Two-Steps Hybrid Iterative Schemes With Errors For Generalized Equilibrium Problems And Common Fixed Point Problems

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Abstract

In this paper, we consider a two-steps hybrid iterative scheme with errors for generalized equilibrium problems and common fixed point problems, and prove the weak limits of sequences \( \{x_n\} \) and \( \{v_n\} \) obtained under the given scheme for \( N \) finite asymptotically \( k_i \)-strictly pseudo-contractive mappings \( \{T_i\}_{i=1}^N \) and a firmly nonexpansive mapping \( S_r \) are the same and hence the point is a common fixed point of \( \{T_i\}_{i=1}^N \) and \( S_r \).

1 Introduction

The applications of equilibrium problems and fixed point theory to many branches have been well-known for a long time in nonlinear analysis including optimization theory, economics, etc. (see [2, 3, 5, 7, 9, 12]).

Recently there have been many researches on approximating convergence of fixed points under iteration schemes with errors concerning equilibrium problems and variational inequalities, etc. (see [2, 3, 4, 5, 9]).

On the other hand, Qin et al. [11] and Kumam et al. [6] considered equilibrium problems with fixed point problems under one-step hybrid iterative schemes and two-step hybrid iterative schemes, respectively, in Hilbert spaces.

Inspired by those results, we consider the following two-step hybrid iterative scheme with errors for generalized equilibrium problems and common fixed point problems, and obtain a result that the weak limits of sequences \( \{x_n\} \) and \( \{v_n\} \) obtained under the given scheme for \( N \) finite asymptotically \( k_i \)-strictly pseudo-contractive mappings \( \{T_i\}_{i=1}^N \) and a firmly nonexpansive mapping \( S_r \) are the same and that the same point is a common fixed point of \( \{T_i\}_{i=1}^N \) and \( S_r \).

ALGORITHM 1.1. Let \( C \) be a closed convex subset of a Hilbert space \( H, T_i, \psi : C \to C \) \( (i = 1, 2, \cdots, N) \) be mappings and \( \phi : C \times C \to \mathbb{R} \) be a bifunction. For any

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First of all, we recall some definitions and results needed in the main results.

REMARK 1.1. (a) Putting $\psi \equiv 0$ in (1), we obtain an algorithm

\[
\begin{aligned}
\phi(v_{n-1}, y) + \frac{1}{r_{n-1}}(y - v_{n-1}, v_{n-1} - x_{n-1}) &\geq 0 & \text{for all } y \in C, \\
x_n = a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^h v_{n-1} + c_{n-1}u_{n-1} &\text{for each } n \in \mathbb{N}. \\
\end{aligned}
\]  

(2)

(b) Putting $c_n = 0$ for all $n \in \mathbb{N}$ in (2), we obtain the algorithm considered in [6]

\[
\begin{aligned}
\phi(v_{n-1}, y) + \frac{1}{r_{n-1}}(y - v_{n-1}, v_{n-1} - x_{n-1}) &\geq 0 & \text{for all } y \in C, \\
x_n = a_{n-1}v_{n-1} + (1 - a_{n-1})T_{i(n)}^h v_{n-1} &\text{for each } n \in \mathbb{N}. \\
\end{aligned}
\]  

(3)

(c) Putting $\phi \equiv 0$ and $v_n = x_n \ (n \in \mathbb{N})$ in (3), we obtain the algorithm considered in [11]

\[
x_n = a_{n-1}x_{n-1} + (1 - a_{n-1})T_{i(n)}^h x_{n-1} &\text{for each } n \in \mathbb{N}. \\
\]  

(4)

2 Preliminaries

First of all, we recall some definitions and results needed in the main results.

DEFINITION 2.1. Let $\phi : C \times C \to \mathbb{R}$ be a function and $\psi : C \to C$ be a nonlinear mapping. (a) $\phi$ is said to be monotone if $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$. (b) $\psi$ is said to be monotone if $\langle \psi x - \psi y, x - y \rangle \geq 0$ for all $x, y \in C$.

