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Some Fixed Point Theorems For Mappings Involving Rational Type Expressions In Partial Metric Spaces*

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Abstract

The aim of this paper is to establish some fixed point theorems for mappings involving rational expressions in a complete partial metric space using a class of pairs of functions satisfying certain assumptions. Our result extends and generalizes some well known results of [6, 7, 9] in (usual) metric spaces.

1 Introduction

It is well known that, there are lots of literature dealing with fixed points and common fixed points for Banach contraction principle in different spaces. One of the most interesting space is Partial metric space introduced by Matthews ([16, 17]) in 1994. In fact, a partial metric space is a generalization of usual metric spaces in which the self distance for any point need not be equal to zero. The partial metric space has wide applications in many branches of mathematics as well as in the field of computer domain and semantics. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see, e.g. [1-3, 11-17, 22, 24]).

On the other hand, Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambient spaces. Das and Gupta [4] were the pioneers in proving fixed point theorems using contractive conditions involving rational expressions. They prove the following fixed point theorem in the setting of complete metric space.

THEOREM 1 [6]. Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx, Ty) \le \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \tag{1}$$

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for all $x, y \in X$. Then T has a unique fixed point.

Recently, one of the most important ingredients of a contractivity condition is to study the kind of involved functions, like *altering distance functions* introduced by Khan et al. [17] as follows.

DEFINITION 1 [17]. An *altering distance functions* is a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if and only if $t = 0$.

In [5], Berzig et al. introduced the notion of *pair of generalized altering distance functions* as follows.

DEFINITION 2 [5]. The pair (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$, is called a *pair of generalized altering distance functions* if the following conditions are satisfied:

(a1) φ is continuous;

(a2) φ is nondecreasing;

(a3) $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

The condition (a3) was introduced by Popescu in [20] and Moradi and Farajzadeh in [18]. Notice that the above conditions do not determine the values of $\varphi(0)$ and $\phi(0)$.

In the recent work, Agarwal et al. [4] introduced the following family of function.

DEFINITION 3 [4]. We will denote by $\mathcal{F}$ the family of all pairs (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$ are functions satisfying the following three conditions.

($\mathcal{F}1$) φ is nondecreasing;

($\mathcal{F}2$) if there exists $t_0 \in [0, \infty)$ such that $\phi(t_0) = 0$, then $t_0 = 0$ and $\varphi^{-1}(0) = \{0\}$;

($\mathcal{F}3$) if $\{a_k\}$, $\{b_k\} \subset [0, \infty)$ are sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\varphi(b_k) \le (\varphi - \phi)(a_k)$ for all k, then $L = 0$.

Recently, Karapinar et al. [12] studied the existence and uniqueness of a fixed point for mappings satisfying rational type contractive condition using auxiliary functions. The main purpose of this paper is to present some fixed point theorems for contractions of rational type by using a class of pairs of functions satisfying certain assumptions in the setting of partial metric spaces.

2 Preliminaries

The definition of partial metric space is given by Matthews [19, 20] as follows:

DEFINITION 4. Let X be a nonempty set and let $p : X \times X \to [0, \infty)$ satisfies

(1) $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$;

(2) $p(x, x) \leq p(x, y)$;

(3) $p(x, y) = p(y, x)$;

(4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$

for all $x, y, z \in X$. Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.

It is clear that if $p(x, y) = 0$, then $x = y$. But if $x = y$ then $p(x, y)$ may need not be zero. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$ where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-balls is defined as

$$B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}.$$

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ is given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{2}$$

is a (usual) metric on X.

EXAMPLE 1. Let $X = \mathbb{R}$ and $p(x, y) = e^{\max\{x, y\}}$ for all $x, y \in X$. Then (X, p) is a partial metric space.

DEFINITION 5 [18, 19].

(1) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$, if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(2) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exist and finite.

(3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\} \in X$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

The following lemmas of [12] and [17] will be used in the sequel.

LEMMA 1. The following statements hold.

(1) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, d_p).

(2) A partial metric space (X, p) is said to be complete if a metric space (X, d_p) is complete, i.e.

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

LEMMA 2. Let (X, p) be a partial metric space.

(1) If $p(x, y) = 0$ then $x = y$.

(2) If $x \neq y$ then $p(x, y) > 0$.

