

Generalized Minimax Theorems On Nonconvex Domains*

Byung-Soo Lee[†]

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Abstract

In this paper, the author considers generalized minimax theorems for vector set-valued mappings using Fan-KKM theorem on nonconvex domains of Hausdorff topological vector spaces.

1 Introduction

In 1953, K. Fan [2] considered the following minimax equality which is called Ky Fan minimax inequality,

$$\min_X \sup_Y f(x, y) = \sup_Y \min_X f(x, y)$$

for a convex-concave like function $f : X \times Y \rightarrow \mathbb{R}$ involving no linear structures. Since then, there have been many generalized results on Ky Fan minimax theorems due to the important roles of the theorems to many fields, such as variational inequalities, game theory, mathematical economics, control theory, equilibrium problem and fixed point theory. In generalizing Ky Fan minimax theorems, authors have used famous known theorems. For examples, Zhang and Li [9] considered

$$\text{Min} \bigcup_{x \in X} \text{Max}_w F(x, X) \subset \text{Max} \bigcup_{x \in X} F(x, x) - S$$

and

$$\text{Max} \bigcup_{x \in X} \text{Min}_w F(X, x) \subset \text{Min} \bigcup_{x \in X} F(x, x) + S$$

for a set-valued mapping F defined on $X \times X$ using Kakutani-Fan-Glicksberg fixed point theorem, where X is a convex domain and S is a pointed closed convex cone. Chang et al. [1] obtained a Ky Fan minimax inequality for vector-valued mappings on W -spaces by applying a generalized section theorem and a generalized fixed point theorem.

Li et al. [5] considered the existence of $x_0 \in X_0$ such that

$$\text{Min}_w F(x_0, Y) \subset \text{Min} \cup_{y \in Y} \text{co}(\text{Max}_w F(x_0, y)) + S$$

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[†]Department of Mathematics, Kyungsung University, Busan 608-736, Korea

on convex domains X and Y by using more generalized Ky Fan's section theorem. Very recently, Zhang et al. [10] investigated the existences of

$$z_1 \in \text{Max} \bigcup_{x \in X} \text{Min}_w F(x, Y) \text{ and } z_2 \in \text{Min} \bigcup_{y \in Y} \text{Max}_w F(X, y)$$

such that $z_1 \in z_2 + S$ on compact convex subsets of $X \times Y$ by using Fan-Browder fixed point theorem.

In 2010, Yang et al. [8] considered

$$\emptyset \neq \text{Min}_w \bigcup_{x \in X} \text{Max}_w F(x, X) \subset \text{Max} \bigcup_{x \in X} F(x, x) + Z \setminus \text{int}(S),$$

$$\emptyset \neq \text{Max}_w \bigcup_{x \in X} F(x, x) \subset \text{Min} \bigcup_{x \in X} \text{Max}_w F(x, X) + Z \setminus (-\text{int}(S))$$

and

$$\text{Max}_w \bigcup_{x \in X} F(x, x) \subset \text{Min} \bigcup_{x \in X} \text{Max}_w F(x, X) + S$$

for a vector-valued mapping $F : X \times X \rightarrow Z$ using Fan-Browder type fixed point theorem and maximal element theorem in abstract convex spaces.

In 2010, Li et al. [6] considered

$$\text{Min}_{lex} \bigcup_{y \in Y_0} \text{Max}_{lex} \bigcup_{x \in X_0} f(x, y) = \text{Max}_{lex} \bigcup_{x \in X_0} \text{Min}_{lex} \bigcup_{y \in Y_0} f(x, y)$$

for a vector-valued mapping $f : X_0 \times Y_0 \rightarrow Z$, where X_0 and Y_0 are nonempty compact subsets of metric spaces X and Y , respectively and Z is a nonempty subset of the n -dimensional Euclidean space \mathbb{R}^n with a lexicographic cone C_{lex} .

In the previous cited works, almost all the results on generalized Ky Fan minimax theorem were considered on convex domains. Hence it would be desirable and reasonable to consider generalized Ky Fan minimax theorems on nonconvex domains for many further applications.

