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Absence Of Nontrivial Solutions For A Class Of Elliptic Equations And Systems In Unbounded Domains*

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Abstract

In this paper, we are interested in the study of the nonexistence of nontrivial solutions for a class of ellptic equations, in unbounded domains. This leads us to extend these results to systems of equations. The method used is based on energy type identities.

1 Introduction

In this work we study the absence of nontrivial solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left(p(y)\frac{\partial u}{\partial y_i}\right) + f(y,u) = 0 \text{ in } \mathbb{R}\times\Omega, \\ u + \varepsilon\frac{\partial u}{\partial \nu} = 0 \text{ on } \mathbb{R}\times\partial\Omega. \end{cases} \tag{1}$$

considered in $H^2(\mathbb{R}\times\Omega)\cap L^\infty(\mathbb{R}\times\Omega)$, for a bounded and positive function $p \in C^1(\mathbb{R}\times\Omega)$.

We use the notations

$$t = y_1 \in \mathbb{R},\ x = (x_1, ..., x_{n-1}) = (y_2, ..., y_n) \in \Omega,$$

$$H = L^2(\Omega),$$

$$\|u(t,x)\| = \left(\int_\Omega |u(t,x)|^2\, dx\right)^{\frac{1}{2}},\ \text{the norm of } u \text{ in } H,$$

$$\|\nabla u(t,x)\|^2 = \int_\Omega \sum_{i=1}^{n-1}\left|\frac{\partial u}{\partial x_i}\right|^2 dx,$$

$$F(y,u) = F(t,x,u) = \int_0^u f(t,x,\sigma)d\sigma,\ \forall\ x\in\Omega,\ u\in\mathbb{R}.$$

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Let L be the operator defined by

$$Lu\,(t,x) = -\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(p\,(t,x)\,\frac{\partial u}{\partial x_i}\right), (t,x) \in \mathbb{R}\times\Omega,$$

and $f:\mathbb{R}\times\,\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ a real continuous function, locally Lipschitz in u, such that

$$f(t,x,0) = 0,\ \forall\ x \in \overline{\Omega}.$$

We assume that

$$u \in H^2(\mathbb{R};H) \cap L^\infty(\mathbb{R};L^\infty(\Omega))$$

satisfies the equation

$$-\frac{\partial}{\partial t}\left(p\,(t,x)\,\frac{\partial u}{\partial t}\right) + Lu\,(t,x) + f(t,x,u) = 0 \text{ a.e. } (t,x)\in\mathbb{R}\times\Omega \tag{2}$$

under the boundary conditions

$$(u+\varepsilon\frac{\partial u}{\partial n})\,(t,\sigma) = 0, (t,\sigma)\in\mathbb{R}\times\partial\Omega,\ \text{Robin condition,} \tag{3}$$

$$u(t,\sigma) = 0, (t,\sigma)\in\mathbb{R}\times\partial\Omega,\ \text{Dirichlet condition,} \tag{4}$$

$$\frac{\partial u(t,\sigma)}{\partial\nu} = 0, (t,\sigma)\in\mathbb{R}\times\partial\Omega,\ \text{Neumann condition.} \tag{5}$$

We will extend our result for (1) to the system of m equations of the form

$$\begin{cases} -\sum\limits_{i=1}^{n} \frac{\partial}{\partial y_i}\left(p_k\,(y)\,\frac{\partial u_k}{\partial y_i}\right) + f_k\,(y,u_1,...,u_m) = 0 \text{ in } \mathbb{R}\times\Omega, \\ u_k + \varepsilon\frac{\partial u_k}{\partial\nu} = 0 \text{ on } \mathbb{R}\times\partial\Omega, \end{cases}$$

where $1\le k\le m$, $f_k:\mathbb{R}\times\overline{\Omega}\times\mathbb{R}^m\to\mathbb{R}$, are real continuous functions, locally Lipschitz in u_i, verifying

$$f_k(t,x,u_1,...,0,...,u_m) = 0,\ \forall\ x\in\overline{\Omega},$$

$$\exists F_m:\mathbb{R}\times\overline{\Omega}\times\mathbb{R}^m\to\mathbb{R} \text{ such that } \frac{\partial F_m}{\partial s_j} = f_j\,(t,x,s_1,...,s_m)\,, 1\le j\le m.$$

