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Existence Of Solutions To Nonlinear κth-Order Coupled Klein-Gordon Equations With Nonlinear Sources And Memory Terms*

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Abstract

In this article, we consider a system of κth-order derivatives of the dependent variables of coupled Klein-Gordon equations to improve recent results obtained in [10, 31, 33] using idea in [25]. Using the potential well method, we prove that the solutions of (1) exist globally, under some conditions on the initial datum.

1 Introduction

In this paper, we consider a system of κth-order derivatives of the dependent variables of coupled Klein-Gordon equations

$$\begin{cases} u_1'' + (-1)^\kappa \Delta^\kappa u_1 + m_1^2 u_1 + \alpha_1(t) \int_0^t g_1(t-s)\Delta^\kappa u_1(x,s)ds + |u_1'|^{r-2}u_1' \\ = |u_1|^{p-2}u_1|u_2|^p, \\ u_2'' + (-1)^\kappa \Delta^\kappa u_2 + m_2^2 u_2 + \alpha_2(t) \int_0^t g_2(t-s)\Delta^\kappa u_2(x,s)ds + |u_2'|^{r-2}u_2' \\ = |u_2|^{p-2}u_2|u_1|^p \end{cases} \tag{1}$$

where $m_i, i = 1,2$ are non-negative constants, $r, p \geq 2, \kappa \geq 1$. In a bounded domain $\Omega \subset \mathbb{R}^n$ Yaojun Ye [33] introduced related problem to (1) with $\kappa = 1, \alpha_i = 0, i = 1,2$, supplemented with the initial and Dirichlet boundary conditions. By using the potential well method, global existence is discussed and asymptotic stability is also given, by using multiplied method.

Erhan Piskin and Necat Polat [10] considered a system of class of nonlinear higher-order wave equations (1) with $m_i = g_i = 0, i = 1,2$ and strong nonlinearity in sources. Under suitable conditions on the initial datum, theorems of global existence and decay rate are proved.

In (1), $u_i = u_i(t,x), i = 1,2,$ where $x \in \Omega$ is a bounded domain of $\mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$. Our system is supplemented with the following initial conditions

$$u_i(x,0) = u_{i0}(x) \in H_0^\kappa(\Omega), \ i = 1,2, \tag{2}$$

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$$u_i'(x,0)) = u_{i1}(x) \in L^2(\Omega),\ i = 1,2, \tag{3}$$

and boundary conditions

$$u_i(x) = \frac{\partial u_i}{\partial \nu} = \cdots = \frac{\partial^{\kappa-1} u_i}{\partial \nu^{\kappa-1}} = 0 \text{ for } x \in \partial\Omega \text{ and } i = 1,2, \tag{4}$$

where ν is the outward normal to the boundary.

We mention here that

$$|\nabla^\kappa u|^2 = (\Delta^{\kappa/2} u)^2 \text{ for pair value of } \kappa$$

and

$$|\nabla^\kappa u|^2 = |\nabla(\Delta^{(\kappa-1)/2} u)|^2 \text{ for odd } \kappa$$

where

$$|\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 \text{ and } \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

This kind of systems ($g_i \neq 0, i = 1,2$) appears in the models of nonlinear viscoelasticity. Viscoelastic materials have properties between two types, elastic materials and viscous fluids. This two types of materials are usually considered in basic texts on continuum mechanics. At each material point of an elastic material the stress at the present time depends only on the present value of the strain. On the other hand, for an incompressible viscous fluid the stress at a given point is a function of the present value of the velocity gradient at that point. Such materials have memory: the stress depends not only on the present values of the strain and/or velocity gradient, but also on the entire temporal history of motion.

The systems of nonlinear wave equations go back to Reed [27] who proposed a system in three space dimensions, where this type of system was completely analyzed. Existence and uniqueness of global weak solutions, asymptotic behavior for an analogous hyperbolic-parabolic system of related problems have attracted a great deal of attention in the last decades, and many results have appeared. See in this directions [5, 6, 7, 8, 12, 17, 22, 21, 24] and references therein.

