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On The Annular Regions Containing All The Zeros Of A Polynomial*

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Abstract

In this paper, we obtain some results concerning the location of zeros of a complex polynomial. The results presented here improve upon the earlier results and this is justified through some specific examples, for which we develop MATLAB code to construct polynomials for which the region obtained by our results are sharper than obtainable from some of the known results. A brief comparative analysis on the computational results is also done. It may be noted that the problems dealing with the location of zeros of a polynomial, besides being of theoretical interest, have important applications in many areas, such as signal processing, communication theory and control theory.

1 Introduction

Let $p(z)$ be a polynomial of degree n, having complex coefficients. Since by the Fundamental Theorem of Algebra the polynomial $p(z)$ has exactly n zeros, so it would obviously be of interest to obtain the smallest possible region containing all the zeros of a polynomial. The results related to the location of zeros of a polynomial have significant applications in many areas such as Mathematical Physics, Signal Processing, Communication Theory, Control Theory, Coding Theory, Cryptography, Mathematical Biology, and Computer Engineering, and so there is always a demand for better and better results.

It may be remarked that there are methods, for example, Ehrlich-Aberth's type (see, [1, 13, 21]) for the simultaneous determination of the zeros of algebraic polynomials, and there are studies to accelerate convergence and increase computational efficiency of these methods (for example, see [19, 22]). These methods, which are of course very useful, because of their giving approximations to the zeros of a polynomial, can possibly become more efficient when combined with the results of this paper that provide annulus containing all the zeros of a polynomial.

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The first result concerning the location of zeros of a polynomial is probably due to Gauss, who proved that a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

with all a_k real, has no zeros outside the circle $|z| = R$, where

$$R = \max_{1 \le k \le n} (n\sqrt{2}|a_k|)^{\frac{1}{k}}.$$

The above result of Gauss was improved by Cauchy [5], who proved that if

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$$

is a complex polynomial of degree n, then all the zeros of $p(z)$ lie in the disc

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\},$$

where $A = \max_{1 \le k \le n-1} |a_k|$ and η is the unique positive root of the real-coefficient equation

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|\, z - |a_0| = 0.$$

If one applies the above result of Cauchy to the polynomial $P(z) = z^n p(1/z)$, one easily gets the following Theorem 1.

THEOREM 1 (Cauchy). All the zeros of the polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the annulus $r_1 \le |z| \le r_2$, where r_1 is the unique positive root of the equation

$$|a_n|z^n + |a_{n-1}|z^{n-1} + \cdot + |a_1|z - |a_0| = 0, \tag{1}$$

and r_2 is the unique positive root of the equation

$$|a_0| + |a_1|\, z + \cdot + |a_{n-1}|z^{n-1} - |a_n|z^n = 0. \tag{2}$$

Although the above Theorem of Cauchy gives an annulus containing all the zeros of a polynomial, it is implicit, in the sense, that in order to find the annulus containing all the zeros of a polynomial, one needs to compute the zeros of two other polynomials. The results providing annuli with radii explicitly in terms of coefficients have been given in many papers and books (see [2, 3, 6, 9, 14, 15, 18, 24]), and we begin by stating the following theorem due to Diaz-Barrero [10], which gives an annulus containing all the zeros of a polynomial.

THEOREM 2. If $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ is a non-constant complex polynomial of degree n, with $a_k \neq 0,\ 1 \le k \le n$, then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where

$$r_1 = \frac{3}{2} \min_{1 \le k \le n} \left\{ \frac{2^n F_k C(n,k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \frac{2}{3} \max_{1\le k\le n} \left\{ \frac{F_{4n}}{2^n F_k C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}},$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$, and F_k is the k^{th} Fibonacci number, defined by $F_0 = 0,\ F_1 = 1$ and $F_k = F_{k-1} + F_{k-2},\ k \ge 2$.