Let L_k be the operators defined by

$$L_k u\,(t,x) = -\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(p_k\,(t,x)\,\frac{\partial u}{\partial x_i}\right), (t,x)\in\mathbb{R}\times\Omega,$$

and assume that

$$u_k \in H^2(\mathbb{R};H) \cap L^\infty(\mathbb{R};L^\infty(\Omega))$$

are solutions of the system

$$-\frac{\partial}{\partial t}\left(p_k\,(t,x)\,\frac{\partial u_k}{\partial t}\right) + L_k u_k\,(t,x) + f(t,x,u_1,...,u_m) = 0\ , (t,x)\in\mathbb{R}\times\Omega, \tag{6}$$

where $1 \leq k \leq m$ with boundary conditions

$$(u_k + \varepsilon \frac{\partial u_k}{\partial n})\,(t,\sigma) = 0, (t,\sigma) \in \mathbb{R} \times \partial\Omega, \text{ Robin condition,} \tag{7}$$

$$u_k(t,\sigma) = 0, (t,\sigma) \in \mathbb{R} \times \partial\Omega, \text{ Dirichlet condition,} \tag{8}$$

$$\frac{\partial u_k(t,\sigma)}{\partial \nu} = 0, (t,\sigma) \in \mathbb{R} \times \partial\Omega, \text{ Neumann condition.} \tag{9}$$

The study of the nonexistence of nontrivial solutions of elliptic equations and systems is the subject of several works of many authors, and various methods are used to obtain necessary and sufficient conditions that guarantee that the systems studied admit only trivial solutions. The works of Esteban & Lions [4], Pohozaev [9] and Van Der Vorst [10], contain results concerning semilinear elliptic equations and systems. A similar result can be found in [7] and [1], where equations of the form

$$\begin{cases} \frac{\partial}{\partial t}\left(\lambda\,(t)\,\frac{\partial u}{\partial t}\right) - \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p\,(t,x)\,\frac{\partial u}{\partial x_i}\right) + f\,(x,u) = 0 \text{ in } \mathbb{R}\times\Omega, \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \mathbb{R}\times\partial\omega, \end{cases}$$

are considered in $H^2(\mathbb{R}\times\omega) \cap L^\infty(\mathbb{R}\times\omega)$, where Ω is a bounded domain of $\mathbb{R}^n$, $0 < \varepsilon < +\infty$, $\lambda \in \mathbb{R}^*$, $p : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is continuous and positive function in Ω, and f $: \Omega\times\mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous function such that

$$\lambda\,(t) > 0 \;\; (\text{resp} < 0)\,, \;\; \forall t \in \mathbb{R},$$

$$2F\,(x,u) - uf\,(x,u) \leq 0 \;\; (\text{resp} \geq 0)\,,$$

where F is the primitive of the function f.

Our proof is based on energy type identities established in section 2, which make it possible to obtain the main nonexistence result in section 3. In section 4 we apply the results to some examples.

2 Identities of Energy Type

In this section, we give essential lemmas for showing the main result of this paper.

LEMMA 1. Let F and p *satisfy*

$$\begin{array}{l} \frac{\partial F}{\partial t}\,(t,x,u) \leq 0 \;\; (\text{resp } \geq 0)\,, \forall\,(t,x) \in \mathbb{R}\times\Omega, \\ \frac{\partial p}{\partial t}(t,x) \leq 0 \;\; (\text{resp } \geq 0)\,, \forall\,(t,x) \in \mathbb{R}\times\Omega. \end{array} \tag{10}$$

Then the following energy identity,

$$-\frac{1}{2}\int_\Omega p\,(t,x)\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_\Omega p(t,x)\,|\nabla u|^2\,dx + \int_\Omega F(t,x,u)dx$$
$$+\frac{1}{2\varepsilon}\int_{\partial\Omega} p(t,s)u^2\,(t,s)\,ds = 0 \tag{11}$$

holds for any solution of the Robin problem of (2) and (3).