We mention the work of [2], where authors studied the following system:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{5}$$

in $\Omega \times (0,T)$ with initial and boundary conditions and the nonlinear functions f_1 and f_2 satisfying appropriate conditions. They proved under some restrictions on the parameters and the initial data many results on the existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time using the same techniques as in [11].

In [20], authors considered the nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{6}$$

for $x \in \Omega$ and $t > 0$ where

$$f_1(u,v) = a\,|u+v|^{2(\rho+1)}\,(u+v) + b\,|u|^{\rho}\,u\,|v|^{(\rho+2)}\,,$$
$$f_2(u,v) = a\,|u+v|^{2(\rho+1)}\,(u+v) + b\,|u|^{(\rho+2)}\,|v|^{\rho}\,v,$$

and they prove a global nonexistence theorem for certain solutions with positive initial energy, the main tool of the proof is a method used in [28].

The non-critical case of (1) where $g_i = 0, m = 2, i = 1,2$, has been studied recently in [26]. M. A. Rammaha and Sawanya Sakuntasathien focus on the global well-posedness of the system of nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u + \left(d|u|^k + e|v|^l\right)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \left(d'\,|v|^{\theta} + e'\,|u|^{\rho}\right)|v_t|^{r-1}v_t = f_2(u,v), \end{cases} \tag{7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$, and $0 < r, m < 1$, with Dirichlet boundary conditions. The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ act as strong sources in the system. Under some restriction on the parameters in the system, they obtained several results on the existence and uniqueness of solutions. In addition, they proved that weak solutions blow up in finite time whenever the initial energy is negative and the exponent of the source term is more dominant than the exponents of both damping terms. This last result was extended by A. Benaissa et al. in [3] with positive initial energy, $r, m > 1$ and for $n > 0$.

The gender of our systems ($m_i \neq 0$, $i = 1,2$) has been proposed first by Segal [30] in the next coupled Klein-Gordon equations which is considered in the study of the quantum field theory and defines the motion of a charged meson in an electromagnetic field

$$\begin{cases} u_1'' - \Delta u_1 + m_1^2 u_1 + h_1 u_1 u_2^2 = 0, \\ u_2'' - \Delta u_2 + m_2^2 u_2 + h_2 u_1^2 u_2 = 0, \end{cases} \tag{8}$$

where m_1, m_2, h_1 and h_2 are non-negative constants.

When $g_i = 0$, $i = 1,2$, Yaojun Ye generalized the problem (8), where author studied coupled nonlinear Klein-Gordon equations with nonlinear damping and source terms, in a bounded domain with the initial and Dirichlet boundary conditions

$$\begin{cases} u_1'' - \Delta u_1 + m_1^2 u_1 + a|u_1'|^{\alpha}u_1' = b|u_1|^{\beta}u_1|u_2|^{\beta+2}, \\ u_2'' - \Delta u_2 + m_2^2 u_2 + a|u_2'|^{\alpha}u_2' = b|u_2|^{\beta}u_2|u_1|^{\beta+2}, \end{cases} \tag{9}$$

where m_1, m_2, a, b are non-negative constants, $\alpha > 0$ and $\beta \geq 0$. The existence of global solutions is discussed by using the potential well method and the asymptotic stability is also given by applying a Lemma due to V. Komornik [14].

REMARK 1.1. Noting here that our contribution is: We investigate the same system in [33] with the presence of the viscoelastic terms and potential functions, under additional condition (18), we prove that the solutions stay in the stable set (13).

2 Preliminaries

From now on, we denote by c_i, $i = 0, 1, 2, ...,$ used throughout this paper, various positive constants which may be different at different occurrences and in the sequel, for the sake of simplicity we will denote the t derivative value dv/dt by v' and d^2v/dt^2 by v''.