LEMMA 3. Let $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p), where $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

3 Main Results

We start this section presenting the following class of pairs of functions F. A pair of functions (φ, ϕ) is said to belong to the class F, if they satisfy the following conditions:

(i) $\varphi, \phi : [0, \infty) \to [0, \infty)$;

(ii) for $t, s \in [0, \infty)$, if $\varphi(t) \leq \phi(s)$, then $t \leq s$;

(iii) for $\{t_n\}$ and $\{s_n\}$ sequences in $[0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$, if $\varphi(t_n) \leq \phi(s_n)$ for any $n \in \mathbb{N}$, then $a = 0$.

Notice that, if a pair φ, ϕ verifies $(\mathcal{F}1)$ and $(\mathcal{F}2)$, then the pair $(\varphi, \phi = \varphi - \phi)$ satisfies (i) and (ii). Furthermore, if $(\varphi, \phi = \varphi - \phi)$ satisfies (iii), then (φ, ϕ) satisfies $(\mathcal{F}3)$.

EXAMPLE 2. The conditions (i)–(iii) of the above definition are fulfilled for the functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ defined by

$$\varphi(t) = \ln \frac{1 + 2t}{2} \text{ and } \phi(t) = \ln \frac{1 + t}{2} \text{ for all } t \in [0, \infty).$$

REMARK 1. Note that, if $(\varphi, \phi) \in F$ and $\varphi(t) \leq \phi(t)$, then $t = 0$. Since we can take $t_n = s_n = t$ for any $n \in \mathbb{N}$ and by condition (iii), we deduce that $t = 0$.

Now we present some interesting examples of pairs of functions belonging to the class F.

EXAMPLE 3. Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function such that $\varphi(t) = 0$ if and only if $t = 0$ (these functions are known as the altering distance function in the literature).

Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi \leq \varphi$. Then the pair $(\varphi, \varphi - \phi) \in F$. In fact, it is clear that $(\varphi, \varphi - \phi)$ satisfies condition (i).

Now we will prove condition (ii). For this, let us suppose that $t, s \in [0,\infty)$ and $\varphi(t) \leq (\varphi - \phi)(s)$. Then from $\varphi(t) \leq \varphi(s) - \phi(s) \leq \varphi(s)$ and since φ is increasing, we can deduce that $t \leq s$. Finally, to prove (iii), we assume that

$$\varphi(t_n) \leq \varphi(s_n) - \phi(s_n), \tag{3}$$

for any $n \in \mathbb{N}$, where $t_n, s_n \in [0,\infty)$ and

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a.$$

Taking limit as $n \to \infty$ in (3), we infer that $\lim_{n\to\infty} \phi(s_n) = 0$.

Now, let us suppose that $a > 0$. Since $\lim_{n\to\infty} s_n = a > 0$, we can find $\varepsilon > 0$ and a sequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} > \varepsilon$ for any $k \in \mathbb{N}$. As ϕ is non-decreasing, we have $\phi(s_{n_k}) > \phi(\varepsilon)$ for any $k \in \mathbb{N}$ and consequently, $\lim_{k\to\infty} \phi(s_{n_k}) \geq \phi(\varepsilon)$. This contradicts the fact that $\lim_{k\to\infty} \phi(s_{n_k}) = 0$. Therefore, $a = 0$. This proves that $(\varphi, \varphi - \phi) \in F$.

An interesting particular case is, when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$ and $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \leq t$ for any $t \in [0,\infty)$.

EXAMPLE 4. Let S be the class of functions defined by

$$S = \left\{\alpha : [0,\infty) \to [0,1) : \text{If } \lim_{n\to\infty} \alpha(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0\right\}.$$

Let us consider the pair of functions $\left(1_{[0,\infty)}, \alpha 1_{[0,\infty)}\right)$, where $\alpha \in S$ and $\alpha 1_{[0,\infty)}$ is defined by $\left(\alpha 1_{[0,\infty)}\right)(t) = \alpha(t)\, t$ for $t \in [0,\infty)$. Then it is easy to prove that $\left(1_{[0,\infty)}, \alpha 1_{[0,\infty)}\right) \in F$ for $\alpha \in S$.

REMARK 2. Suppose that $g : [0,\infty) \to [0,\infty)$ is an increasing function and $(\varphi, \phi) \in F$. Then it is easily seen that the pair $(g \circ \varphi, g \circ \phi) \in F$.

Now we are in a position to prove our main results.

