Open Neighborhood Chromatic Number Of An Antiprism Graph

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Abstract

An open neighborhood $k$-coloring of a simple connected undirected graph $G(V,E)$ is a $k$-coloring $c : V \rightarrow \{1, 2, \ldots, k\}$, such that, for every $w \in V$ and for all $u, v \in N(w)$, $c(u) \neq c(v)$. The minimum value of $k$ for which $G$ admits an open neighborhood $k$-coloring is called the open neighborhood chromatic number of $G$ denoted by $\chi_{onc}(G)$. In this paper, we obtain the open neighborhood chromatic number of the Petersen graph. Also, we determine this number for a family of graphs called antiprism graphs.

1 Introduction

All the graphs considered in this paper are simple, non-trivial, undirected, finite and connected. For standard terminologies, we refer [2] and [7]. A vertex coloring, or simply a coloring, of a graph $G = (V,E)$ is an assignment of colors to the vertices of $G$. A $k$-coloring of $G$ is a surjection $c : V \rightarrow \{1, 2, \ldots, k\}$. A proper coloring of $G$ is an assignment of colors to the vertices of $G$ so that adjacent vertices are colored differently. A proper $k$-coloring of $G$ is a surjection $c : V \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ if $u$ and $v$ are adjacent in $G$. The minimum $k$ for which there is a proper $k$-coloring of $G$ is called the chromatic number of $G$ denoted by $\chi(G)$.

As seen in Fig. 3, the Petersen graph [10] is an undirected graph with 10 vertices and 15 edges and serves as a useful example and counterexample for many problems in graph theory. It is a cubic symmetric graph and is non-planar. The chromatic number and the domination number of the Petersen graph are both equal to 3. The generalized Petersen graph $GP(n,k)$, $n \geq 3$ and $k < n/2$, is a graph consisting of an inner star polygon $\{n,k\}$ and an outer regular polygon $C_n$ with corresponding vertices in the inner and outer polygons connected with edges. The Petersen graph can be obtained from this graph by choosing $n = 5$ and $k = 2$.

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The graph obtained by replacing the faces of a polyhedron with its edges and vertices is called the skeleton [3] of the polyhedron. For example, the polyhedral graphs corresponding to the skeletons of tetrahedron and octahedron are illustrated in Fig. 1 and 2.

An \( n \)-antiprism [4], \( n \geq 3 \), is a semiregular polyhedron constructed with \( 2n \)-gons and \( 2n \) triangles. It is made up of two \( n \)-gons on top and bottom, separated by a ribbon of \( 2n \) triangles, with the two \( n \)-gons being offset by one ribbon segment. The graph corresponding to the skeleton of an \( n \)-antiprism is called the \( n \)-antiprism graph, denoted by \( Q_n \), \( n \geq 3 \) as shown in Fig. 4. As seen from the figure, \( Q_n \) has \( 2n \) vertices and \( 4n \) edges, and is isomorphic to the circulant graph \( C_{2n}(1,2) \). In particular, the 3-antiprism graph \( Q_3 \) is isomorphic to the octahedral graph in Fig. 2.

An open neighborhood coloring [5] of a graph \( G(V, E) \) is a coloring \( c : V \to \mathbb{Z}^+ \), such that for each \( w \in V \) and \( \forall u, v \in N(w) \), \( c(u) \neq c(v) \). An open neighborhood \( k \)-coloring of a graph \( G(V, E) \) is a \( k \)-coloring \( c : V \to \{1, 2, \ldots, k\} \) which admits the conditions of an open neighborhood coloring. The minimum value of \( k \) for which \( G \) admits an open neighborhood \( k \)-coloring is called the open neighborhood chromatic number of \( G \) denoted by \( \chi_{onc}(G) \).

In [5], we have established some bounds on the open neighborhood chromatic number of a graph. We have also obtained this parameter for an infinite triangular lattice. Further, in [6], we have determined the open neighborhood chromatic number of prism graph which is obtained from the generalized Petersen graph \( GP(n, k) \) by choosing \( k = 1 \) and \( n \geq 3 \).
We recall some of the definitions and results on the open neighborhood chromatic number discussed in [5].

THEOREM 1.1. If $f$ is an open neighborhood $k$-coloring of $G(V, E)$ with $\chi_{onc}(G) = k$, then $f(u) \neq f(v)$ holds where $u, v$ are the end vertices of a path of length 2 in $G$.

THEOREM 1.2. For any graph $G(V, E)$, $\chi_{onc}(G) \geq \Delta(G)$.