PROOF. The assumptions $f \in W^{1,\infty}_{loc}(\mathbb{R}\times\Omega\times\mathbb{R})$, $p \in L^{\infty}(\mathbb{R}\times\Omega)$ and $f(t,x,0) = 0$, for $x \in \Omega$, allow us to deduce the existence of two positive constants C_1 and C_2, such that

$$|p(t,x)| \leq C_1, \quad |F(t,x,u)| \leq C_2 \left|u\left(t,x\right)\right|^2.$$

In addition, consider the functions

$$A\left(t\right) = \frac{1}{2}\int_{\Omega} p(t,x)\left|\nabla u\right|^2 dx \text{ and } B\left(t\right) = \int_{\Omega} F(t,x,u)dx \text{ for } t \in \mathbb{R},$$

where A and B are of class C^1, and

$$|A\left(t\right)| \leq C_1 \left\|\nabla u(t,x)\right\|^2 \text{ and } |B\left(t\right)| \leq C_2 \left\|u(t,x)\right\|^2 \text{ for } t \in \mathbb{R}.$$

Then,

$$B'\left(t\right) = \int_{\Omega}\left(f(t,x,u)\frac{\partial u}{\partial t} + \frac{\partial F}{\partial t}\right)dx \text{ for } t \in \mathbb{R},$$

and

$$\begin{aligned} A'\left(t\right) &= \int_{\Omega}\left(\sum_{i=1}^{n-1} p(t,x)\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial t} + \frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial p}{\partial t}(t,x)\left|\frac{\partial u}{\partial t}\right|^2\right)dx \\ &= -\int_{\Omega}\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial t}dx + \frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial p}{\partial t}(t,x)\left|\frac{\partial u}{\partial x_i}\right|^2 \\ &\quad + \int_{\partial\Omega}\sum_{i=1}^{n-1} p(t,s)\frac{\partial u}{\partial x_i}\nu_i\frac{\partial u}{\partial t}\left(t,s\right)ds. \end{aligned}$$

Define the function $M : \mathbb{R} \to \mathbb{R}$ by

$$M\left(t\right) = -\frac{1}{2}\int_{\Omega} p\left(t,x\right)\left|\frac{\partial u}{\partial t}\right|^2 dx + A(t) + B(t).$$

Obviously M is absolutely continuous and differentiable in $\mathbb{R}$, and

$$\begin{aligned} M'(t) &= -\frac{1}{2}\int_{\Omega}\frac{\partial P}{\partial t}(t,x)\left|\frac{\partial u}{\partial t}\right|^2 dx - \int_{\Omega} p\left(t,x\right)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}dx + A'(t) + B'(t) \\ &= \frac{1}{2}\int_{\Omega}\frac{\partial P}{\partial t}(t,x)\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2 dx \\ &\quad + \int_{\Omega}\frac{\partial F}{\partial t}\left(t,x,u\right)dx + \int_{\partial\Omega}\sum_{i=1}^{n-1} p(t,s)\frac{\partial u}{\partial x_i}\nu_i\frac{\partial u}{\partial t}\left(t,s\right)ds \\ &\quad + \int_{\Omega}\left(-\frac{\partial}{\partial t}\left(p\left(t,x\right)\frac{\partial u}{\partial t}\right) - \sum_{i=2}^{n}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right) + f(t,x,u)\right)\frac{\partial u}{\partial t}dx. \end{aligned}$$

Because u is solution of (2) and (3), we deduce that

$$\begin{aligned} M'(t) &= \int_\Omega \frac{\partial P}{\partial t}(t,x)\left(\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2\right)dx + \int_\Omega \frac{\partial F}{\partial t}(t,x,u)\,dx \\ &\quad + \int_{\partial\Omega}\sum_{i=1}^{n-1} p(t,s)\frac{\partial u}{\partial x_i}\nu_i\frac{\partial u}{\partial t}(t,s)\,ds, \end{aligned}$$