We assume that, for $i = 1, 2$, the relaxation functions $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and the potential $\alpha_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are nonincreasing differentiable satisfying:

$$g_i(0) > 0,\ +\infty > \int_0^{+\infty} g_i(s)ds,\ 1 - \alpha_i(t) \int_0^t g_i(s)ds \geq l_i > 0, \text{ and } \alpha_i(t) > 0. \tag{10}$$

The following notation will be used throughout this paper

$$(\Phi \circ \Psi)(t) = \int_0^t \Phi(t - \tau) \|\Psi(t) - \Psi(\tau)\|_2^2 \, d\tau. \tag{11}$$

The following technical Lemma will play an important role.

LEMMA 2.1. For any $v \in C^1(0, T, H^\kappa(\Omega))$ we have

$$\begin{aligned}
& \int_\Omega \alpha(t) \int_0^t g(t-s)\Delta^\kappa v(s)v'(t)dsdx \\
= \ & \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g \circ \nabla^\kappa v\right)(t) - \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_\Omega |\nabla^\kappa v(t)|^2 \, dxds\right] \\
& -\frac{1}{2}\alpha(t)\left(g' \circ \nabla^\kappa v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_\Omega |\nabla^\kappa v(t)|^2 \, dxds \\
& -\frac{1}{2}\alpha'(t)\left(g \circ \nabla^\kappa v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_\Omega |\nabla^\kappa v(t)|^2 \, dxds.
\end{aligned}$$

PROOF. It's not hard to see

$$\begin{aligned}
& \int_\Omega \alpha(t) \int_0^t g(t-s)\Delta^\kappa v(s)v'(t)dsdx \\
= \ & -\alpha(t)\int_0^t g(t-s)\int_\Omega \nabla^\kappa v'(t)\nabla^\kappa v(s)dxds \\
= \ & -\alpha(t)\int_0^t g(t-s)\int_\Omega \nabla^\kappa v'(t)\left[\nabla^\kappa v(s) - \nabla^\kappa v(t)\right]dxds \\
& -\alpha(t)\int_0^t g(t-s)\int_\Omega \nabla^\kappa v'(t)\nabla^\kappa v(t)dxds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_\Omega \alpha(t) \int_0^t g(t-s)\Delta^\kappa v(s) v'(t) ds dx \\
= &\frac{1}{2}\alpha(t)\int_0^t g(t-s)\frac{d}{dt}\int_\Omega \left|\nabla^\kappa v(s)-\nabla^\kappa v(t)\right|^2 dx ds \\
&-\alpha(t)\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega \left|\nabla^\kappa v(t)\right|^2 dx\right) ds,
\end{aligned}$$

which implies,

$$\begin{aligned}
&\int_\Omega \alpha(t) \int_0^t g(t-s)\Delta^\kappa v(s) v'(t) ds dx \\
= &\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(t-s)\int_\Omega \left|\nabla^\kappa v(s)-\nabla^\kappa v(t)\right|^2 dx ds\right] \\
&-\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_\Omega \left|\nabla^\kappa v(t)\right|^2 dx ds\right] \\
&-\frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_\Omega \left|\nabla^\kappa v(s)-\nabla^\kappa v(t)\right|^2 dx ds \\
&+\frac{1}{2}\alpha(t)g(t)\int_\Omega \left|\nabla^\kappa v(t)\right|^2 dx ds - \frac{1}{2}\alpha'(t)\int_0^t g(t-s)\int_\Omega \left|\nabla^\kappa v(s)-\nabla^\kappa v(t)\right|^2 dx ds \\
&+\frac{1}{2}\alpha'(t)\int_0^s g(s) ds\int_\Omega \left|\nabla^\kappa v(t)\right|^2 dx ds.
\end{aligned}$$

This completes the proof.