DEFINITION 2.2. A mapping $T : C \to C$ is asymptotically $k$-strictly pseudo-contractive if there exist $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

\[
\|T^n x - T^n y\|^2 \leq k_n^2\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}. \\
\]

LEMMA 2.1 ([7, 10]). Let $H$ be a real Hilbert space. Then we have the following identities:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ for all $x, y \in H$. 


(ii) For all \( x, y \in H \) and \( a, b, c \in [0, 1] \) with \( a + b + c = 1 \),
\[
\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - \|b||x - y\|^2 - \|bc||y - z\|^2 - \|ca||z - x\|^2.
\]

(iii) If \( \{x_n\} \) is a sequence in \( H \) weakly converging to \( z \), then
\[
\lim_{n \to \infty} \sup \|x_n - y\|^2 = \lim_{n \to \infty} \sup \|x_n - z\|^2 + \|z - y\|^2 \quad \text{for all } y \in H.
\]

**Lemma 2.2** ([10]). Let \( \{a_n\}, \{c_n\} \) and \( \{\delta_n\} \) be nonnegative real sequences satisfying the condition \( a_{n+1} \leq (1 + \delta_n)a_n + c_n \) for each \( n \in \mathbb{N} \). If
\[
\sum_{n=1}^{\infty} \delta_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty,
\]
then \( \lim_{n \to \infty} a_n \) exists.

### 3 Main Results

We assume that the mapping \( \phi : C \times C \to \mathbb{R} \) satisfies the following conditions:

(i) \( \phi(x, x) = 0 \) for all \( x \in C \);

(ii) \( \phi \) is monotone;

(iii) \( \phi \) is upper hemi-continuous in the first variable;

(iv) \( \phi \) is convex and lower semi-continuous in the second variable.

We have the following theorems.

**Theorem 3.1.** Let \( C \) be a closed convex subset of a Hilbert space \( H \). Let \( \psi : C \to C \) be a monotone nonlinear mapping. For \( r > 0 \) and \( x \in H \), define a mapping \( S_r : H \to 2^C \) by
\[
S_r x = \left\{ z \in C : \phi(z, y) + \langle \psi z, y - z \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C \right\}.
\]

Then the following statements (i)–(iii) hold.

(i) \( S_r x \) is a singleton for each \( x \in H \).

(ii) \( S_r \) is firmly nonexpansive, i.e.,
\[
\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle \quad \text{for } x, y \in H.
\]

(iii) The set \( F(S_r) \) of all fixed points of \( S_r \) is a closed and convex subset of \( C \) as a solution set of the following equilibrium problem considered in [9]: finding \( x \in C \) such that \( \phi(x, y) + \langle \psi x, y - x \rangle \geq 0 \) for all \( y \in C \).
PROOF. (i) We put \( \zeta(x, y) = \phi(x, y) + \langle \psi x, y-x \rangle \) for all \( x, y \in C \). By the conditions of \( \phi \) and [1, Theorem 1], we see that \( S_x \neq \emptyset \) for any \( x \in C \). Next, we show that \( S_x \) is a singleton for \( x \in C \). Suppose that \( z_1, z_2 \in S_x \). Then
\[
\begin{align*}
\phi(z_1, y) + \langle \psi z_1, y - z_1 \rangle + \frac{1}{r} \langle y - z_1, z_1 - x \rangle & \geq 0 \quad \text{for } y \in C, \\
\phi(z_2, y) + \langle \psi z_2, y - z_2 \rangle + \frac{1}{r} \langle y - z_2, z_2 - x \rangle & \geq 0 \quad \text{for } y \in C.
\end{align*}
\]
Putting \( y = z_2 \) in the first inequality and \( y = z_1 \) in the second inequality (5), respectively and adding them, we have
\[
\phi(z_1, z_2) + \phi(z_2, z_1) + \langle \psi z_1 - \psi z_2, z_2 - z_1 \rangle \geq \frac{1}{r} \|z_1 - z_2\|^2.
\]
Since \( \phi(z_1, z_2) + \phi(z_2, z_1) \leq 0 \) and \( \langle \psi z_1 - \psi z_2, z_2 - z_1 \rangle \leq 0 \), we have \( z_1 = z_2 \). So we prove statement (i).