THEOREM 2. Let (X, p) be a complete partial metric space and $T : X \to X$ a self map such that there exists a pair of functions $(\varphi, \phi) \in F$ such that

$$\varphi(p(Tx, Ty)) \leq \max\left\{\phi(p(x,y)), \phi\left(p(y,Ty)\frac{1 + p(x,Tx)}{1 + p(x,y)}\right)\right\}$$

for all $x, y \in X$. Then T has a unique fixed point.

PROOF. Let $x_0 \in X$ be arbitrary. We construct a sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n \text{ for } n \geq 0. \tag{4}$$

Now applying contractive condition, we have

$$\begin{aligned}\varphi(p(x_{n+1}, x_n)) &= \varphi(p(Tx_n, Tx_{n-1})) \\ &\leq \max\left\{\phi(p(x_n, x_{n-1})), \phi\left(p(x_{n-1}, Tx_{n-1})\frac{1 + p(x_n, Tx_n)}{1 + p(x_n, x_{n-1})}\right)\right\} \\ &\leq \max\left\{\phi(p(x_n, x_{n-1})), \phi\left(p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})}\right)\right\}. \end{aligned} \tag{5}$$

Now, let us assume that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0+1}, x_{n_0}) = 0$. In this case, $x_{n_0+1} = x_{n_0}$ and consequently, by (4)

$$Tx_{n_0} = x_{n_0+1} = x_{n_0},$$

i.e. x_{n_0} would be the fixed point of T. So we may assume that $p(x_{n+1}, x_n) \neq 0$ for any $n \in \mathbb{N}$. Now we consider two cases.

Case 1. Consider

$$\max\left\{\phi(p(x_n, x_{n-1})), \phi\left(p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})}\right)\right\} = \phi(p(x_n, x_{n-1})). \tag{6}$$

Then from (5), we have

$$\varphi(p(x_{n+1}, x_n)) \leq \phi(p(x_n, x_{n-1})), \tag{7}$$

and since $(\varphi, \phi) \in F$, we deduce that $p(x_{n+1}, x_n) \leq p(x_n, x_{n-1})$.

Case 2. If

$$\begin{aligned} &\max\left\{\phi(p(x_n, x_{n-1})), \phi\left(p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})}\right)\right\} \\ = \quad &\phi\left(p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})}\right), \end{aligned} \tag{8}$$

then from (5), we obtain

$$\varphi(p(x_{n+1}, x_n)) \leq \phi\Big(p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1})}\Big) \tag{9}$$

and since $(\varphi, \phi) \in F$, we deduce that

$$p(x_{n+1}, x_n) \leq p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1}},$$

which implies that $p(x_{n+1}, x_n) \leq p(x_n, x_{n-1})$.

Therefore, from both the cases we conclude that $\{p(x_{n+1}, x_n)\}$ is a decreasing sequence of non-negative real numbers.

Put $r = \lim_{n\to\infty} p(x_{n+1}, x_n)$, where $r \geq 0$, and denote

$$A = \{n \in \mathbb{N} : n \text{ satisfies (6)}\} \text{ and } B = \{n \in \mathbb{N} : n \text{ satisfies (8)}\}.$$

Now, we make the following remark.

(1). If Card $A = \infty$, then from (5), we can find infinitely many natural numbers satisfying (7) and since $\lim_{n\to\infty} p(x_{n+1}, x_n) = \lim_{n\to\infty} p(x_n, x_{n-1}) = r$ and $(\varphi, \phi) \in F$, we deduce that $r = 0$.

(2). If Card $B = \infty$, then from (5), we can find infinitely many natural numbers satisfying (9) and since $(\varphi, \phi) \in F$, and using the same argument one used in Case 2, we obtain

$$p(x_{n+1}, x_n) \leq p(x_{n-1}, x_n)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, x_{n-1}}.$$

Taking limit as $n \to \infty$ in the last inequality and taking into account that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = r,$$

one can deduce that $r = 0$.