The energy functional $E(t)$ associated with our system is given by

$$E(t) = \frac{1}{2}\sum_{i=1}^{2}\|u_i'\|_2^2 + J(t) \tag{12}$$

where

$$\begin{aligned}
J(t) = &\frac{1}{2}\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 \\
&+\frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i) + \frac{1}{2}\sum_{i=1}^{2} m_i^2\|u_i\|_2^2 - \frac{1}{p}\|u_1u_2\|_p^p.
\end{aligned}$$

Now, we introduce the stable set as follows:

$$W = \left\{(u_1,u_2)\in (H_0^\kappa(\Omega))^2 : I(t)>0 \text{ and } J(t)<d\right\}\cup\{(0,0)\} \tag{13}$$

where

$$
\begin{aligned}
I(t) &= \sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 \\
&\quad +\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i)+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2-\|u_1u_2\|_p^p.
\end{aligned}
$$

REMARK 2.2. We notice that the mountain pass level d given in (13) defined by

$$d=\inf\left\{\sup_{(u_1,u_2)\in\left(H_0^\kappa(\Omega)\right)^2\setminus\{(0,0)\}\mu\geq 0} J\left(\mu\left(u_1,u_2\right)\right)\right\}.$$

Also, by introducing the so called "Nehari manifold"

$$\mathcal{N}=\left\{(u_1,u_2)\in\left(H_0^\kappa\left(\Omega\right)\right)^2\setminus\{(0,0)\}:I\left(t\right)=0\right\}.$$

It is readily seen that the potential depth d is also characterized by

$$d=\inf_{(u_1,u_2)\in\mathcal{N}}J(t).$$

This characterization of d shows that

$$dist\left((0,0),\mathcal{N}\right)=\min_{(u_1,u_2)\in\mathcal{N}}\|(u_1,u_2)\|_{\left(H_0^\kappa(\Omega)\right)^2}.$$

The notation $\|.\|$ stands for the norm in L^2 and we denote by $\|.\|_X$ the norm in the space X. Also, the following imbedding will be used frequently without mention $\|u\|_p\leq C\|\nabla^\kappa u\|_2$ for $u\in H_0^\kappa(\Omega)$ where

$$
\begin{cases}
2\leq p<+\infty & \text{if } n=\kappa,2\kappa,\\
2\leq p\leq\frac{2n}{n-2\kappa} & \text{if } ,n\geq 3\kappa.
\end{cases}
\tag{14}
$$

We introduce the following definition of weak solution to (1)-(4)

DEFINITION 2.3. A pair of functions (u_1,u_2) is said to be a weak solution of (1)-(4) on $[0,T]$ if $u_1,u_2\in C_w([0,T],H_0^\kappa(\Omega))$, $u_1',u_2'\in C_w([0,T],L^2(\Omega))$, $(u_{10},u_{20})\in H_0^\kappa(\Omega)\times H_0^\kappa(\Omega)$, $(u_{11},u_{21})\in L^2(\Omega)\times L^2(\Omega)$ and (u_1,u_2) satisfies

$$
\begin{aligned}
\int_0^t\int_\Omega|u_1|^{p-2}u_1|u_2|^p\phi dxds &= \int_0^t\int_\Omega u_1''\phi dxds+m_1^2\int_0^t\int_\Omega u_1\phi dxds \\
&\quad +\int_0^t\int_\Omega\nabla^\kappa u_1\nabla^\kappa\phi dxds+\int_0^t\int_\Omega|u_1'|^{r-2}u_1'\phi dxds \\
&\quad -\int_0^t\int_\Omega\alpha_1(t)\int_0^s g_1(t-\tau)\nabla^\kappa u_1(x,\tau)\nabla^\kappa\phi d\tau dxds
\end{aligned}
$$

and

$$\begin{aligned}\int_0^t\int_\Omega |u_2|^{p-2}u_2|u_1|^p\psi dxds &= \int_0^t\int_\Omega u_2''\psi dxds + m_2^2\int_0^t\int_\Omega u_2\psi dxds\\ &+\int_0^t\int_\Omega \nabla^\kappa u_2\nabla^\kappa\psi dxds + \int_0^t\int_\Omega |u_2'|^{r-2}u_2'\psi dxds\\ &-\int_0^t\int_\Omega \alpha_2(t)\int_0^s g_2(t-\tau)\nabla^\kappa u_2(x,\tau)\nabla^\kappa\psi d\tau dxds\end{aligned}$$

for all test functions $\phi,\psi\in H_0^\kappa(\Omega)\cap L^2(\Omega)$ and almost all $t\in[0,T]$ where $C_w([0,T],X)$ denotes the space of weakly continuous functions from $[0,T]$ into Banach space X.