(ii) Let \( z = S_r x \) and \( z' = S_r x' \). Then
\[
\begin{align*}
\phi(z, z') + \langle \psi z, z' - z \rangle + \frac{1}{r} \langle z' - z, z - x \rangle & \geq 0, \\
\phi(z', z) + \langle \psi z', z - z' \rangle + \frac{1}{r} \langle z - z', z' - x' \rangle & \geq 0.
\end{align*}
\]
Adding two inequalities and applying the monotonicity of \( \phi \) and \( \psi \), we have
\[
\langle S_r x - S_r x', x - x' \rangle = \langle z - z', x - x' \rangle \geq \|z - z'\|^2 = \|S_r x - S_r x'\|^2.
\]
Hence, \( S_r \) is a firmly nonexpansive mapping. So we prove statement (ii).

(iii) If \( x \in F(S_r) \), then
\[
\phi(x, y) + \langle \psi x, y - x \rangle = \phi(x, y) + \langle \psi x, y - x \rangle + \frac{1}{r} \langle y - x, x - x \rangle \geq 0
\]
for all \( y \in C \). So \( x \) is a solution of the equilibrium problem in [9]. Next, let \( \{x_n\} \) be a convergent sequence in \( F(S_r) \) with a limit \( x \in H \). Since \( F(S_r) \subset C \) and \( C \) is closed, we have \( x \in C \). Also, \( S_r \) is continuous. Then we have
\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} S_r x_n = S_r x.
\]
It means that \( x \in F(S_r) \), that is, \( F(S_r) \) is closed.

To show that \( F(S_r) \) is convex, we let \( z = \lambda x + (1 - \lambda)y \) for \( x, y \in F(S_r) \) and \( \lambda \in [0, 1] \). By Lemma 2.1(ii) and the nonexpansiveness of \( S_r \), we have
\[
\begin{align*}
\|z - S_r z\|^2 & = \|\lambda(x - S_r z) + (1 - \lambda)(y - S_r z)\|^2 \\
& = \lambda \|x - S_r z\|^2 + (1 - \lambda)\|y - S_r z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \\
& \leq \lambda \|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \\
& = \lambda \|x - (\lambda x + (1 - \lambda)y)\|^2 + (1 - \lambda)\|y - (\lambda x + (1 - \lambda)y)\|^2 \\
& \quad - \lambda(1 - \lambda)\|x - y\|^2 \\
& = \lambda(1 - \lambda)^2\|x - y\|^2 + (1 - \lambda)\lambda^2\|x - y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 = 0.
\end{align*}
\]
Hence, $S_r z = z$ and $z \in F(S_r)$. Therefore $F(S_r)$ is convex. So we prove statement (iii).

The proof of Theorem 3.1 is complete.

REMARK 3.1. By putting $\psi \equiv 0$ in Theorem 3.1, we obtain [4, Lemma 2.12].

Next, we consider our main result.

THEOREM 3.2. Assume that the mappings $T_i : C \to C$ for $i = 1, \cdots, N$ satisfy the following conditions:

(i) $C$ is a closed $k_i$-strictly pseudo-contractive for $k_i \in [0, 1)$, $i = 1, 2, \cdots, N$

and for each $i \in \{1, 2, \cdots, N\}$, $\{k_n, i\}$ is a sequence in $[1, \infty)$ such that $\sum_{n=1}^{\infty} (k^2_n - 1) < \infty$;

(ii) $k = \max\{k_i : 1 \leq i \leq N\}$ and $k'_n = \max\{k_{n, i} : 1 \leq i \leq N\}$ for each $n \in \mathbb{N}$.

Let $\psi : C \to C$ be a monotone nonlinear mapping with

$$F := \left( \bigcap_{i=1}^{N} F(T_i) \right) \cap F(S_r) \neq \emptyset.$$

For any $x_0 \in C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 1.1. Then $\{x_n\}$ and $\{v_n\}$ converge weakly to the unique same element of $F$.