Therefore,

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{10}$$

Due to inequality (2), we have

$$d_p(x_{n+1}, x_n) \leq 2p(x_{n+1}, x_n),$$

therefore,

$$\lim_{n\to\infty} d_p(x_{n+1}, x_n) = 0. \tag{11}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in X, i.e. we prove that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

In the contrary case, since $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, we can find $\varepsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \quad p(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{12}$$

This means that

$$p(x_{n(k)-1}, x_{m(k)}) \leq \varepsilon. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned} \varepsilon &\leq p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) < \varepsilon + p(x_{n(k)}, x_{n(k)-1}). \end{aligned}$$

Taking the limit as $k \to \infty$ and using (10), we get

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{14}$$

By the triangular inequality, we have

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\quad -p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} P(x_{n(k)-1}, x_{m(k)-1}) &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) - p(x_{m(k)}, x_{m(k)-1}) \\ &\quad -p(x_{m(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) - p(x_{m(k)}, x_{m(k)-1}). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above two inequalities and using (10) and (14), we get

$$\lim_{n\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{15}$$

Now applying the contractive condition, we have

$$\begin{aligned} \varphi(p(x_{m(k)}, x_{n(k)})) &= \varphi(p(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \max\Big\{\phi(p(x_{m(k)-1}, x_{n(k)-1})), \\ &\quad \phi\Big(p(x_{n(k)-1}, Tx_{n(k)-1})\frac{1 + p(x_{m(k)-1}, Tx_{m(k)-1})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big)\Big\} \\ &= \max\Big\{\phi(p(x_{m(k)-1}, x_{n(k)-1})), \\ &\quad \phi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big)\Big\} \text{ for } k \in \mathbb{N}. \end{aligned} \tag{16}$$

Let us put

$$C = \{k \in \mathbb{N} : \varphi(p(x_{m(k)}, x_{n(k)})) \leq \phi(p(x_{m(k)-1}, x_{n(k)-1}))\},$$

$$D = \left\{k \in \mathbb{N} : \varphi(p(x_{m(k)}, x_{n(k)})) \leq \phi\left(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\right)\right\}.$$

By (16), we have $\operatorname{Card} C = \infty$ or $\operatorname{Card} D = \infty$.

Let us suppose that $\operatorname{Card} C = \infty$. Then there exists infinitely many $k \in \mathbb{N}$ such that

$$\varphi(p(x_{m(k)}, x_{n(k)})) \leq \phi(p(x_{m(k)-1}, x_{n(k)-1})),$$

and since $(\varphi, \phi) \in F$, we get

$$p(x_{m(k)}, x_{n(k)}) \leq p(x_{m(k)-1}, x_{n(k)-1}).$$

Taking the limit as $k \to \infty$ in the above inequality, we get $\varepsilon = 0$ a contradiction.

Let us suppose that $\mathrm{Card}\, D = \infty$. In this case, we can find infinitely many $k \in \mathbb{N}$ such that

$$\varphi(p(x_{m(k)}, x_{n(k)})) \leq \phi\left(p(x_{n(k)-1}, x_{n(k)}) \frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\right),$$

and since $(\varphi, \phi) \in F$, we infer

$$p(x_{m(k)}, x_{n(k)}) \leq p(x_{n(k)-1}, x_{n(k)}) \frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}.$$

Taking limit as $k \to \infty$ and in view of (10) and (15), it follows that $\varepsilon \leq 0$ and we get a contradiction.

Therefore, in both possibilities, we obtain a contradiction and so we have

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0.$$

Since $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and finite, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, p). Due to inequality (2), we have $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore

$$\lim_{n,m \to \infty} d_p(x_n, x_m) = 0.$$

Thus, by Lemma 1, $\{x_n\}$ is a Cauchy sequence in (X, d_p) and (X, p). Since (X, p) is a complete partial metric space, there exists $x \in X$ such that $\lim_{n \to \infty} p(x_n, x) = p(x, x)$. Since $\lim_{n,m \to \infty} p(x_n, x_m) = 0$, then again by Lemma 1, we have $p(x, x) = 0$.

Next, we will prove that x is a fixed point of T. Suppose that $Tx \neq x$. Now applying contractive condition and Lemma 3, we have

$$\varphi(p(Tx, Tx_n)) \leq \max\left\{\phi(p(x, x_n)), \phi\left(p(x_n, Tx_n) \frac{1 + p(x, Tx)}{1 + p(x, x_n)}\right)\right\}.$$

We can distinguish two cases again.

Case 1. There exist infinitely many $n \in \mathbb{N}$ such that

$$\varphi(p(Tx, Tx_n)) \leq \phi(p(x, x_n)).$$

Since $(\varphi, \phi) \in F$, we obtain

$$p(Tx, Tx_n) \leq p(x, x_n).$$

Letting $n \to \infty$, we get

$$\lim_{n \to \infty} Tx_n = Tx, \tag{17}$$

where to simplify our consideration, we will denote the subsequence by the same symbol $\{Tx_n\}$. By (4)

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1}. \tag{18}$$

Since $x_n \to x$ in X, this means that $\limsup p(x_n, x) \to 0$ and consequently,

$$\lim_{n\to\infty} x_{n+1} = x.$$

From this last result and (18), we deduce that $Tx = x$, which means that x is a fixed point of T.