In order to state the local existence result, we introduce the following complete metric space (the proof is similar to that in [29, 32, 31])

$$\begin{aligned}Y_T &= \{(u,v): u,v\in C\left([0,T]; H_0^\kappa(\Omega)\times H_0^\kappa(\Omega)\right),\\ &\quad u',v'\in C\left([0,T]; L^2(\Omega)\times L^2(\Omega)\right)\}\end{aligned}$$

THEOREM 2.4. Let $(u_{10},u_{20})\in(H_0^\kappa(\Omega))^2$ and $(u_{11},u_{21})\in(L^2(\Omega))^2$ for $i=1,2$ be given. Suppose that $r>2$ and p satisfies

$$\begin{cases} 1\le p<+\infty & \text{if } n=\kappa,2\kappa,\\ 1\le p\le \frac{4\kappa-n}{n-2\kappa} & \text{if } n\ge 3\kappa.\end{cases} \tag{15}$$

Then, under assumptions on two functions g_i, $i=1,2$, the problem (1)-(4) has a unique local solution $(u_1(t,x),u_2(t,x))\in Y_T$ for T small enough.

3 Global Existence Result

LEMMA 3.1. Suppose that (10) and (15) hold. Let (u_1,u_2) be the solution of the system (1)-(4). Then the energy functional is a non-increasing function, that is for all $t\ge 0$,

$$\begin{aligned}E'(t) &= \frac{1}{2}\sum_{i=1}^2\alpha_i(t)(g_i'\circ\nabla^\kappa u_i) - \frac{1}{2}\sum_{i=1}^2\alpha_i(t)g_i(t)\|\nabla^\kappa u_i\|_2^2\\ &+\frac{1}{2}\sum_{i=1}^2\alpha'(t)(g_i\circ\nabla^\kappa u_i) - \frac{1}{2}\sum_{i=1}^2\alpha'(t)\left(\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2\\ &\le \frac{1}{2}\sum_{i=1}^2\alpha'(t)(g_i\circ\nabla^\kappa u_i) - \frac{1}{2}\sum_{i=1}^2\alpha'(t)\left(\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2.\end{aligned} \tag{16}$$

We will prove the invariance of the set W. That is for some $t_0>0$ if $(u_1(t_0),u_2(t_0))\in W$, then $(u_1(t),u_2(t))\in W$ for $t\ge t_0$ and $i=1,2$. We begin with by the existence of the potential depth in the next Lemma.

LEMMA 3.2. d is a positive constant.

PROOF. We have

$$
\begin{aligned}
J(\mu(u_1,u_2)) &= \frac{\mu^2}{2}\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2+\frac{\mu^2}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i)\\
&\quad+\frac{\mu^2}{2}\sum_{i=1}^{2}m_i^2\|u_i\|_2^2-\frac{\mu^{2p}}{p}\|u_1u_2\|_p^p.
\end{aligned}
$$

Using (10) to get

$$J(\mu(u_1,u_2))\geq K(\mu),$$

where

$$K(\mu)=\frac{\mu^2}{2}\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2+\frac{\mu^2}{2}\sum_{i=1}^{2}m_i^2\|u_i\|_2^2-\frac{\mu^{2p}}{p}\|u_1u_2\|_p^p.$$

By differentiating the second term in the last equality with respect to μ, to get

$$\frac{d}{d\mu}K(\mu)=\mu\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2+\mu\sum_{i=1}^{2}m_i^2\|u_i\|_2^2-2\mu^{2p-1}\|u_1u_2\|_p^p.$$