PROOF. Let $p \in F$. By Algorithm 1.1 and Theorem 3.1(i), we see that $v_{n-1} = S_{r_{n-1}} x_{n-1}$ and

$$\|v_{n-1} - p\| = \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p\| \leq \|x_{n-1} - p\|$$

for each $n \in \mathbb{N}$. By Algorithm 1.1 and Lemma 2.1(ii), we have

$$\|x_n - p\|^2 = \|a_{n-1} (v_{n-1} - p) + b_{n-1} (T_{i(n)}^{(h(n))} v_{n-1} - p) + c_{n-1} (v_{n-1} - p)\|^2$$

$$\leq \|a_{n-1} (v_{n-1} - p)\|^2 + b_{n-1} \left\| T_{i(n)}^{(h(n))} v_{n-1} - T_{i(n)}^{(h(n))} p \right\|^2 + c_{n-1} \|v_{n-1} - p\|^2$$

$$- a_{n-1} b_{n-1} \left\| T_{i(n)}^{(h(n))} v_{n-1} - v_{n-1} \right\|^2$$

$$\leq \|a_{n-1} (v_{n-1} - p)\|^2 + b_{n-1} \left\{ (k_{h(n)})^2 \|v_{n-1} - p\|^2 + k \| (I - T_{i(n)}^{(h(n))}) v_{n-1} \right\}$$

$$- (I - T_{i(n)}^{(h(n))}) p \| v_{n-1} - p \|^2 + c_{n-1} \|u_{n-1} - p\|^2 - a_{n-1} b_{n-1} \left\| T_{i(n)}^{(h(n))} v_{n-1} - v_{n-1} \right\|^2$$

$$\leq (k_{h(n)})^2 \|v_{n-1} - p\|^2 - b_{n-1} (a_{n-1} - k) \left\| T_{i(n)}^{(h(n))} v_{n-1} - v_{n-1} \right\|^2$$
\[ + c_{n-1} \|u_{n-1} - p\|^2 \]
\[ \leq \left[ 1 + (k_{h(n)}')^2 - 1 \right] \|x_{n-1} - p\|^2 + c_{n-1} \|u_{n-1} - p\|^2. \]

(6)

(7)

Since \( \sum_{n=1}^{\infty} (k_{n,i}^2 - 1) < \infty \), and by Lemma 2.2, we see that \( \lim_{n \to \infty} \|x_n - p\| \) exists. On the other hand, since \( a_n \geq k + \varepsilon \) and \( b_n \geq \varepsilon \) for \( n \in \mathbb{N} \) and some \( \varepsilon \in (0,1) \), we have

\[ (k_{h(n)}')^2 \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + c_{n-1} \|u_{n-1} - p\|^2 \]
\[ \geq b_{n-1}(a_n - k) \|T_{i(n)}^h v_{n-1} - v_{n-1}\|^2 \]
\[ \geq \varepsilon^2 \|T_{i(n)}^h v_{n-1} - v_{n-1}\|^2. \]

Since \( \lim_{n \to \infty} k_{h(n)}' = 1 \) and \( \lim_{n \to \infty} c_n = 0 \), taking the limits as \( n \to \infty \) in the above inequality, we have

\[ \lim_{n \to \infty} \left\| T_{i(n)}^h v_{n-1} - v_{n-1} \right\|^2 = 0. \]

(8)

Observe that

\[ \|x_n - v_{n-1}\| = \left\| a_{n-1} v_{n-1} + b_{n-1} T_{i(n)}^h v_{n-1} + c_{n-1} u_{n-1} - v_{n-1} \right\| \]
\[ \leq \left\| - (1 - a_{n-1}) (v_{n-1} - T_{i(n)}^h v_{n-1}) + c_{n-1} \left( u_{n-1} - T_{i(n)}^h v_{n-1} \right) \right\| \]
\[ \leq (1 - a_{n-1}) \left\| v_{n-1} - T_{i(n)}^h v_{n-1} \right\| + c_{n-1} \left\| u_{n-1} - T_{i(n)}^h v_{n-1} \right\|. \]

By (8), we see that

\[ \lim_{n \to \infty} \|x_n - v_{n-1}\| = 0. \]

(9)