In this direction, Kim [16] also gave the following

THEOREM 3. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \ne 0,\ 1 \le k \le n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where

$$r_1 = \min_{1\le k\le n} \left\{ \frac{C(n,k)}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\le k\le n} \left\{ \frac{2^n - 1}{C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}},$$

where $C(n,k)$ are the binomial coefficients.

Another result in this direction, providing annulus containing all the zeros of a polynomial is due to Diaz-Barrero and Egozcue [11].

THEOREM 4. If $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ is a non-constant complex polynomial of degree n, with $a_k \ne 0,\ 1 \le k \le n$, then for $j \ge 2$, all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where

$$r_1 = \min_{1\le k\le n} \left\{ \frac{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}}{A_{jn}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\le k\le n} \left\{ \frac{A_{jn}}{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}},$$

where $C(n,k)$ are the binomial coefficients,

$$B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k} \text{ and } \sum_{k=0}^{n} C(n,k)(bB_{j-1})^{n-k} B_j^k A_k = A_{jn} \text{ for } j \ge 2,$$

where $A_n = cr^n + ds^n,\ c, d$ are real constants and r, s are the roots of the equation $x^2 - ax - b = 0$, in which a, b are strictly positive real numbers.

Recently Dalal and Govil [7] (also, see [8]) proved the following theorem which unifies and includes all the above Theorems 2, 3 and 4 as special cases.

THEOREM 5. Let $A_k > 0$ for $1 \leq k \leq n$ and be such that $\sum_{k=1}^{n} A_k = 1$. If $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ is a non-constant complex polynomial of degree n, with $a_k \neq 0$ for $1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \text{ and } r_2 = \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

As an application of Theorem 5, Dalal and Govil [7] also gave the following.

THEOREM 6. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0, \ \ 1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C_{k-1} C_{n-k}}{C_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{C_n}{C_{k-1} C_{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}},$$

where $C_k = \frac{C(2k,k)}{k+1}$ is the k^{th} Catalan number in which $C(2k, k)$ are the binomial coefficients.

As mentioned in the paper of Dalal and Govil [7], the above Theorem 5, besides including Theorems 2, 3, and 4, as special cases, is also capable of generating many new results by making appropriate choice of the numbers A_k. Recently Bidkham et al. [4] and Rather and Matto [23] obtained results on annulus containing all the zeros of a polynomial involving Fibonacci numbers and generalized Fibonacci numbers respectively. Although not mentioned in their papers, but as is easy to see, the results obtained by them can also be obtained as special cases of Theorem 5.

In this paper, we use Theorem 5 to obtain the following theorems, which provide annuli containing all the zeros of a polynomial. Also, we show that for some polynomials our theorems sharpen some of the known results in this direction, and this has been done in Section 4 where we develop MATLAB code to generate examples of polynomials for which our results give better bounds than obtainable from the known results, such as Theorems 2, 3 and 6. As we will see, in some cases improvement has come out to be quite significant.

Our first result, stated below gives annulus in terms of Narayana numbers [20].

THEOREM 7. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0, \ \ 1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{N(n,k)}{C_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}, \ \ r_2 = \max_{1 \leq k \leq n} \left\{ \frac{C_n}{N(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}, \tag{3}$$

$C_n = \frac{C(2n,n)}{n+1}$ is the n^{th} Catalan number, $N(n,k)$, $(1 \le k \le n)$ are Narayana numbers given for any natural number n, by $N(n,k) = \frac{1}{n}C(n,k)C(n,k-1)$, and $C(n,k)$ are binomial coefficients.

In the next result, we will make use of Motzkin numbers (see [12]) to evaluate the radii of two circles involved in the annular region containing all the zeros of a polynomial.