Case 2 There exist infinitely many $n \in \mathbb{N}$ such that

$$\varphi(p(Tx, Tx_n)) \le \phi\left(p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)}\right).$$

To simplify our consideration, we will denote the subsequence by the same symbol $\{Tx_n\}$. Since $(\varphi, \phi) \in F$ and $Tx_n = x_{n+1}$, we have

$$p(Tx, Tx_n) \le p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)} \ \text{ for any } \ n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$ and by (11), $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we infer (17). From the above Case 1, we deduce that x is a fixed point of T.

Therefore, in both the cases we have shown that x is a fixed point of T.

Finally, we will prove the uniqueness of the fixed point. Suppose that y is another fixed point of T such that $x \ne y$. Now using contractive condition, we get

$$\begin{aligned}\varphi(p(x,y)) &= \varphi(p(Tx,Ty)) \\ &\le \max\left\{\phi(p(x,y)), \phi\left(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\right)\right\} \\ &\le \max\{\phi(p(x,y)), \phi(0)\}.\end{aligned} \tag{19}$$

Now there are two cases.

Case 1. Consider $\max\{\phi(p(x,y)), \phi(0)\} = \phi(p(x,y))$. In this case, from (19)

$$\varphi(p(x,y)) \le \phi(p(x,y)).$$

Since $(\varphi, \phi) \in F$ and by Remark 1, we deduce that $p(x,y) = 0$, that is, $x = y$.

Case 2. Consider $\max\{\phi(p(x,y)), \phi(0)\} = \phi(0)$. Then from (19), we obtain

$$\varphi(p(x,y)) \le \phi(0).$$

Since $(\varphi, \phi) \in F$, we infer that $p(x,y) \le 0$. Therefore, $p(x,y) = 0$, that is, $x = y$.

Hence in both the cases $x = y$, that is, the fixed point is unique. This completes the proof of the Theorem 2.

From Theorem 2, we obtain the following corollaries.

COROLLARY 1. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying

$$\varphi(p(Tx, Ty)) \leq \phi(p(x, y)),$$

for all $x, y \in X$. Then T has a unique fixed point in X.

COROLLARY 2. Let (X, p) be a complete partial metric space and $T : X \to X$ a self map such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying

$$\varphi(p(Tx, Ty)) \leq \phi\left(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\right),$$

for all $x, y \in X$. Then T has a unique fixed point in X.

REMARK 3. Notice that the contractive condition appearing in Theorem 1,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)},$$

for all $x, y \in X$, where $\alpha, \beta > 0$ with $\alpha + \beta < 1$, implies that

$$\begin{aligned} d(Tx, Ty) &\leq (\alpha + \beta) \max\left\{d(x, y), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right\} \\ &\leq \max\left\{(\alpha + \beta)d(x, y), (\alpha + \beta)\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right\} \end{aligned}$$

for all $x, y \in X$.

This condition is a particular case of the contractive condition appearing in Theorem 2 with the pair of functions $(\varphi, \phi) \in F$ given by $\varphi = 1_{[0,\infty)}$ and $\phi = (\alpha + \beta)1_{[0,\infty)}$. Therefore, Theorem 1 is a particular case of the following corollary and considered as an extension and generalizations of Theorem 1 in the setting of complete partial metric spaces.

COROLLARY 3. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying

$$d(Tx, Ty) \leq \max\left\{(\alpha + \beta)d(x, y), (\alpha + \beta)\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right\}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Taking into account Example 3, we have the following corollary.

COROLLARY 4. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} \varphi(p(Tx, Ty)) \leq \max\Big\{&\varphi(p(x, y)) - \phi(p(x, y)), \\ &\varphi\left(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\right) - \phi\left(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\right)\Big\} \end{aligned}$$

for any $x, y \in X$, where φ is a continuous and increasing function satisfying $\varphi(t) = 0$ if and only if $t = 0$ and ϕ is a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi \leq \varphi$. Then T has a unique fixed point in X.

Corollary 4 has the following consequences.