By the firm nonexpansiveness of \( S_{r_{n-1}} \) and Lemma 2.1(i), we have

\[ \|v_{n-1} - p\|^2 = \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p\|^2 \leq \langle S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p, x_{n-1} - p \rangle \]
\[ = \langle v_{n-1} - p, x_{n-1} - p \rangle = - \langle (x_{n-1} - v_{n-1}) - (x_{n-1} - p), x_{n-1} - p \rangle \]
\[ = - \frac{1}{2} \left( \|x_{n-1} - v_{n-1}\|^2 - \|x_{n-1} - p\|^2 - \|v_{n-1} - p\|^2 \right), \]

and hence

\[ \|v_{n-1} - p\|^2 \leq \|x_{n-1} - p\|^2 - \|x_{n-1} - v_{n-1}\|^2. \]

Applying this inequality to (6), we have

\[ \|x_n - p\|^2 \leq \left( k_{h(n)}' \right)^2 \left( \|x_{n-1} - p\|^2 - \|x_{n-1} - v_{n-1}\|^2 \right) + c_{n-1} \|u_{n-1} - p\|^2. \]

Since \( \lim_{n \to \infty} \|x_n - p\| \) exists and \( \lim_{n \to \infty} k_{h(n)}' = 1 \), we see that

\[ \lim_{n \to \infty} \|x_{n-1} - v_{n-1}\| = 0. \]

(10)
Applying (9) and (10) to the triangle inequality, we have
\[ \|v_n - v_{n-1}\| \leq \|v_n - x_n\| + \|x_n - v_{n-1}\| \to 0 \text{ as } n \to \infty, \]
which implies that
\[ \lim_{n \to \infty} \|v_n - v_{n+j}\| = 0 \text{ for } j \in \{1, \cdots, N\}. \]  

(11)

Similarly, applying (10) and (11) to the triangle inequality, we obtain
\[ \|x_n - x_{n-1}\| \leq \|x_n - v_n\| + \|v_n - v_{n-1}\| + \|v_{n-1} - x_{n-1}\| \to 0 \text{ as } n \to \infty, \]
which implies that \( \lim_{n \to \infty} \|x_n - x_{n+j}\| = 0 \text{ for } j \in \{1, \cdots, N\}. \) On the other hand,
\[ \|v_{n-1} - T_n v_{n-1}\| \leq \|v_{n-1} - T_i h^{-1}(n) v_{n-1}\| + \|T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n) v_{n-1}\| \leq \|v_{n-1} - T_i h^{-1}(n) v_{n-1}\| + \|T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n) v_{n-1}\| \]
\[ + \left( L \left( \|T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n) v_{n-1}\| + \|v_{n-1} - v_{n-1}\| \right) \right), \]
\[ \text{where} \]
\[ L = \sup \left\{ \frac{k + \sqrt{1 + (n^2 - 1)(1 - k)}}{1 - k} : n \in \mathbb{N} \right\}. \]  

(12)

Since, for each \( n > N, n = (h(n) - 1)N + i(n), i(n - N) = i(n) \) and \( h(n - N) = h(n) - 1, \)
\[ \left\| T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n - N) v_{n-1}\right\| = \left\| T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n) v_{n-1}\right\| \leq L \|v_{n-1} - v_{n-1}\| \]  

(13)

and
\[ \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\| \leq \left\| T_i h^{-1}(n - N) v_{n-1} - T_i h^{-1}(n - N) v_{n-1}\right\| + \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\| \leq L \|v_{n-1} - v_{n-1}\| + \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\|. \]  

(14)