while on the boundary,

$$\begin{aligned} &\int_{\partial\Omega}\sum_{i=1}^{n-1} p(t,s)\frac{\partial u}{\partial x_i}\nu_i\frac{\partial u}{\partial t}(t,s)\,ds \\ =\ &\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu}(t,s)\,ds = -\frac{1}{\varepsilon}\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial t}u\,(t,s)\,ds \\ =\ &-\frac{1}{2\varepsilon}\frac{d}{dt}\left(\int_{\partial\Omega} p(t,s)u^2\,(t,s)\,ds\right) + \frac{1}{2\varepsilon}\int_{\partial\Omega}\frac{\partial p}{\partial t}(t,s)u^2\,(t,s)\,ds. \end{aligned}$$

Also

$$\begin{aligned} \frac{d}{dt}\left(M(t) + \frac{1}{2\varepsilon}\int_{\partial\Omega} p(t,s)u^2\,(t,s)\,ds\right) &= \frac{1}{2}\int_\Omega \frac{\partial P}{\partial t}(t,x)\left(\left|\frac{\partial u}{\partial t}\right|^2 + |\nabla u|^2\right)dx \\ &\quad + \int_\Omega \frac{\partial F}{\partial t}(t,x,u)\,dx \\ &\quad + \frac{1}{2\varepsilon}\int_{\partial\Omega}\frac{\partial p}{\partial t}(t,s)u^2\,(t,s)\,ds. \end{aligned}$$

We set

$$N\,(t) = M(t) + \frac{1}{2\varepsilon}\int_{\partial\Omega} p(t,s)u^2\,(t,s)\,ds.$$

Conditions (2.1) imply that

$$N'\,(t) \le 0 \ \ (\text{resp} \ge 0)\,,\ \ \forall t \in \mathbb{R},$$

i.e., M is monotone. However, this function also satisfies

$$\lim_{|t|\to+\infty} M\,(t) = 0$$

because $M \in L^2\,(\mathbb{R})$. Hence, $M\,(t) = 0$ for $t \in \mathbb{R}$, and this gives the desired result.

LEMMA 2. Let F and p verify (2.1). The solution of the Dirichlet problem (2) and (4) or the Neumann problem (2) and (5), satisfies the following energy identity

$$-\frac{1}{2}\int_\Omega p\,(t,x)\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_\Omega p(t,x)\,|\nabla u|^2\,dx + \int_\Omega F(t,x,u)dx = 0.$$

PROOF. For the problem (2) and (4), the fact that $u=0$ on the boundary implies that

$$\begin{aligned}\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}(t,s)\,ds &= \frac{d}{dt}\left(\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}u(t,s)\,ds\right)\\ &\quad -\int_{\partial\Omega}\frac{\partial p}{\partial t}\frac{\partial u}{\partial \nu}u(t,s)\,ds-\int_{\partial\Omega}p(t,s)\frac{\partial^2 u}{\partial t\partial \nu}u(t,s)\,ds\\ &= 0.\end{aligned}$$

For the problem (2) and (5), the fact that $\frac{\partial u}{\partial \nu}=0$ on the boundary implies that

$$\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}(t,s)\,ds=0.$$

The remainder of the proof is similar to that of Lemma 1.

LEMMA 3. Let λ and p_k satisfy

$$\frac{\partial F_m}{\partial t}(t,x,u_1,...,u_m)\leq 0 \;\; (\text{resp } \;\geq 0)\,,\;\forall\,(t,x)\in\mathbb{R}\times\Omega,$$

$$\frac{\partial p_k}{\partial t}(t,x)\leq 0 \;\; (\text{resp } \;\geq 0)\,,\;1\leq k\leq m,\;\forall\,(t,x)\in\mathbb{R}\times\Omega.$$

Then any solution of the system (6) and (7) satisfies the following energy identity

$$\begin{aligned}&-\frac{1}{2}\sum_{k=1}^{m}\int_{\Omega}p_k(t,x)\left|\frac{\partial u_k}{\partial t}(t,x)\right|^2dx+\frac{1}{2}\sum_{k=1}^{m}\int_{\Omega}p_k(t,x)\left|\nabla u_k\right|^2dx\\ &+\int_{\Omega}F_m(t,x,u_1,...,u_m)dx+\frac{1}{2\varepsilon}\sum_{k=1}^{m}\int_{\partial\Omega}p_k(t,s)u_k^2(t,s)\,ds=0. \qquad (12)\end{aligned}$$