For $\mu_1=0$ and

$$\mu_2=2^{-\frac{1}{2(p-1)}}\left(\frac{\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2}{\|u_1u_2\|_p^p}\right)^{\frac{1}{2(p-1)}},$$

then we have

$$\frac{d}{d\mu}K(\mu)=0.$$

As

$$\frac{d}{d\mu}K(\mu_2)=0,\quad K(\mu_1)=0,$$

and since

$$\frac{d^2}{d\mu^2}K(\mu)\Big|_{\mu=\mu_2}<0,$$

we see that

$$\begin{aligned}
\sup_{\mu\geq 0} j(\mu) &\geq \sup_{\mu\geq 0} K(\mu)=K(\mu_2)\\
&= 2^{\frac{-2p}{2(p-1)}}\left(\frac{\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2}{\|u_1u_2\|_p^p}\right)^{\frac{2}{2(p-1)}}\times\\
&\quad\left(\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right)\\
&\quad -\frac{1}{p}2^{\frac{-2p}{2(p-1)}}\left(\frac{\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2}{\|u_1u_2\|_p^p}\right)^{\frac{2p}{2(p-1)}}\|u_1u_2\|_p^p\\
&= 2^{\frac{-2p}{2(p-1)}}\left(\frac{p-1}{p}\right)\left(\frac{\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2}{\|u_1u_2\|_p}\right)^{\frac{2p}{2(p-1)}}.
\end{aligned}$$

It follows from the Holder inequality for some $C>0$ and assumptions (10)

$$\begin{aligned}
\|u_1u_2\|_p &\leq \|u_1\|_{2p}\|u_2\|_{2p}\leq C^2\|\nabla^\kappa u_1\|_2\|\nabla^\kappa u_2\|_2\\
&\leq \frac{1}{2}C^2\left(\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2\right)\leq \frac{1}{2}C^2\left(\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right),
\end{aligned}$$

which implies that

$$\frac{\|u_1u_2\|_p}{\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2}\leq \frac{1}{2}C^2.$$

Since $p>1$, we obatin that

$$\begin{aligned}
\sup_{\mu\geq 0} j(\mu) &\geq 2^{\frac{-2p}{2(p-1)}}\left(\frac{p-1}{p}\right)\left[\frac{\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2+\sum_{i=1}^{2} m_i^2\|u_i\|_2^2}{\|u_1u_2\|_p}\right]^{\frac{2p}{2(p-1)}}\\
&\geq \frac{(p-1)}{p}C^{\frac{-2p}{(p-1)}}=d>0.
\end{aligned}$$

Then, by the definition of d, we conclude that $d>0$ for $p>1$.

LEMMA 3.3. W is a bounded neighborhood of 0 in $H_0^\kappa(\Omega)$.

PROOF. For $u \in W$, and $u \neq 0$, we have

$$\begin{aligned}
J(t) &= \frac{1}{2}\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i \circ \nabla^\kappa u_i) \\
&\quad + \frac{1}{2}\sum_{i=1}^{2} m_i^2\|u_i\|_2^2 - \frac{1}{p}\|u_1u_2\|_p^p \\
&= \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2\right. \\
&\quad \left. + \sum_{i=1}^{2}\alpha_i(t)(g_i \circ \nabla^\kappa u_i) + \sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right] + \frac{1}{p}I(t) \\
&\geq \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2\right. \\
&\quad \left. + \sum_{i=1}^{2}\alpha_i(t)(g_i \circ \nabla^\kappa u_i) + \sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right]. \qquad (17)
\end{aligned}$$

By using (10), (17) becomes

$$\begin{aligned}
J(t) &\geq \left(\frac{p-2}{2p}\right)\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 \\
&\geq \left(\frac{p-2}{2p}\right)\sum_{i=1}^{2} l_i\|\nabla^\kappa u_i\|_2^2 \\
&\geq \left(\frac{p-2}{2p}\right)\min(l_1,l_2)\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2.
\end{aligned}$$

It follows that

$$\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2 \leq \frac{1}{\min(l_1,l_2)}\left(\frac{2p}{p-2}\right)J(t) < \frac{1}{\min(l_1,l_2)}\left(\frac{2p}{p-2}\right)d = R.$$

Consequently, $\forall (u_1,u_2) \in W$, we have $(u_1,u_2) \in B$, where

$$B = \left\{(u_1,u_2) \in (H_0^\kappa(\Omega))^2 : \sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2 < R\right\}.$$

This completes the proof.