So by (12)–(14), we see that
\[ \|v_{n-1} - T_n v_{n-1}\| \leq \|v_{n-1} - T_i h^{-1}(n) v_{n-1}\| + L \cdot \left\{ \left\| T_i h^{-1}(n) v_{n-1} - T_i h^{-1}(n - N) v_{n-1}\right\| \right. \]
\[ + \left. \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\| \right\} + \|v_{n-1} - v_{n-1}\| + \|v_{n-1} - v_{n-1}\| \]
\[ + \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\| + \left\| T_i h^{-1}(n - N) v_{n-1} - v_{n-1}\right\|. \]
By (8) and (11), we have that $\lim_{n \to \infty} \|v_{n-1} - T_n v_{n-1}\| = 0$. Since
\[
\|v_n - T_n v_n\| \leq \|v_n - v_{n-1}\| + \|v_{n-1} - T_n v_{n-1}\| + \|T_n v_{n-1} - T_n v_n\|
\leq (1 + L) \cdot \|v_n - v_{n-1}\| + \|v_{n-1} - T_n v_{n-1}\| \to 0 \text{ as } n \to \infty,
\]
for any $j = 1, \cdots, N$, we have
\[
\|v_n - T_{n+j} v_n\| \leq \|v_n - v_{n+j}\| + \|v_{n+j} - T_{n+j} v_{n+j}\| + \|T_{n+j} v_{n+j} - T_{n+j} v_n\|
\leq (1 + L) \cdot \|v_n - v_{n+j}\| + \|v_{n+j} - T_{n+j} v_{n+j}\| \to 0 \text{ as } n \to \infty,
\]
which gives that $\lim_{n \to \infty} \|v_n - T_l v_n\| = 0$ for $l \in \{1, \cdots, N\}$. Moreover, for each $l \in \{1, \cdots, N\}$, we have
\[
\|x_n - T_l x_n\| \leq \|x_n - v_n\| + \|v_n - T_l v_n\| + \|T_l v_n - T_l x_n\|
\leq (1 + L) \cdot \|x_n - v_n\| + \|v_n - T_l v_n\| \to 0 \text{ as } n \to \infty.
\]
Put
\[
W(x_n) = \{x \in H : x_n \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.
\]
Then $W(x_n) \neq \emptyset$ by the fact that $\{x_n\}$ is bounded in $H$. Next, we claim that $W(x_n) \subset F$. Let $w \in W(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $w$. Since $\lim_{n \to \infty} \|x_n - v_n\| = 0$, we can obtain that $v_{n_i} \rightharpoonup w$ as $i \to \infty$. By the fact that $\lim_{n \to \infty} \|v_n - T_l v_n\| = 0$, we have $T_l v_{n_i} \rightharpoonup w$ for $l \in \{1, \cdots, N\}$. Now, we show that $w$ is a fixed point of $S_r$. Since $v_n = T_{r_n} v_n$ for each $n \in \mathbb{N}$, we have
\[
\phi(v_n, y) + \langle \psi v_n, y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, v_n - x_n \rangle \geq 0 \text{ for all } y \in C \text{ and } n \in \mathbb{N}.
\]
By the monotonicity of $\phi$, we have
\[
\langle y - v_{n_i}, \frac{v_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq \phi(y, v_{n_i}) + \langle \psi v_{n_i}, v_{n_i} - y \rangle \text{ for } i \in \mathbb{N}.
\]
Since $\frac{v_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $v_{n_i} \rightharpoonup w$ as $i \to \infty$, and by the condition (iv) of $\phi$, we have
\[
\phi(y, w) + \langle \psi w, w - y \rangle \leq 0 \text{ for } y \in C.
\]
By the conditions (i) and (iv) of $\phi$, we see that
\[
0 = \phi(y_t, y_t) \leq t \phi(y_t, y) + (1 - t) \phi(y_t, w)
\leq t \phi(y_t, y) + (1 - t) \langle \psi w, y_t - w \rangle = t \phi(y_t, y) + (1 - t) \langle \psi w, y_t - w \rangle
\leq \phi(y_t, y) + (1 - t) \langle \psi w, y_t - w \rangle,
\]
where $t \in (0, 1)$, $y \in C$, and $y_t = ty + (1 - t)w$. By the condition (iii) of $\phi$,
\[
0 \leq \phi(w, y) + \langle \psi w, y - w \rangle \text{ for all } y \in C,
\]
which shows that \( w \in F(S_r) \). Moreover, \( w \in \bigcap_{i=1}^{N} F(T_i) \). In fact, if \( w \notin F(T_i) \) for some \( l \in \{1, \cdots, N\} \), then from the Opial’s condition and the fact that \( \lim_{n \to \infty} \|x_n - T_l x_n\| = 0 \),

\[
\liminf_{i \to \infty} \|x_{n_i} - w\| < \liminf_{i \to \infty} \|x_{n_i} - T_l w\| \leq \liminf_{i \to \infty} \left\{ \|x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - T_l w\| \right\} \\
\leq \liminf_{i \to \infty} L \cdot \|x_{n_i} - w\|
\]

which derives a contradiction. Consequently, we have

\[
w \in F = \left( \bigcap_{i=1}^{N} F(T_i) \right) \bigcap F(S_r).
\]