In this paper, the author considers the following generalized minimax theorems

$$\text{Min} \bigcup_{x \in X, y \in Y} G(x, y) \subset \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) - S$$

and

$$\text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) \subset \text{Min} \bigcup_{x \in X, y \in Y} G(x, y) + S.$$

for a vector set-valued mapping G defined on $X \times Y$ using Fan-KKM theorem on a nonconvex domain X and a convex domain Y of Hausdorff topological vector spaces E and F , respectively.

2 Preliminaries

Throughout this paper, let E, F and V be real Hausdorff topological vector spaces and X and Y be subsets of E and F , respectively. Assume that S is a pointed closed convex cone in V with the nonempty interior $\text{int}S$. We recall the following definitions and lemmas in [3–9].

DEFINITION 2.1. Let $A \subset V$ be a nonempty set. Then

- (i) A point $z \in A$ is a minimal point of A if $A \cap (z - S) = \{z\}$ and $\text{Min}A$ denotes the set of all minimal points of A ;
- (ii) A point $z \in A$ is a weakly minimal point of A if $A \cap (z - \text{int}S) = \emptyset$ and $\text{Min}_w A$ denotes the set of all weakly minimal points of A ;
- (iii) A point $z \in A$ is a maximal point of A if $A \cap (z + S) = \{z\}$ and $\text{Max}A$ denotes the set of all maximal points of A ;
- (iv) A point $z \in A$ is a weakly maximal point of A if $A \cap (z + \text{int}S) = \emptyset$ and $\text{Max}_w A$ denotes the set of all weakly maximal points of A .

REMARK 2.1. It is known that

$$\text{Min}A \subset \text{Min}_w A \quad \text{and} \quad \text{Max}A \subset \text{Max}_w A.$$

LEMMA 2.1 Let $A \subset V$ be a nonempty compact subset. Then $\text{Min}A \neq \emptyset$, $A \subset \text{Min}A + S$, $\text{Max}A \neq \emptyset$ and $A \subset \text{Max}A - S$.

DEFINITION 2.2.

- (i) G is said to be lower semicontinuous (l.s.c.) at a point \bar{x} if for any open set $W \subset E$ with $W \cap G(\bar{x}) \neq \emptyset$, there exists a neighborhood $N(\bar{x})$ of \bar{x} such that $G(x) \cap W \neq \emptyset$ for all $x \in N(\bar{x})$.
- (ii) G is said to be upper semicontinuous (u.s.c.) at \bar{x} if for every open set $W \subset E$ with $G(\bar{x}) \subset W$, there exists a neighborhood $N(\bar{x})$ of \bar{x} such that $G(x) \subset W$ for all $x \in N(\bar{x})$.
- (iii) G is said to be l.s.c. (resp., u.s.c.) on $X \subset E$ if it is l.s.c. (resp., u.s.c.) at every point $x \in X$.

PROPOSITION 2.1 ([4, 7]). The following statements (i) and (ii) hold:

- (i) G is l.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \rightarrow x$ and any $y \in G(x)$, there exists $y_\alpha \in G(x_\alpha)$ such that $y_\alpha \rightarrow y$.
- (ii) If G has compact set-values (i.e., $G(x)$ is a compact set for each $x \in X$), then the following (a) and (b) are equivalent

- (a) G is u.s.c. at $x \in X$.
- (b) for any net $\{x_\alpha\} \subset X$ with $x_\alpha \rightarrow x$ and any $y_\alpha \in G(x_\alpha)$, there exists $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \rightarrow y$.

LEMMA 2.2. Let X_0 be a nonempty subset of E , and $G : X \rightarrow 2^V$ be a set-valued mapping. If X is compact and G is upper semicontinuous and compact set-valued, then $G(X) = \bigcup_{x \in X} G(x)$ is compact.

Now we define a g -KKM mapping $G : X \rightarrow 2^F$.

DEFINITION 2.3. Let X and Y be nonempty subsets of E and F , respectively, and $g : X \rightarrow Y$ be a mapping. A set-valued mapping $H : X \rightarrow 2^F$ is said to be a g -KKM mapping if $co(g(A)) \subset \bigcup_{x \in A} H(x)$ for every finite subset A of X , where $co(A)$ is the convexhull of A .

