta^2},\frac{2KM}{\delta^2}\right\}.
$$

Then, taking maximum of all entries of the corresponding matrices and taking supremum over $(x,y)\in D$, we get

$$
\left\|\Phi_{mn}(F)-F\right\|_{s\times t}\leq\varepsilon K+C\left\{\sum_{j=1}^{s}\sum_{k=1}^{t}\sum_{i=0}^{3}\left\|\Phi_{mn}\left(E_{ijk}\right)-E_{ijk}\right\|_{s\times t}\right\}. \qquad (13)
$$

Now, for a given $\varepsilon^{'} > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \frac{\varepsilon^{'}}{K}$. Then, define

$$\Gamma := \left\{ (m,n) \in \mathbb{N}^2 : \ \ \|\Phi_{mn}(F) - F\|_{s\times t} \geq \varepsilon^{'} \right\}$$

and

$$\Gamma_{ijk} := \left\{ (m,n) \in \mathbb{N}^2 : \|\Phi_{mn}\left(E_{ijk}\right) - E_{ijk}\|_{s\times t} \geq \frac{\varepsilon^{'} - \varepsilon K}{4stC} \right\},$$

where $i = 0,1,2,3$ and $1 \leq j \leq s$, $1 \leq k \leq t$. Then it is easy to see that $\Gamma \subseteq \bigcup_{j=1}^{s} \bigcup_{k=1}^{t} \bigcup_{i=0}^{3} \Gamma_{ijk}$. Thus, we may write, for every $(p,r) \in \mathbb{N}^2$, that

$$\sum_{(m,n)\in\Gamma} a_{prmn} \leq \sum_{j=1}^{s} \sum_{k=1}^{t} \sum_{i=0}^{3} \sum_{(m,n)\in\Gamma_{ijk}} a_{prmn}.$$

Letting $p,r \to \infty$, using (4), we obtain (5). The proof is complete.

3 Concluding Remarks

In Theorem 1 if we replace the matrix A by the double Cesáro matrix $C(1,1)$, then we immediately get the following statistical result.

COROLLARY 1. Let (Φ_{mn}) be a double sequence of $mPLOs$ acting from $C(D,\mathbb{C}^{s\times t})$ into itself. Then for all $(j,k) \in \{1,2,...,s\} \times \{1,2,...,t\}$ and for each $i = 0,1,2,3$,

$$st^2 - \lim_{m,n} \|\Phi_{mn}(E_{ijk}) - E_{ijk}\|_{s\times t} = 0,$$

where E_{ijk} is given by (2) and (3), if and only if for every $F \in C(D,\mathbb{C}^{s\times t})$ as in (1),

$$st^2 - \lim_{m,n} \|\Phi_{mn}(F) - F\|_{s\times t} = 0.$$

In Theorem 1 if we replace the matrix A by the double identity matrix I, then we immediately get the following classical result.

COROLLARY 2. Let (Φ_{mn}) be a double sequence of $mPLOs$ acting from $C(D,\mathbb{C}^{s\times t})$ into itself. Then for all $(j,k) \in \{1,2,...,s\} \times \{1,2,...,t\}$ and for each $i = 0,1,2,3$,

$$P - \lim_{m,n} \|\Phi_{mn}(E_{ijk}) - E_{ijk}\|_{s\times t} = 0,$$

where E_{ijk} is given by (2) and (3), if and only if for every $F \in C(D,\mathbb{C}^{s\times t})$ as in (1),

$$P - \lim_{m,n} \|\Phi_{mn}(F) - F\|_{s\times t} = 0.$$

Now we present an example such that our new approximation result works but its classical case (Corollary 2) does not work. Let $a,\ b, c, d$ be fixed real numbers and $D=[a,b]\times[c,d]$. First consider the following the matrix-valued Bernstein-type operators:

$$\begin{aligned} B_{mn}\left(F;x,y\right) &= \sum_{l=0}^{m}\sum_{r=0}^{n} F\left(a+\frac{l}{m}\left(b-a\right),c+\frac{r}{n}\left(d-c\right)\right)\binom{m}{l}\binom{n}{r} \\ &\quad\times\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r}, \qquad (14) \end{aligned}$$

where $(x,y)\in D$, $F\in C\left(D,\mathbb{C}^{s\times t}\right)$ such that $F(x,y):=\left[f_{jk}(x,y)\right]_{s\times t}$, $1\le j\le s$, $1\le k\le t$. Also, observe that the matrix-valued Bernstein-type polynomials B_{mn} can be also written as follows:

$$\begin{aligned} B_{mn}\left(F;x,y\right) &= \sum_{l=0}^{m}\sum_{r=0}^{n}\sum_{j=1}^{s}\sum_{k=1}^{t} f_{jk}\left(a+\frac{l}{m}\left(b-a\right),c+\frac{r}{n}\left(d-c\right)\right)\binom{m}{l}\binom{n}{r} \\ &\quad\times\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r}E_{jk}, \qquad (15) \end{aligned}$$

where E_{jk} denotes the matrix of the canonical basis of $\mathbb{C}^{s\times t}$ being 1 in the position (j,k) and zero otherwise. Then, by (14) or (15), we obtain that, for $(j,k)\in\{1,2,...,s\}\times\{1,2,...,t\}$,

$$\begin{aligned} & B_{mn}\left(E_{0jk};x,y\right) \\ =\ & \sum_{l=0}^{m}\sum_{r=0}^{n}\binom{m}{l}\binom{n}{r}\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r} \\ & \times E_{0jk}\left(a+\frac{l}{m}\left(b-a\right),c+\frac{r}{n}\left(d-c\right)\right) \\ =\ & E_{jk}\sum_{l=0}^{m}\sum_{r=0}^{n}\binom{m}{l}\binom{n}{r}\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r} \\ =\ & E_{0jk}\left(x,y\right). \end{aligned}$$

Similarly, we get that, for $(j,k)\in\{1,2,...,s\}\times\{1,2,...,t\}$,

$$\begin{aligned} & B_{mn}\left(E_{1jk};x,y\right) \\ =\ & \sum_{l=0}^{m}\sum_{r=0}^{n}\binom{m}{l}\binom{n}{r}\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r} \\ & \times E_{1jk}\left(a+\frac{l}{m}\left(b-a\right),c+\frac{r}{n}\left(d-c\right)\right) \\ =\ & xE_{jk}=E_{1jk}\left(x,y\right) \end{aligned}$$

and

$$B_{mn}\left(E_{2jk};x,y\right)=yE_{jk}=E_{2jk}\left(x,y\right).$$

Finally, we obtain that, for $(j,k)\in\{1,2,...,s\}\times\{1,2,...,t\}$,

$$\begin{aligned}
&B_{mn}\left(E_{3jk};x,y\right)\\
=\ &\sum_{l=0}^{m}\sum_{r=0}^{n}\binom{m}{l}\binom{n}{r}\left(\frac{x-a}{b-a}\right)^{l}\left(\frac{y-c}{d-c}\right)^{r}\left(\frac{b-x}{b-a}\right)^{m-l}\left(\frac{d-y}{d-c}\right)^{n-r}\\
&\times E_{3jk}\left(a+\frac{l}{m}\left(b-a\right),c+\frac{r}{n}\left(d-c\right)\right)\\
=\ &\left\{x^{2}-\frac{1}{m}\left(x-a\right)^{2}+\frac{\left(b-a\right)\left(x-a\right)}{m}\right.\\
&\left.+y^{2}-\frac{1}{n}\left(y-c\right)^{2}+\frac{\left(d-c\right)\left(y-c\right)}{n}\right\}E_{jk}\\
=\ &\left(\frac{\left(b-a\right)\left(x-a\right)}{m}-\frac{1}{m}\left(x-a\right)^{2}-\frac{1}{n}\left(y-c\right)^{2}+\frac{\left(d-c\right)\left(y-c\right)}{n}\right)E_{jk}\\
&+E_{3jk}\left(x,y\right).
\end{aligned}$$

Now take $A=C(1;1)$ and define a double sequence $u=(u_{mn})$ by

$$u_{mn}:=\begin{cases}1 & \text{if } m \text{ and } n \text{ are squares,}\\ 0 & \text{otherwise.}\end{cases}$$

Using polynomials given by (14) and the double sequence $u=(u_{mn})$, we introduce the following $mPLOs$ on $C\left(D,\mathbb{C}^{s\times t}\right)$:

$$\Phi_{mn}(F;x,y)=(1+u_{mn})B_{mn}\left(F;x,y\right), \tag{16}$$

where $(m,n)\in\mathbb{N}^{2}$, $(x,y)\in D$ and $F\in C\left(D,\mathbb{C}^{s\times t}\right)$ such that $F(x,y):=\left[f_{jk}(x,y)\right]_{s\times t}$, $1\leq j\leq s$, $1\leq k\leq t$. So, using the properties of matrix-valued the above operators given by (14), for each $1\leq j\leq s$, $1\leq k\leq t$, one can obtain the following results at once:

$$\left\|\Phi_{mn}(E_{ijk})-E_{ijk}\right\|_{s\times t}=u_{mn},\ i=0,1,2,$$

and

$$\begin{aligned}
\left\|\Phi_{mn}(E_{3jk})-E_{3jk}\right\|_{s\times t}\ \leq\ &\frac{1}{m}\left(\alpha-a\right)^{2}+\frac{\left(b-a\right)\left(\alpha-a\right)}{m}+\frac{1}{n}\left(\beta-c\right)^{2}\\
&+\frac{\left(d-c\right)\left(\beta-c\right)}{n}\\
&+\frac{u_{mn}}{m}\left(\alpha-a\right)^{2}+\frac{\left(b-a\right)\left(\alpha-a\right)}{m}u_{mn}\\
&+\frac{u_{mn}}{n}\left(\beta-c\right)^{2}+\frac{\left(d-c\right)\left(\beta-c\right)}{n}u_{mn},
\end{aligned}$$

where $\alpha=\max\left|x\right|$, $\beta=\max\left|y\right|$. Since $st-\lim_{m,n}u_{mn}=0$, we conclude that

$$st^{2}-\lim_{m,n}\left\|\Phi_{mn}(E_{ijk})-E_{ijk}\right\|_{s\times t}=0,\quad i=0,1,2,3,$$

for each $1 \leq j \leq s$, $1 \leq k \leq t$. So, by Theorem 1, we immediately see that

$$st^2 - \lim_{m,n} \|\Phi_{mn}(F) - F\|_{s\times t} = 0$$

for all $F \in C\left(D, \mathbb{C}^{s\times t}\right)$. However, since u is not ordinary convergent to zero, the double sequence (Φ_{mn}) given by (16) does not satisfy the conditions of Corollary 2.

4 Rate of Convergence

Various ways of defining rates of convergence in the A-statistical sense for four-dimensional summability matrices were introduced in [2]. In this section, we compute the corresponding rates of A-statistical convergence in Theorem 1 by means of two different ways.

DEFINITION 1 ([2]). Let $A = [a_{prmn}]$ be a non-negative RH-regular summability matrix and let (α_{mn}) be a positive non-increasing double sequence. A double sequence $x = (x_{mn})$ is A-statistically convergent to a number L with the rate of $o(\alpha_{mn})$ if for every $\varepsilon > 0$,

$$P - \lim_{p,r} \frac{1}{\alpha_{pr}} \sum_{(m,n)\in K(\varepsilon)} a_{prmn} = 0,$$

where

$$K(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\right\}.$$

In this case, we write

$$x_{mn} - L = st_A^{(2)} - o(\alpha_{mn}) \quad \text{as} \quad m,n \to \infty.$$

DEFINITION 2 ([2]). Let $A = [a_{prmn}]$ and (α_{mn}) be the same as in Definition 1. Then, a double sequence $x = \{x_{mn}\}$ is A-statistically convergent to a number L with the rate of $o_{mn}(\alpha_{mn})$ if for every $\varepsilon > 0$,

$$P - \lim_{p,r} \sum_{(m,n)\in M(\varepsilon)} a_{prmn} = 0,$$

where

$$M(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\ \alpha_{mn}\right\}.$$

In this case, we write

$$x_{mn} - L = st_A^{(2)} - o_{mn}(\alpha_{mn}) \quad \text{as} \quad m,n \to \infty.$$

We see from the above statements that, in Definition 1 the rate sequence (α_{mn}) directly effects the entries of the matrix $A = [a_{prmn}]$ although, according to Definition 2, the rate is more controlled by the terms of the sequence $x = (x_{mn})$ (see for details, [5]) Let $F \in C(D, \mathbb{C}^{s\times t})$ such that

$$F(x,y) := [f_{jk}(x,y)]_{s\times t}\,,\ 1\le j\le s,\ 1\le k\le t.$$

Consider the the following modulus of continuity $\omega(f_{jk};\delta)$:

$$\omega\left(f_{jk};\delta\right) := \sup\left\{\left|f_{jk}\left(u,v\right)-f_{jk}\left(x,y\right)\right| : \left(u,v\right),\left(x,y\right)\in D, \sqrt{\left(u-x\right)^2+\left(v-y\right)^2}\le\delta\right\}$$

where f_{jk} are scalar valued functions continuous on D and $\delta > 0$. Then, we define the matrix modulus of continuity of F as follows:

$$\omega_{s\times t}\left(F;\delta\right) := \max_{1\le j\le s,\ 1\le k\le t}\omega\left(f_{jk};\delta\right).$$

In order to obtain our result, we will make use of the elementary inequality, for all $F \in C(D, \mathbb{C}^{s\times t})$ and for $\lambda, \delta > 0$,

$$\omega_{s\times t}\left(F;\lambda\delta\right) \le \left(1+[\lambda]\right)\omega_{s\times t}\left(F;\delta\right), \tag{17}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Then we have the following result.