THEOREM 8. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0,\ 1 \le k \le n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$ with

$$r_1 = \min_{1\le k\le n} \left\{ \frac{M_{k-1}M_{n-1-k}}{M_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{4}$$

and

$$r_2 = \max_{1\le k\le n} \left\{ \frac{M_n}{M_{k-1}M_{n-1-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}, \tag{5}$$

where M_n is the n^{th} Motzkin number given by $M_0 = M_1 = M_{-1} = 1$ and

$$M_{n+1} = \frac{2n+3}{n+3}M_n + \frac{3n}{n+3}M_{n-1},\ n \ge 1.$$

Finally, we present the following result which involves the special combination of binomial coefficients.

THEOREM 9. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0,\ \ 1 \le k \le n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where

$$r_1 = \min_{1\le k\le n} \left\{ \frac{C(2(k-1),k-1)\ C(2(n-k),n-k)}{4^{n-1}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\le k\le n} \left\{ \frac{4^{n-1}}{C(2(k-1),k-1)\ C(2(n-k),n-k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}, \tag{6}$$

where $C(2(k-1),k-1)$ and $C(2(n-k),n-k)$ are binomial coefficients.

As mentioned above, these theorems are of interest because for some polynomials, they yield sharper bounds than obtainable from some of the known results and this has been shown in Section 4.

2 Lemmas

We will need the following lemmas to prove our results. Our first lemma connects Narayana numbers with Catalan numbers

LEMMA 1. If $N(n,k)$ are Narayana numbers for the given positive integer n, then

$$\sum_{k=1}^{n} N(n,k) = C_n, \tag{7}$$

where C_n is the nth Catalan number.

PROOF. Even though, it is a fundamental identity in the field of combinatorics, for the sake of completeness, we provide brief outlines of the proof. Note that, if $C(n,k)$ denote the binomial coefficients, then Narayana numbers are given by the formula

$$N(n,k) = \frac{1}{n} C(n,k)\, C(n,k-1),$$

and Catalan numbers, by the formula $C_n = \frac{1}{n+1} C(2n,n)$. Then

$$\begin{aligned} \sum_{k=1}^{n} N(n,k) &= \frac{1}{n} \sum_{k=1}^{n} C(n,k) C(n,k-1) = \frac{1}{n} C(2n, n-1) \\ &= \frac{1}{n} \frac{(2n)!}{(n-1)!(2n-(n-1))!} = \frac{1}{n+1} C(2n,n) = C_n. \end{aligned}$$

Next lemma provides an identity involving Motzkin numbers, which we will use to prove Theorem 8. Although, this identity is known, however for the sake of completeness we will present brief outlines of the proof.

LEMMA 2. If M_n is the n^{th} Motzkin number, then

$$M_n = \sum_{k=1}^{n} M_{k-1} M_{n-1-k}, \tag{8}$$

where $M_0 = M_1 = M_{-1} = 1$.

PROOF. It is known that, Motzkin number M_n is the number of different ways of drawing non-intersecting chords on a circle between n points. Take any one point on the circle and join it to any one of the other points on the circle. If one side of the chord contains i points then the other side of the chord contains $(n-2-i)$ points. Hence the number of different ways of drawing non-intersecting chords in which the selected point always forms a chord is $M_0 M_{n-2} + M_1 M_{n-3} + \cdots + M_{n-2} M_0$. By removing selected point from the circle, the number of ways drawing non-intersecting chords in which the selected point never forms a chord is M_{n-1}. In other words, the total number of ways of drawing non-intersecting chords on the circle is the sum of those two numbers; that is,

$$M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}.$$

Proper re-indexing gives the equality (8), thus completing the proof.

Our next result is a nice consecutive binomial coefficient identity, which is well-known. We omit its proof since it can be found in [17, p. 77].