LEMMA 4. Let λ and p_k verify (12). *Then the solutions of the systems* (6) and (8), or (6) and (9) satisfy the following equality

$$\begin{aligned}&-\frac{1}{2}\sum_{k=1}^{m}\int_{\Omega}p_k(t,x)\left|\frac{\partial u_k}{\partial t}(t,x)\right|^2dx+\frac{1}{2}\sum_{k=1}^{m}\int_{\Omega}p_k(t,x)\left|\nabla u_k\right|^2dx\\ &+\int_{\Omega}F_m(t,x,u_1,...,u_m)dx=0. \qquad (13)\end{aligned}$$

PROOF. Let us define the function $M_m:\mathbb{R}\to\mathbb{R}$ by

$$M_m(t)=-\frac{1}{2}\sum_{k=1}^{m}\int_{\Omega}p_k(t,x)\left|\frac{\partial u_k}{\partial t}(t,x)\right|^2dx+A_m(t)+B_m(t),$$

where the functions A_m and B_m are defined as follows

$$A_m(t) = \frac{1}{2}\int_\Omega \sum_{k=1}^{m} p_k(t,x)\left|\nabla u_k\right|^2 dx \text{ for } t \in \mathbb{R},$$

$$B_m(t) = \int_\Omega F_m(t,x,u_1,...,u_m)dx \text{ for } t \in \mathbb{R}.$$

The rest of the proof is similar to the proofs of the preceding lemmas.

3 The Main Results

In this section, we present three main theorems.

THEOREM 1. Let us suppose that F and f verify

$$2F(t,x,u) - uf(t,x,u) \leq 0, \tag{14}$$

and (10) holds. Then the problem (2) and (3) admit only the null solution.

PROOF. Let us define the function E by

$$E(t) = \int_\Omega p(t,x)\,u^2(x,t)\,dx.$$

Multiplying equation (2) by u and integrating the new equation on Ω, we obtain

$$\begin{aligned}
&\int_\Omega \left[-\frac{\partial}{\partial t}\left(p(t,x)\frac{\partial u}{\partial t}\right) - \sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right) + f(t,x,u)\right] u dx \\
= &\int_\Omega \left[-\frac{1}{2}\left(\frac{\partial p}{\partial t}\frac{\partial\left(u^2\right)}{\partial t} + p(t,x)\frac{\partial^2\left(u^2\right)}{\partial t^2}\right) + p(t,x)\left|\frac{\partial u}{\partial t}\right|^2\right. \\
&\left. + p(t,x)\left|\nabla u\right|^2 + uf(x,u)\right] dx - \sum_{i=1}^{n-1}\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial x_i}u(t,s)\,\nu_i ds \\
= &-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega p(t,x)\frac{\partial\left(u^2\right)}{\partial t}dx\right) + \int_\Omega p(t,x)\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\nabla u\right|^2\right) dx \\
&+ \int_\Omega uf(x,u)\,dx - \int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}u(t,s)\,ds = 0 \\
= &-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega p(t,x)\frac{\partial\left(u^2\right)}{\partial t}dx\right) + \int_\Omega p(t,x)\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\nabla u\right|^2\right) dx \\
&+ \int_\Omega uf(x,u)\,dx + \frac{1}{\varepsilon}\int_{\partial\Omega} p(t,s)\,u^2(t,s)\,ds = 0\ .
\end{aligned}$$

Using identity (11), we have

$$\frac{d}{dt}\left(\int_{\Omega} p(t,x)\,\frac{\partial\left(u^{2}\right)}{\partial t}dx\right)=4\int_{\Omega} p(t,x)\left|\frac{\partial u}{\partial t}\right|^{2}dx-2\int_{\Omega}\left(2F(x,u)-uf(x,u)\right)dx.$$