THEOREM 2. Let $A = [a_{prmn}]$ be a nonnegative RH-regular summability matrix, let (α_{mn}) be a positive non-increasing double sequence. and let (Φ_{mn}) be a double sequence of $mPLOs$ acting from $C(D, \mathbb{C}^{s\times t})$ into itself. Then for all $(j,k) \in \{1,2,...,s\}\times\{1,2,...,t\}$ and for each $i = 0,1,2,3$,

(a) $\left\|\Phi_{mn}(E_{0jk}) - E_{0jk}\right\|_{s\times t} = st_A^{(2)} - o(\alpha_{mnjk})$ as $m,n\to\infty$,

(b) $\omega_{s\times t}\left(F;\delta_{mn}\right) = st_A^{(2)} - o(\delta_{mn})$ as $m,n\to\infty$ where

$$F \in C(D, \mathbb{C}^{s\times t}) \text{ and } \delta_{mn} := \sqrt{\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}(\Psi_{jk})\right\|_{s\times t}}$$

where $\Psi_{jk}(u,v) = (u-x)^2 + (v-y)^2$ for each $(x,y)\,,(u,v)\in D$.

Then, we get, for each $F \in C(D, \mathbb{C}^{s\times t})$ as in (1),

$$\left\|\Phi_{mn}(F) - F\right\|_{s\times t} = st_A^{(2)} - o(\gamma_{mn}),$$

where

$$\gamma_{m,n} := \max_{1\le j\le s,\ 1\le k\le t}\{\alpha_{mnjk}, \delta_{mn}\}$$

for all $(m,n)\in\mathbb{N}^2$. Furthermore, similar conclusions hold with the symbol "o" replaced by "o_{mn}"

PROOF. To see this, we first assume that $(x,y) \in D$ and $F \in C(D, \mathbb{C}^{s\times t})$ be fixed, and that (a) and (b) hold. Since Φ_{mn} is a $mPLO$, we get

$$\begin{aligned} & \left|\Phi_{mn}(F(u,v);x,y) - F(x,y)\right| \\ \leq\ & K\Phi_{mn}(\left|F(u,v) - F(x,y)\right|;x,y) + \left|\Phi_{mn}(F(x,y);x,y) - F(x,y)\right|, \end{aligned}$$

where K is a positive constant. Also,

$$\begin{aligned} \left|F(u,v) - F(x,y)\right| &\leq \omega_{s\times t}\left(F; \sqrt{(u-x)^2 + (v-y)^2}\right)E \\ &\leq \left(1 + \frac{(u-x)^2 + (v-y)^2}{\delta^2}\right)\omega_{s\times t}(F;\delta)\,E, \end{aligned} \tag{18}$$

where E is the $s \times t$ matrix such that all entires 1. As in the proof Theorem 1, we may write

$$\left|\Phi_{mn}(F(x,y);x,y) - F(x,y)\right| \leq M \sum_{j=1}^{s}\sum_{k=1}^{t} \left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right) - E_{0jk}(x,y)\right|, \tag{19}$$

where

$$M := \max_{1\leq j\leq s,\ 1\leq k\leq t} M_{jk} = \|F\|_{s\times t}\,.$$

By (18) and (19), we obtain

$$\begin{aligned} & \left|\Phi_{mn}(F(u,v);x,y) - F(x,y)\right| \\ \leq\ & K\omega_{s\times t}(F;\delta)\,\Phi_{mn}(E) + \frac{K}{\delta^2}\omega_{s\times t}(F;\delta)\sum_{j=1}^{s}\sum_{k=1}^{t}\Phi_{mn}(\Psi_{jk};x,y) \\ & + M\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right) - E_{0jk}(x,y)\right| \\ \leq\ & K\omega_{s\times t}(F;\delta)\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right) - E_{0jk}(x,y)\right| \\ & + M\sum_{j=1}^{s}\sum_{k=1}^{t}\left|\Phi_{mn}\left(E_{0jk}(u,v);x,y\right) - E_{0jk}(x,y)\right| \\ & + \frac{K}{\delta^2}\omega_{s\times t}(F;\delta)\sum_{j=1}^{s}\sum_{k=1}^{t}\Phi_{mn}(\Psi_{jk};x,y) + K\omega_{s\times t}(F;\delta)\,E. \end{aligned}$$