In the following Lemma, we will see that if the initial data (or for some $t_0 > 0$) is in the set W, then the solution stays there forever.

LEMMA 3.4. Suppose that (10), (15) and

$$\left(\frac{C^2}{(2\min(l_1,l_2))}\right)^p \left(\frac{2pE(0)}{p-2}\right)^{(p-1)} < 1. \tag{18}$$

hold, where C is the best Poincare's constant. If $(u_{10},u_{20}) \in W$ and $(u_{11},u_{21}) \in \left(L^2(\Omega)\right)^2$, then the solution $(u_1(t),u_2(t)) \in W$ for $t \geq 0$.

PROOF. Since $(u_{10},u_{20}) \in W$, we see that

$$I(t) = \sum_{i=1}^{2} \|\nabla^\kappa u_{i0}\|_2^2 + \sum_{i=1}^{2} m_i^2 \|u_{i0}\|_2^2 - \|u_{10}u_{20}\|_p^p > 0.$$

Consequently, by continuity, there exists $T_m \leq T$ such that

$$\begin{aligned} I(u(t)) &= \sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i) \\ &\quad + \sum_{i=1}^{2} m_i^2\|u_i\|_2^2 - \|u_1u_2\|_p^p \geq 0 \text{ for } t\in[0,T_m]. \end{aligned}$$

This gives

$$\begin{aligned} J(t) &= \frac{1}{2}\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i) \qquad (19) \\ &\quad +\frac{1}{2}\sum_{i=1}^{2} m_i^2\|u_i\|_2^2 - \frac{1}{p}\|u_1u_2\|_p^p \\ &= \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i)\right. \\ &\quad \left. + \sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right] + \frac{1}{p}I(u(t)) \\ &\geq \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i)\right. \\ &\quad \left. + \sum_{i=1}^{2} m_i^2\|u_i\|_2^2\right]. \end{aligned}$$

By using (10) and the fact that $\int_0^t g_i(s)ds \leq \int_0^\infty g_i(s)ds$, we easily see that, for $t \in [0,T_m]$,

$$\begin{aligned} \sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2 &\leq \frac{1}{\min(l_1,l_2)}\left(\frac{2p}{p-2}\right)J(t) \leq \frac{1}{\min(l_1,l_2)}\left(\frac{2p}{p-2}\right)E(t) \\ &\leq \frac{1}{\min(l_1,l_2)}\left(\frac{2p}{p-2}\right)E(0). \end{aligned}$$

We then exploit (10), (15) and from the Holder inequality for some $C > 0$. So we have

$$\|u_1u_2\|_p \leq \|u_1\|_{2p}\|u_2\|_{2p} \leq C^2\|\nabla^\kappa u_1\|_2\|\nabla^\kappa u_2\|_2 \leq \frac{1}{2}C^2\left(\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2\right).$$

for $C = C(n, p, \Omega)$.

Consequently, we have

$$\begin{aligned}
\|u_1u_2\|_p^p &\leq 2^{-p}C^{2p}\left(\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2\right)^p \\
&\leq 2^{-p}C^{2p}\left(\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2\right)^{p-1}\left(\sum_{i=1}^{2}\|\nabla^\kappa u_i\|_2^2\right) \\
&\leq C^{2p}\left(2\min(l_1,l_2)\right)^{-p}\left(\frac{2p}{p-2}\right)^{(p-1)}E(0)^{(p-1)}\left(\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2\right) \\
&\leq \beta\left(\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2\right),
\end{aligned}$$

where

$$\beta = C^{2p}\left(2\min(l_1,l_2)\right)^{-p}\left(\frac{2p}{p-2}\right)^{(p-1)}E(0)^{(p-1)}.$$