LEMMA 3. With the standard binomial coefficient notations,

$$\sum_{k=1}^{n} C(2(k-1), k-1)C(2(n-k), n-k) = 4^{n-1}. \tag{9}$$

3 Proofs of the Theorems

PROOF OF THEOREM 7. If $N(n,k)$ are Narayana numbers and C_n is the n^{th} Catalan number, then by Lemma 1,

$$\sum_{k=1}^{n} \frac{N(n,k)}{C_n} = 1.$$

Thus, if we take $A_k = \frac{N(n,k)}{C_n}$, then each A_k is positive and $\sum_{k=1}^{n} A_k = 1$, and hence applying Theorem 5 for this set of A_k, $(1 \leq k \leq n)$, we get the required annulus C given by (3), that contains all the zeros of the polynomial $p(z)$.

PROOF OF THEOREM 8. Note that if M_n is the n^{th} Motzkin number, then from Lemma 2, we have

$$\sum_{k=1}^{n} \frac{M_{k-1}M_{n-1-k}}{M_n} = 1,$$

where $M_0 = M_1 = M_{-1} = 1$. If we take $A_k = \frac{M_{k-1}M_{n-1-k}}{M_n}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 5 for this set of values of A_k, $(1 \leq k \leq n)$, we get (4), and Theorem 8 is proved.

PROOF OF THEOREM 9. By Lemma 3, we have

$$\sum_{k=1}^{n} \frac{C(2(k-1), k-1)\; C(2(n-k), n-k)}{4^{n-1}} = 1.$$

Now, if we take

$$A_k = \frac{C(2(k-1), k-1)\; C(2(n-k), n-k)}{4^{n-1}},$$

then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence applying Theorem 5 for this set of values of A_k, we get the required annulus given by (6), and the proof of Theorem 9 is thus complete.

4 Computational Results and Analysis

In this section, we present two examples of polynomials, for which the annuli obtained by our results are significantly smaller than the annuli obtainable from the other stated results.

Our first example is the one given in the paper due to Dalal and Govil [7].

EXAMPLE 1. Let $p(z) = z^3 + 0.1z^2 + 0.1z + 0.7$.

As one can observe from the Table 1 given below, our Theorem 8 is giving significantly better bound than obtainable from any of the known Theorems 2, 3 and 6. In fact the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 8 is about 2.3701, which is about 68.24% of the area of the annulus obtained by Theorem 2, about 28.27% of the area of the annulus obtained by Theorem 3, and about 73.69% of the area of the annulus obtained by Theorem 6.

Table 1:

Result	r_1	r_2	*Area of the annulus*
THEOREM 2	0.6402	1.2312	3.4730
THEOREM 3	0.4641	1.6984	8.382
THEOREM 6	0.6542	1.2050	3.2159
THEOREM 7	0.5192	1.5182	6.3950
THEOREM 8	0.7047	1.1186	2.3701
THEOREM 9	0.6403	1.2313	3.4748
Actual bound	0.8840	0.8899	0.0328

Our next example has been constructed by using MATLAB code.

EXAMPLE 2. Let $p(z) = z^5 + 0.06z^4 + 0.29z^3 + 0.29z^2 + 0.29z + 0.001$.

For the polynomial $p(z)$ given in this example, it is clear from the Table 2 below that Theorem 7 gives the best upper bound of the annular region containing all the zeros of the polynomial $p(z)$. Theorem 7 also provides considerably good result for the estimation of the area of the annular region containing all the zeros of $p(z)$, which is very close to the actual area of the annulus. In fact the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 7 is about 2.2167 which differs from the actual area by only about 1.089%. Also, this area is about 24.67% of the area obtained from Theorem 2, about 52.61% of the area obtained by Theorem 3, and about 28.96 % of the area obtained by Theorem 6.

Table 2:

Result	r_1	r_2	*Area of the annulus*
$THEOREM$ 2	0.00012233	1.6912	8.986
$THEOREM$ 3	0.00055617	1.158	4.2125
$THEOREM$ 6	0.0011	1.5608	7.6529
$THEOREM$ 7	0.00024631	0.84	2.2167
$THEOREM$ 8	0.0015	1.2339	4.7831
$THEOREM$ 9	0.000942895	1.362351	5.83078
Actual bound	0.0034602	0.83544	2.1927

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