Taking supremum over $(x,y) \in D$ on the both-sides of the above inequality and

$$\delta := \delta_{mn} := \sqrt{\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}(\Psi_{jk})\right\|_{s\times t}},$$

then we obtain

$$\begin{aligned}\left\|\Phi_{mn}(F)-F\right\|_{s\times t} &\leq 2K\omega_{s\times t}\left(F;\delta_{mn}\right)\\ &\quad +K\omega_{s\times t}\left(F;\delta_{mn}\right)\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\\ &\quad +M\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}.\end{aligned}$$

Hence, we get

$$\begin{aligned}&\left\|\Phi_{mn}(F)-F\right\|_{s\times t}\\ \leq\ & B\left\{\omega_{s\times t}\left(F;\delta_{mn}\right)+\omega_{s\times t}\left(F;\delta_{mn}\right)\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\right.\\ &\left.+\sum_{j=1}^{s}\sum_{k=1}^{t}\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\right\} \qquad (20)\end{aligned}$$

where $B=\max\{2K,M\}$. Now, given $\varepsilon>0$, define the following sets:

$$\Gamma:=\left\{(m,n)\in\mathbb{N}^2:\left\|\Phi_{mn}(F)-F\right\|_{s\times t}\geq\varepsilon\right\},$$

$$\Gamma_1:=\left\{(m,n)\in\mathbb{N}^2:\omega_{s\times t}\left(F;\delta_{mn}\right)\geq\frac{\varepsilon}{(2st+1)\,B}\right\},$$

$$\Delta_{jk}:=\left\{(m,n)\in\mathbb{N}^2:\omega_{s\times t}\left(F;\delta_{mn}\right)\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\geq\frac{\varepsilon}{(2st+1)\,B}\right\},$$

$$\Theta_{jk}:=\left\{(m,n)\in\mathbb{N}^2:\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\geq\frac{\varepsilon}{(2st+1)\,B}\right\},$$

where $1\leq j\leq s$, $1\leq k\leq t$. Then, it follows from (20) that

$$\Gamma\subseteq\Gamma_1\cup\left(\bigcup_{j=1}^{s}\bigcup_{k=1}^{t}\Delta_{jk}\right)\cup\left(\bigcup_{j=1}^{s}\bigcup_{k=1}^{t}\Theta_{jk}\right).$$

Also, defining

$$U:=\left\{(m,n)\in\mathbb{N}^2:\omega_{s\times t}\left(F;\delta_{mn}\right)\geq\sqrt{\frac{\varepsilon}{(2st+1)\,B}}\right\}$$

and

$$U_{jk}:=\left\{(m,n)\in\mathbb{N}^2:\left\|\Phi_{mn}\left(E_{0jk};x,y\right)-E_{0jk}\right\|_{s\times t}\geq\sqrt{\frac{\varepsilon}{(2st+1)\,B}}\right\},$$

we have $\Delta_{jk} \subset U \cup U_{jk}$, which yields

$$\Gamma \subseteq \Gamma_1 \cup U \cup \left(\bigcup_{j=1}^{s} \bigcup_{k=1}^{t} U_{jk} \right) \cup \left(\bigcup_{j=1}^{s} \bigcup_{k=1}^{t} \Theta_{jk} \right).$$

Therefore, since $\gamma_{mn} := \max_{1\leq j\leq s,\ 1\leq k\leq t} \{\alpha_{mnjk}, \delta_{mn}\}$, we conclude that, for all $(p, r) \in \mathbb{N}^2$,

$$\begin{aligned} & \frac{1}{\gamma_{pr}} \sum_{(m,n)\in\Gamma} a_{prmn} \\ \leq\ & \frac{1}{\delta_{pr}} \sum_{(m,n)\in\Gamma_1} a_{prmn} + \sum_{j=1}^{s}\sum_{k=1}^{t} \left(\frac{1}{\alpha_{mnjk}} \sum_{(m,n)\in U_{jk}} a_{prmn} \right) \\ & + \frac{1}{\delta_{pr}} \sum_{(m,n)\in U} a_{prmn} + \sum_{j=1}^{s}\sum_{k=1}^{t} \left(\frac{1}{\alpha_{mnjk}} \sum_{(m,n)\in\Theta_{jk}} a_{prmn} \right). \qquad (21) \end{aligned}$$

Letting $p, r \to \infty$ (in any manner) on both sides of (21), from (18) and (19), we get

$$P - \lim_{p,r} \frac{1}{\gamma_{pr}} \sum_{(m,n)\in\Gamma} a_{prmn} = 0.$$

Therefore, the proof is completed.

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