COROLLARY 5. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ such that

$$\varphi(p(Tx, Ty)) \leq \varphi(p(x, y)) - \phi(p(x, y))$$

for any $x, y \in X$, where φ is an increasing function and ϕ is a nondecreasing function and they satisfy $\varphi(t) = \phi(t) = 0$ if and only if $t = 0$ and φ is continuous with $\phi \leq \varphi$. Then T has a unique fixed point in X.

Corollary 5 can be considered as an extension of the following result about fixed point theorems which appears in [7] in the setting of complete partial metric space.

THEOREM 3. Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \phi(d(x, y))$$

for any $x, y \in X$, where φ and ϕ satisfy the same conditions as in Corollary 5. Then T has a unique fixed point.

COROLLARY 6. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ satisfying the same conditions as in Corollary 5 such that

$$\varphi(p(Tx, Ty)) \leq \varphi\left(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\right) - \phi\left(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\right)$$

for any $x, y \in X$. Then T has a unique fixed point.

Taking into account Example 4, we have the following corollary.

COROLLARY 7. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exist $\alpha \in S$ (see Example 4) satisfying

$$\begin{aligned} p(Tx, Ty) \leq \max \Big\{ & \alpha(p(x, y))p(x, y), \\ & \alpha\Big(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\Big)\Big(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\Big)\Big\} \end{aligned}$$

for any $x, y \in X$. Then T has a unique fixed point.

Corollary 7 has the following consequence.

COROLLARY 8. Let (X, p) be a complete partial metric space and $T : X \to X$ be a self map such that there exist $\alpha \in S$ (see Example 4) satisfying

$$p(Tx, Ty) \leq \alpha(p(x, y))p(x, y)$$

for any $x, y \in X$. Then T has a unique fixed point.

Corollary 8 can be considered as an extension of the following result about fixed point theorems which appears in [9] in the setting of complete partial metric spaces.

THEOREM 4. Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for any $x, y \in X$, where $\alpha \in S$ (see Example 4). Then T has a unique fixed point.

4 Example

In this section, we give an example in support of our main result.

EXAMPLE 5. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$, then (X, p) is a partial metric space. Suppose $T : X \to X$ such that $Tx = \frac{x}{2}$ for all $x \in X$. Define the function $\varphi, \phi : [0, \infty) \to [0, \infty)$ as follows:

$$\varphi(x) = \ln\left(\frac{1}{12} + \frac{5x}{12}\right) \text{ and } \phi(x) = \ln\left(\frac{1}{12} + \frac{3x}{12}\right) \text{ for all } \ x \in [0, \infty).$$

Without loss of generality, assume that $x \geq y$. Then we have

$$\begin{aligned}\varphi(p(Tx, Ty)) &= \ln\left(\frac{1}{12} + \frac{5}{12}p(Tx, Ty)\right) = \ln\left(\frac{1}{12} + \frac{5}{12}\max\{x, y\}\right)\\ &= \ln\left(\frac{1}{12} + \frac{1}{12}x\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi(p(x, y)) &= \ln\left(\frac{1}{12} + \frac{3}{12}p(x, y)\right) = \ln\left(\frac{1}{12} + \frac{3}{12}\max\{x, y\}\right)\\ &= \ln\left(\frac{1}{12} + \frac{3}{12}x\right)\end{aligned}$$

and

$$\begin{aligned}\phi\Big(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\Big) &= \ln\Big(\frac{1}{12} + \frac{3}{12}\Big(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\Big)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}\Big(y\frac{1 + x}{1 + x}\Big)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}y\Big).\end{aligned}$$

Therefore,

$$\max\Big\{\phi(p(x,y)),\phi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)\Big\} = \max\Big\{\ln\Big(\frac{1}{12}+\frac{3}{12}x\Big),\ln\Big(\frac{1}{12}+\frac{3}{12}y\Big)\Big\}$$
$$= \ln\Big(\frac{1}{12}+\frac{3}{12}x\Big).$$

Combining the observations above, we get

$$\varphi(p(Tx,Ty)) = \ln\Big(\frac{1}{12}+\frac{1}{12}x\Big) \leq \ln\Big(\frac{1}{12}+\frac{3}{12}x\Big)$$
$$= \max\Big\{\phi(p(x,y)),\phi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)\Big\}.$$

Thus all the conditions of Theorem 2 are satisfied. Hence T has a unique fixed point, indeed $x = 0$ is the required fixed point.

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