THEOREM 1.3. If $H$ is a connected subgraph of $G$, then $\chi_{onc}(H) \leq \chi_{onc}(G)$.

THEOREM 1.4. The open neighborhood chromatic number of a connected graph $G$ is 1 if and only if $G \cong K_1$ or $K_2$.

THEOREM 1.5. Let $G(V, E)$ be a connected graph on $n \geq 3$ vertices. Then $\chi_{onc}(G) = n$ if and only if $N(u) \cap N(v) \neq \emptyset$ holds for every pair of vertices $u, v \in V(G)$.

THEOREM 1.6. For a path $P_n$, $n \geq 2$,

$$\chi_{onc}(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3. \end{cases}$$

THEOREM 1.7. For a cycle $C_n$, $n \geq 3$,

$$\chi_{onc}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4}, \\ 3 & \text{otherwise.} \end{cases}$$

DEFINITION 1.8. In a graph $G$, a subset $V_1$ of $V(G)$ such that no two vertices of $V_1$ are end vertices of a path of length two in $G$ is called a $P_3$-independent set of $G$.

In this paper, we obtain the open neighborhood chromatic number of the Petersen graph. Also we determine this number for the $n$-antiprism graph $Q_n$.

2 Open Neighborhood Chromatic Number of Petersen Graph

OBSERVATION 2.1. For any graph $G$ of order $n$, if $\chi_{onc}(G) = n$, then $\text{diam}(G) \leq 2$.

PROOF. Consider a graph $G$ with $V(G) = \{v_1, v_2, \cdots, v_n\}$ with $\chi_{onc}(G) = n$. Suppose $\text{diam}(G) \geq 3$. Without loss in generality, let $d(v_1, v_2) \geq 3$. We define a coloring $c : V(G) \to \{1, 2, \cdots, n - 1\}$ as follows.

$$c(v_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = 2, \\ i - 1 & \text{otherwise.} \end{cases}$$
Clearly, $c$ is an open neighborhood $(n - 1)$-coloring of $G$ so that $\chi_{one}(G) \leq n - 1$, a contradiction.

**THEOREM 2.2.** If $G$ is any graph of order $n$, $\chi_{one}(G) = 2$ if and only if $G \cong P_n$, $n \geq 3$ or $G \cong C_n$, $n \equiv 0 \pmod{4}$.

**PROOF.** Consider a graph $G$ of order $n$. Suppose $\chi_{one}(G) = 2$. By Theorem 1.2, we have $\chi_{one}(G) \geq \Delta(G)$ so that $\Delta(G) \leq 2$. Thus, $G$ is either a path or a cycle. However, by Theorem 1.7, we know that $\chi_{one}(C_n) = 2$ only when $n \equiv 0 \pmod{4}$. Also by Theorem 1.6, $\chi_{one}(P_n) = 2$ for any $n \geq 3$. Thus, if $\chi_{one}(G) = 2$, then $G \cong P_n$, $n \geq 3$ or $G \cong C_n$, $n \equiv 0 \pmod{4}$. The converse is a direct consequence of Theorem 1.6 and Theorem 1.7.

**THEOREM 2.3.** The open neighborhood chromatic number of the Petersen graph $GP(5, 2)$ is 5.

**PROOF.** Let $u$ be any vertex of $G = GP(n, 2)$. Then in any open neighborhood coloring $c$, $c(u) \neq c(v)$ for any $v \notin N(u)$ as every such vertex is connected by a path of length two from $u$. Further at most one vertex in $N(u)$ can be given the same color as that of $u$ since there is a path of length two between every $v, w \in N(u)$. Thus, one color can be given to at most two vertices in any open neighborhood coloring $c$ of $G$ so that $\chi_{one}(G) \geq 5$. To prove the reverse inequality, consider a coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ as

$$c(v) = \begin{cases} 
1, & \text{if } v = v_0 \text{ or } v = v_4, \\
2, & \text{if } v = v_1 \text{ or } v = v_2, \\
3, & \text{if } v = u_3 \text{ or } v = v_3, \\
4, & \text{if } v = u_0 \text{ or } v = u_2, \\
5, & \text{otherwise}.
\end{cases}$$

It is easy to verify that $c$ is an open neighborhood 5-coloring of $G$ so that $\chi_{one}(G) \leq 5$. Hence, $\chi_{one}(G) = 5$.

### 3 Open Neighborhood Chromatic Number of an Antiprism Graph

In this section, we determine the open neighborhood chromatic number of an $n$-antiprism graph $Q_n$.