The assumption (3.1) implies that

$$\frac{d}{dt}\left(\int_{\Omega} p(t,x)\,\frac{\partial\left(u^{2}\right)}{\partial t}dx\right)\geq 0,\ \ \forall t\in\mathbb{R}.$$

We conclude that the function

$$K(t)=\int_{\Omega} p(t,x)\,\frac{\partial\left(u^{2}\right)}{\partial t}dx$$

is monotone. But, this function verifies

$$\lim_{|t|\rightarrow+\infty} K(t)=0,$$

witch implies that

$$K(t)=0,\forall t\in\mathbb{R},$$

In addition

$$\begin{aligned}E'(t) &= \int_{\Omega}\frac{\partial p}{\partial t}(t,x)\,u^{2}(t,x)\,dx+\int_{\Omega} p(t,x)\,\frac{\partial\left(u^{2}\right)}{\partial t}dx\\ &= \int_{\Omega}\frac{\partial p}{\partial t}(t,x)\,u^{2}(t,x)\,dx+K(t)\\ &= \int_{\Omega}\frac{\partial p}{\partial t}(t,x)\,u^{2}(t,x)\,dx.\end{aligned}$$

The condition (10) implies that

$$E'(t)\leq 0\ \ (\text{resp}\ \ \geq 0),$$

i.e. E is monotone. But, this function verifies

$$\lim_{|t|\rightarrow+\infty} E(t)=0,$$

witch implies that

$$E(t)=0,\ \ \forall t\in\mathbb{R},$$

We deduce that $u\equiv 0$ in $\mathbb{R}\times\Omega$.

THEOREM 2. Let F and f verify (14) and assume (10) holds. Then the only solution of the problems (2) and (4), or (2) and (5) is the null solution.

PROOF. The proof is identical to that of Theorem 1 using appropriate lemmas.

THEOREM 3. Let us suppose that F_m and f_k, $1 \leq k \leq m$, satisfy

$$2F_m(x, u_1, ..., u_m) - \sum_{k=1}^{m} u_k f_k(x, u_1, ..., u_m) \leq 0,$$

and (12) holds. Then the system (6) and (7) admits only the null solutions.

PROOF. Let us define the functions E_m and K_m by

$$E_m = \int_\Omega p_k(t,x)\, u_k^2(t,x)\, dx \text{ and } K_m = \int_\Omega p_k(t,x) \frac{\partial}{\partial t}\left(u_k^2(t,x)\right) dx.$$

Multiplying equation (6) by u_k, integrating the new equation on Ω, and summing on k from 1 to m, one obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega p_k(t,x)\frac{\partial\left(u_k^2\right)}{\partial t} dx\right) + \sum_{k=1}^{m}\int_\Omega p_k(t,x)\left|\frac{\partial u_k}{\partial t}(t,x)\right|^2 dx \\
&+\sum_{k=1}^{m}\int_\Omega p_k(t,x)\left|\nabla u_k\right|^2 dx + \sum_{k=1}^{m}\int_\Omega u_k(t,x)\, f_k(x, u_1, ..., u_m)\, dx \\
&+\frac{1}{\varepsilon}\sum_{k=1}^{m}\int_{\partial\Omega} p_k(t,s)\, u_k^2(t,s)\, ds = 0.
\end{aligned}$$

By using identity (13), we deduce that

$$\begin{aligned}
K_m' &= 4\sum_{k=1}^{m}\int_\Omega p_k(t,x)\left|\frac{\partial u_k}{\partial t}(t,x)\right|^2 dx \\
&\quad -2\int_\Omega \left(2F_m(x, u_1, ..., u_m) - \sum_{k=1}^{m} u_k(t,x)\, f_k(x, u_1, ..., u_m)\right) dx.
\end{aligned}$$

Then, the assumption (3.4) implies that

$$K_m(t) = 0,\ \forall t \in \mathbb{R}.$$

So

$$E_m(t) = 0,\ \forall t \in \mathbb{R}.$$

and this gives the desired result.