Finally, we show that \( \{x_n\} \) and \( \{v_n\} \) converge weakly to the unique same element of \( F \). Indeed, it is sufficient to show that \( W(x_n) \) is a singleton. We take any \( w_1, w_2 \in W(x_n) \) and let \( \{x_{n_i}\} \) and \( \{x_{n_j}\} \) be subsequences of \( \{x_n\} \) such that \( x_{n_i} \to w_1 \) and \( x_{n_j} \to w_2 \). Since \( \lim_{n \to \infty} \|x_n - p\| \) exists for each \( p \in F \) and \( w_1, w_2 \in F \), by Lemma 2.1(iii), we obtain

\[
\limsup_{n \to \infty} \|x_n - w_1\|^2 = \limsup_{j \to \infty} \|x_{n_j} - w_1\|^2 = \limsup_{j \to \infty} \|x_{n_j} - w_2\|^2 + \|w_2 - w_1\|^2 \\
= \limsup_{i \to \infty} \|x_{n_i} - w_2\|^2 + \|w_2 - w_1\|^2 \\
= \limsup_{i \to \infty} \|x_{n_i} - w_1\|^2 + 2\|w_2 - w_1\|^2 \\
= \limsup_{n \to \infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|^2.
\]

Hence \( w_1 = w_2 \), which shows that \( W(x_n) \) is a singleton. The proof of Theorem 3.2 is complete.

We have the following theorems in \([6, 11]\) as corollaries of Theorem 3.2.

**THEOREM 3.3** ([6]). Assume that the conditions (i)–(iii) in Theorem 3.2 hold and that \( \phi \) satisfies

\[
F := \left( \bigcap_{i=1}^{N} F(T_i) \right) \bigcap S(\phi) \neq \emptyset.
\]

For any \( x_0 \in C \), let \( \{x_n\} \) and \( \{v_n\} \) be sequences generated by (3), where \( n = (h - 1)N + i(\geq 1), \ i = i(n) \in \{1, 2, \cdots, N\}, \ h = h(n) \geq 1 \) is a positive integer and \( h(n) \to \infty \) as \( n \to \infty \). Let \( \{a_n\} \) and \( \{r_n\} \) be sequences satisfying \( \{a_n\} \subset [a, \beta] \) for some \( \alpha, \beta \in (k, 1), \ \{r_n\} \subset (0, \infty) \) and \( \liminf_{n \to \infty} r_n > 0 \). Then \( \{x_n\} \) and \( \{v_n\} \) converge weakly to an element of \( F \).

**THEOREM 3.4** ([11]). Assume that the conditions (i)–(iii) in Theorem 3.2 hold and

\[
F := \left( \bigcap_{i=1}^{N} F(T_i) \right) \neq \emptyset.
\]
For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (4), where $\{a_n\}$ is a sequence in $(0, 1)$ such that $k + \varepsilon \le a_n \le 1 - \varepsilon$ for some $\varepsilon \in (0, 1)$, $n = (h - 1)N + i (n \ge 1)$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \ge 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to an element of $F$.

REMARK 3.2. Our result is a weak convergence under Algorithm 1.1 for a finite family of asymptotically $k_i$-strictly pseudo-contractive mappings in Hilbert spaces. The convergences, mappings and spaces need to be more weakened, for examples, strongly convergences, asymptotically nonexpansive mappings and $CAT(0)$-spaces, respectively. Till now, many kinds of strong convergence results are well-known, but the weak convergence results are few. So, we suggest the following open problem.

Open problem. Do $\{x_n\}$ and $\{v_n\}$ weakly converge for a finite family of asymptotically nonexpansive mappings with Algorithm 1.1 under suitable conditions?

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References