**OBSERVATION 3.1.** Every integer $n \geq 8$ with $n \not\equiv 0 \pmod{5}$ can be expressed as $n = 3k + 5m$ for some integers $m \geq 0$ and $k \geq 1$.

**LEMMA 3.2.** For any integer $n \geq 3$, $\chi_{one}(Q_n) \geq 5$. 

PROOF. For each $n \geq 3$, $Q_n$ contains a subgraph $H$ as in Fig. 5. Further, in $H$, there is a path of length two between every pair of vertices so that $\chi_{\text{onc}}(H) = 5$. Hence by Theorem 1.3, $\chi_{\text{onc}}(Q_n) \geq 5$.

Figure 5: A subgraph of $Q_n$

**Observation 3.3.** In the antiprism graph $Q_n$, the only vertices that are connected to a vertex $u_i, 0 \leq i \leq n-1$ by a path of length two are $u_{i\pm 1}, u_{i\pm 2}, v_i, v_{i\pm 1}, v_{i+2}$ where the suffix is under modulo $n$. Similarly, the vertices that are connected to a vertex $v_i, 0 \leq i \leq n-1$ by a path of length two are $u_i, u_{i\pm 1}, u_{i-2}, v_{i\pm 1}, v_{i\pm 2}$ where the suffix is under modulo $n$.

**Lemma 3.4.** Let $Q_n$ be an antiprism graph and let

$$S_k = \{u_i, v_j \mid i \equiv (k+2)(\text{mod } 5) \text{ and } j \equiv k(\text{mod } 5)\} \text{ for } 0 \leq k \leq 4.$$  \hspace{1cm} (1)

Then each $S_k$ is a $P_3$-independent set if and only if $n \equiv 0(\text{mod } 5)$.

**Proof.** Let $n \equiv 0(\text{mod } 5)$. We see that $i \equiv (k+2)(\text{mod } 5)$. It follows that $i+1 \equiv (k+3)(\text{mod } 5), i-1 \equiv (k+1)(\text{mod } 5), i+2 \equiv (k+4)(\text{mod } 5)$ and $i-2 \equiv k(\text{mod } 5)$ so that $u_{i\pm 1}, u_{i\pm 2}, v_i, v_{i\pm 1}, v_{i+2} \notin S_k$. Also, $j \equiv k(\text{mod } 5)$ implies $j+1 \equiv (k+1)(\text{mod } 5), j-1 \equiv (k+4)(\text{mod } 5), j+2 \equiv (k+2)(\text{mod } 5)$ and $j-2 \equiv (k+3)(\text{mod } 5)$ so that $u_j, u_{j\pm 1}, u_{j-2}, v_{j\pm 1}, v_{j+2} \notin S_k$. Hence, by Observation 3.3, $S_k$ is a $P_3$-independent set for $0 \leq k \leq 4$. Next, we assume that $S_k$ is a $P_3$-independent set and we prove the converse by contraposition.

**Case 1.** Suppose $n \equiv 1(\text{mod } 5)$. Then $v_0, v_{n-1} \in S_0$. But $v_0$ and $v_{n-1}$ are end vertices of a path of length 2 so that $S_0$ is not a $P_3$-independent set.

**Case 2.** Suppose $n \equiv 2(\text{mod } 5)$. Then $u_0, u_2 \in S_0$. But $u_0$ and $u_2$ are end vertices of a path of length 2 so that $S_0$ is not a $P_3$-independent set.

**Case 3.** Suppose $n \equiv 3(\text{mod } 5)$. Then $u_{n-1}, v_0 \in S_0$. But $u_{n-1}$ and $v_0$ are end vertices of a path of length 2 so that $S_0$ is not a $P_3$-independent set.

**Case 4.** Suppose $n \equiv 4(\text{mod } 5)$. Then $u_{n-2}, v_0 \in S_0$. But $u_{n-2}$ and $v_0$ are end vertices of a path of length 2 so that $S_0$ is not a $P_3$-independent set.

So by Cases 1–4, we obtain $n \equiv 0(\text{mod } 5)$. Therefore, the proof of Lemma 3.4 is complete.

**Lemma 3.5.** For any positive integer $n \geq 5$, $\chi_{\text{onc}}(Q_n) = 5$ if and only if $n \equiv 0(\text{mod } 5)$. 