Which means, by the definition of $l_i, i = 1, 2$,

$$\begin{aligned}
\|u_1u_2\|_p^p &\leq \beta\left(\sum_{i=1}^{2}l_i\|\nabla^\kappa u_i\|_2^2\right) \\
&\leq \sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 \\
&\leq \sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^\kappa u_i\|_2^2 + \sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^\kappa u_i) + \sum_{i=1}^{2}m_i^2\|u_i\|_2^2.
\end{aligned}$$

Therefore, $I(t) > 0$ for all $t \in [0, T_m]$, by taking the fact that

$$\lim_{t\mapsto T_m} C^{2p}\left(2\min(l_1,l_2)\right)^{-p}\left(\frac{2p}{p-2}\right)^{(p-1)}E(0)^{(p-1)} \leq \beta < 1.$$

This shows that the solution $(u_1(t), u_2(t)) \in W$, for all $t \in [0, T_m]$. By repeating this procedure T_m extends to T.

THEOREM 3.5. Suppose that (10), (15) and (18) hold. If $(u_{10}, u_{20}) \in W$, $(u_{11}, u_{21}) \in \left(L^2(\Omega)\right)^2$. Then the local solution (u_1, u_2) is global in time such that $(u_1, u_2) \in G_T$ where

$$\begin{aligned}
G_T = &\left\{(u,v) : u, v \in L^\infty\left(\mathbb{R}^+; H_0^\kappa(\Omega)\times H_0^\kappa(\Omega)\right)\right. \\
&\left.\text{and } u', v' \in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\times L^2(\Omega)\right)\right\}.
\end{aligned}$$

PROOF. In order to prove Theorem 3.5, it suffices to show that the following norm

$$\sum_{i=1}^{2}\|u_i'\|_2^2+\sum_{i=1}^{2}\|\nabla^{\kappa}u_i\|_2^2+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2$$

is bounded independently of t. To achieve this, we use (12), (16) and (19) to get

$$\begin{aligned}
E(0) \quad &\geq \quad E(t)=J(t)+\frac{1}{2}\sum_{i=1}^{2}\|u_i'(t)\|_2^2\\
&\geq \quad \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla^{\kappa}u_i\|_2^2+\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^{\kappa}u_i)\right.\\
&\qquad \left.+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2\right]+\frac{1}{2}\sum_{i=1}^{2}\|u_i'(t)\|_2^2+\frac{1}{p}I(t)\\
&\geq \quad \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}l_i\|\nabla^{\kappa}u_i\|_2^2+\sum_{i=1}^{2}\alpha_i(t)(g_i\circ\nabla^{\kappa}u_i)+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2\right]\\
&\qquad +\frac{1}{2}\sum_{i=1}^{2}\|u_i'(t)\|_2^2+\frac{1}{p}I(t)\\
&\geq \quad \left(\frac{p-2}{2p}\right)\left[\sum_{i=1}^{2}l_i\|\nabla^{\kappa}u_i\|_2^2+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2\right]+\frac{1}{2}\sum_{i=1}^{2}\|u_i'(t)\|_2^2.
\end{aligned}$$

Since $I(t)$ and $\alpha_i(t)(g\circ\nabla u)(t)$ are positive, hence

$$\sum_{i=1}^{2}\|u_i'(t)\|_2^2+\sum_{i=1}^{2}\|\nabla^{\kappa}u_i\|_2^2+\sum_{i=1}^{2}m_i^2\|u_i\|_2^2\leq CE(0),$$

where C is a positive constant depending only on p and l_i.

This completes the proof.

Open problem Let us mention here that, it will be interesting to discuss the asymptotic stability of this problem where also, one can establish a general decay rate estimate for the energy, which will depend on the behavior of both α and g under following assumption

There exists a non-increasing differentiable function $\xi_i, i=1,2:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ satisfying $\xi_i(t)>0, g_i'(t)\leq\xi_i(t)g_i(t), \forall t\geq 0$, and perhaps other conditions imposed by the nature of our system.

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