4 Applications

EXAMPLE 1. Let $p, q, \geq 1$, $m \in \mathbb{R}$ and

$$\varphi_0, \varphi_1, \varphi_2 : \overline{\Omega} \to \mathbb{R},$$

be nonnegative functions of class $C^1(\mathbb{R})$ such that

$$m\frac{\partial \varphi_i}{\partial x_1} \geq 0,\ \forall i = 0, 1, 2,$$

$$f(x,u) = mu + \varphi_1(x)\,|u|^{p-1}\,u + \varphi_2(x)\,|u|^{q-1}\,u.$$

Then the problem defined by

$$\begin{cases} -\sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\varphi_0(x)\,\frac{\partial u}{\partial x_i}\right) + f(x,u) = 0 \text{ in } \mathbb{R}\times\Omega, \\ \left(u + \varepsilon\frac{\partial u}{\partial n}\right)(y,\sigma) = 0 \text{ on } \mathbb{R}\times\partial\Omega, \end{cases}$$

admits only the null solution.

In this case we have

$$F(x,u) = \frac{1}{2}mu^2 + \frac{1}{p+1}\varphi_1(x)\,|u|^{p-1}\,u + \frac{1}{q+1}\varphi_2(x)\,|u|^{q-1}\,u,$$

$$\frac{\partial F}{\partial x_1}(x,u) = \frac{1}{2}mu^2 + \frac{1}{p+1}\frac{\partial \varphi_1}{\partial x_1}(x)\,|u|^{p-1}\,u + \frac{1}{q+1}\frac{\partial \varphi_2}{\partial x_1}(x)\,|u|^{q-1}\,u.$$

It suffices to check that

$$\frac{\partial F}{\partial x_1}(x,u) \geq 0 \text{ if } m \geq 0,$$

$$\frac{\partial F}{\partial x_1}(x,u) \leq 0 \text{ if } m \leq 0,$$

$$2F(x,u) - uf(x,u) = \varphi_1(x)\,(\frac{2}{p+1} - 1)\,|u|^{p+1} + \varphi_2(x)\,(\frac{2}{q+1} - 1)\,|u|^{q+1} \leq 0,$$

and apply Theorem 1.

EXAMPLE 2. Let Ω be a bounded open of set $\mathbb{R}^n$, $p, q \geq 1$,Then, the system

$$\begin{cases} -\Delta u + (p+1)\,\theta(x)\,u\,|u|^{p-1}\,|v|^{q+1} = 0 \text{ in } \mathbb{R}\times\Omega, \\ -\Delta v + (q+1)\,\theta(x)\,v\,|v|^{q-1}\,|u|^{p+1} = 0 \text{ in } \mathbb{R}\times\Omega, \\ \left(u + \varepsilon\frac{\partial u}{\partial n}\right)(t,\sigma) = \left(v + \varepsilon\frac{\partial v}{\partial n}\right)(t,\sigma) = 0 \text{ on } \mathbb{R}\times\partial\Omega, \end{cases}$$

where $\theta : \overline{\Omega} \to \mathbb{R}$, is nonnegative,

$$\frac{\partial \theta}{\partial x_1} \geq 0\ \text{ (resp } \leq 0)$$

admits only the trivial solutions, $u \equiv v \equiv 0$.

Indeed, there exist a function F defined as follows

$$F(x,u,v) = \theta(x)\,|u|^{p+1}\,|v|^{q+1},$$

which satisfies

$$\frac{\partial F}{\partial u} = f_1(x,u,v) = (p+1)\,\theta(x)\,u\,|u|^{p-1}\,|v|^{q+1},$$

$$\frac{\partial F}{\partial v} = f_2(x,u,v) = (q+1)\,\theta\,(x)\,v\,|v|^{q-1}\,|u|^{p+1}\,,$$
$$\frac{\partial F}{\partial x_1} = \frac{\partial \theta}{\partial x_1}\,(x)\,|u|^{p+1}\,|v|^{q+1} \geq 0 \ \ (\text{resp } \ \leq 0)\,,$$
$$2F(x,u,v) - uf_1(x,u,v) - vf_2\,(x,u,v) = -\theta\,(x)\,(p+q)\,|u|^{p+1}\,|v|^{q+1} \leq 0.$$

Theorem 3 gives the result.

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