**PROOF.** Consider an \( n \)-antiprism graph \( Q_n \) as in Fig. 4 such that \( n \equiv 0(\text{mod} \ 5) \).

By Lemma 1.1, \( \chi_{\text{one}}(Q_n) \geq 5 \). We recall the set \( S_k \) defined by (1). By Lemma 3.4, each \( S_k \) is a \( P_3 \)-independent set which implies that every vertex in any \( S_k \) can be given the same color in any open neighborhood coloring of \( G \). Thus, the coloring \( c: V(Q_n) \to \{1, 2, 3, 4, 5\} \) defined by \( c(v) = k + 1 \) if \( v \in S_k \) for \( 0 \leq k \leq 4 \) is an open neighborhood 5-coloring of \( Q_n \) so that \( \chi_{\text{one}}(Q_n) = 5 \).

We prove the converse by the method of contradiction. Let \( \chi_{\text{one}}(Q_n) = 5 \). Suppose \( n \not\equiv 0(\text{mod} \ 5) \). By Observation 3.3, each of the vertices \( v_0, v_1, v_2, u_0 \) and \( u_1 \) should be in different \( P_3 \)-independent sets. Let \( S_0, S_1, S_2, S_3 \) and \( S_4 \) be mutually disjoint \( P_3 \)-independent sets with \( v_0 \in S_0, v_1 \in S_1, v_2 \in S_2, u_0 \in S_3 \) and \( u_1 \in S_4 \). Now, \( v_3 \) cannot belong to any of the sets \( S_1, S_2, S_3 \) or \( S_4 \). However, it may be in \( S_0, S_3 \) or neither. Also, \( u_2 \) cannot belong to any of the sets \( S_1, S_2, S_3 \) or \( S_4 \). Based on this, we consider the following Cases 1–3.

**Case 1.** Suppose \( v_3 \in S_0 \). Then \( u_2 \) cannot be in \( S_k \) for any \( 0 \leq k \leq 4 \) which means that \( u_2 \in S \), a \( P_3 \)-independent set different from the sets \( S_0, S_1, S_2, S_3 \) and \( S_4 \). Thus, at least six colors are needed to have an open neighborhood coloring of \( Q_n \).

**Case 2.** Suppose \( v_3 \in S_3 \). Then \( u_2 \) may or may not be in \( S_0 \).

**Subcase 2-1.** Assume that \( u_2 \notin S_0 \). Then, \( u_2 \) is not in any of the sets \( S_k, 0 \leq k \leq 4 \). Thus as in Case 1, at least six colors are needed to have an open neighborhood coloring of \( Q_n \).

**Subcase 2-2.** Assume that \( u_2 \in S_0 \). Then, we see that \( v_3 \in S_3, u_3 \in S_1 \) and so on. However, proceeding further in this manner, we get \( v \in S_0 \) with \( v \) being one of \( v_{n-1}, v_{n-2}, u_{n-1} \) or \( u_{n-2} \) according as \( n \equiv 1(\text{mod} \ 5), n \equiv 2(\text{mod} \ 5), n \equiv 3(\text{mod} \ 5) \) or \( n \equiv 4(\text{mod} \ 5) \). In such a case, \( S_0 \) does not remain a \( P_3 \)-independent set. To avoid this, we need to have \( v \in S \), a \( P_3 \)-independent set different from \( S_0, S_1, S_2, S_3 \) and \( S_4 \) so that at least six colors are needed to have an open neighborhood coloring of \( Q_n \).

**Case 3.** Suppose \( v_3 \notin S_0 \) or \( S_3 \). Then, as in Case 1, at least six colors are needed to have an open neighborhood coloring of \( Q_n \).

THEOREM 3.6. Let \( Q_n \) be an antiprism graph. Then

\[
\chi_{\text{one}}(Q_n) = \begin{cases} 
5 & \text{if } n \equiv 0(\text{mod} \ 5), \\
7 & \text{if } n = 7, \\
8 & \text{if } n = 4, \\
6 & \text{otherwise.}
\end{cases}
\]

**PROOF.** We prove the theorem by taking cases for various values of \( n \).

**Case 1.** Suppose \( n = 4 \). Then we have the 4-antiprism graph \( Q_4 \) as in Fig. 6. Since each vertex is connected to every other vertex by a path of length 2, each vertex is to be colored by a different color in any open neighborhood coloring of \( Q_4 \) so that \( \chi_{\text{one}}(Q_4) = 8 \).

**Case 2.** Suppose \( n \geq 5 \) with \( n \equiv 0(\text{mod} \ 5) \). Then, by Lemma 3.5, \( \chi_{\text{one}}(Q_n) = 5 \).

**Case 3.** Suppose \( n = 7 \), then we have the 7-antiprism graph \( Q_7 \) as in Fig. 7. As seen from the figure, in any open neighborhood coloring \( c \), \( c(v_0) \neq c(w) \) for any \( w \) with
$w = u_i, \ i = 0, 1, 5, 6 \text{ or } w = v_j \text{ with } j = 1, 2, 5, 6.$ Further, at most one of the vertices $u_2, u_3, u_4, v_3, v_4$ can be given the same color as that of $v_0.$ Thus, in general, not more than two vertices in $Q_7$ can be given the same color in any open neighborhood coloring of $Q_7.$ This implies that $\chi_{onc}(Q_7) \geq 7.$ To prove the reverse inequality, consider a coloring $c : V(Q_7) \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows.

$$c(v) = \begin{cases} 
1 & \text{if } v = v_0 \text{ or } v = u_3, \\
2 & \text{if } v = v_1 \text{ or } v = u_4, \\
3 & \text{if } v = v_2 \text{ or } v = u_5, \\
4 & \text{if } v = v_3 \text{ or } v = u_6, \\
5 & \text{if } v = v_4 \text{ or } v = u_0, \\
6 & \text{if } v = v_5 \text{ or } v = u_1, \\
7 & \text{otherwise.}
\end{cases}$$

It is easy to verify that $c$ is an open neighborhood 7-coloring of $Q_7$ so that $\chi_{onc}(Q_7) \leq 7.$ Hence, $\chi_{onc}(Q_7) = 7.$

**Case 4.** Suppose $n$ is any other integer, then we take up two subcases as follows.

**Subcase 4-1.** Suppose $n = 3,$ we have the 3-antiprism graph $Q_3$ as in Fig. 2. Since each vertex is connected to every other vertex by a path of length 2, each vertex is to be colored by a different color in any open neighborhood coloring of $Q_3$ so that $\chi_{onc}(Q_3) = 6.$

**Subcase 4-2.** Suppose $n \geq 8.$ Since $n \not\equiv 0(\text{mod} \ 5),$ by Observation 3.1, $n = 3k + 5m$ for some integers $m \geq 0$ and $k \geq 1.$ Also, $\chi_{onc}(Q_n) \geq 6$ by Lemma 1.1 and Lemma 3.5.
To prove the reverse inequality, consider a coloring $c : V(Q_n) \to \{1, 2, 3, 4, 5, 6\}$ as

$$c(v_i) = \begin{cases} 
1, & \text{if } i \equiv 0 \pmod{3} \text{ and } 0 \leq i \leq 3k - 1, \text{ or } i - 3k \equiv (\text{mod } 5) \text{ and } 0 \leq 3k \leq 5m - 1 \\
2, & \text{if } i \equiv 1 \pmod{3} \text{ and } 0 \leq i \leq 3k - 1, \text{ or } i - 3k \equiv 1(\text{mod } 5) \text{ and } 0 \leq 3k \leq 5m - 1 \\
3, & \text{if } i \equiv 2 \pmod{3} \text{ and } 0 \leq i \leq 3k - 1, \text{ or } i - 3k \equiv 2(\text{mod } 5) \text{ and } 0 \leq 3k \leq 5m - 1 \\
4, & \text{if } i - 3k \equiv 3(\text{mod } 5) \text{ and } 0 \leq 3k \leq 5m - 1 \\
5, & \text{otherwise.}
\end{cases}$$

and

$$c(u_i) = \begin{cases} 
1, & \text{if } i - 3k \equiv 2 \pmod{5} \text{ and } 0 \leq 3k \leq 5m - 1 \\
2, & \text{if } i - 3k \equiv 3 \pmod{5} \text{ and } 0 \leq 3k \leq 5m - 1 \\
3, & \text{if } i - 3k \equiv 4 \pmod{5} \text{ and } 0 \leq 3k \leq 5m - 1 \\
4, & \text{if } i \equiv 0 \pmod{3} \text{ and } 0 \leq i \leq 3k - 1, \text{ or } i - 3k \equiv 0 \pmod{5} \text{ and } 0 \leq 3k \leq 5m - 1 \\
5, & \text{if } i \equiv 1 \pmod{3} \text{ and } 0 \leq i \leq 3k - 1, \text{ or } i - 3k \equiv 1 \pmod{5} \text{ and } 0 \leq 3k \leq 5m - 1 \\
6, & \text{otherwise.}
\end{cases}$$

It can be easily seen that $c$ is an open neighborhood coloring of $Q_n$ so that $\chi_{onc}(Q_n) \leq 6$. Hence $\chi_{onc}(Q_n) = 6$.

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