REMARK 2.2. If g is the identity when $X = Y$, then g -KKM mapping reduces to the usual KKM mapping.

THEOREM 2.1 (Fan-KKM Theorem). Let X and Y be nonempty subsets of E and F , respectively, and $g : X \rightarrow Y$ be a mapping. If $H : X \rightarrow 2^F$ is a g -KKM mapping with closed set-values and there exists $x_0 \in X$ such that $H(x_0)$ is compact, then

$$\bigcap_{x \in X} H(x) \neq \emptyset.$$

DEFINITION 2.4 ([5]). A set-valued mapping $G : K \rightarrow 2^Y$ is said to be S -concave if for all $x, y \in K$ and $t \in [0, 1]$, we have

$$G((1-t)x + ty) \subset (1-t)G(x) + tG(y) + S,$$

where K is a convex subset of a vector space X and S is a pointed convex cone in an ordered vector space Y . G is S -convex if $-G$ is S -concave.

3 Main Result

In this section, we show two generalized minimax theorems on nonconvex domains using Fan-KKM theorem.

THEOREM 3.1. Let X be a nonempty compact subset of E and Y a nonempty compact convex subset of F . Let $G : X \times Y \rightarrow 2^V$ be a upper semicontinuous set-valued mapping with compact set-values and $g : X \rightarrow Y$ be a mapping. If G is S -concave in the second variable, then

$$\text{Min} \bigcup_{x \in X, y \in Y} G(x, y) \subset \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) - S$$

PROOF. It is easily shown that $\text{Min}G(x, y) \neq \emptyset$ and $\text{Min} \cup_{x \in X, y \in Y} G(x, y) \neq \emptyset$ by Lemma 2.1 and Lemma 2.2. Let $v \in \text{Min} \cup_{x \in X, y \in Y} G(x, y)$, then $v \in \text{Min}G(x, y)$ for each $x \in X$ and each $y \in Y$. Since

$$G(x, y) \subset \text{Min}G(x, y) + S$$

by Lemma 2.1, we have

$$-v \in -G(x, y) + S \text{ for } x \in X \text{ and } y \in Y. \quad (1)$$

Define a set-valued mapping $H : X \rightarrow 2^F$ by

$$H(x) = \{y \in Y : G(x, y) \subset v + S\} \text{ for } x \in X,$$

then H is a g -KKM mapping. If not, there exists a finite subset $\{x_i ; i = \overline{1, n}\}$ of X such that

$$\sum_{i=1}^n t_i g(x_i) \notin \bigcup_{i=1}^n H(x_i) \text{ for } t_i \in [0, 1] (i = \overline{1, n}) \text{ with } \sum_{i=1}^n t_i = 1.$$

By the definition of H ,

$$G \left(x_i, \sum_{j=1}^n t_j g(x_j) \right) \not\subset v + S \text{ for } i = \overline{1, n}. \quad (2)$$

Hence by the condition that G is S -concave in the second variable,

$$G \left(x_i, \sum_{j=1}^n t_j g(x_j) \right) \subset \sum_{j=1}^n t_j G(x_i, g(x_j)) + S. \quad (3)$$

Thus by (2) and (3),

$$\sum_{j=1}^n t_j G(x_i, g(x_j)) \not\subset v + S,$$

which shows that

$$G(x_i, g(x_j)) \not\subset v + S (i, j = \overline{1, n})$$

leads a contradiction to (1). On the other hand, $H(x) \neq \emptyset$ for all $x \in X$ by (1). Moreover, for all $x \in X$, $H(x)$ is closed. In fact, for any net $\{y_\alpha\}$ in $H(x)$ converging to y and any $z_\alpha \in G(x, y_\alpha)$ with $z_\alpha \in v + S$, there exist $z \in G(x, y)$ with $z \in v + S$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \rightarrow z$ by Proposition 2.1. Hence $y \in H(x)$, which says that $H(x)$ is closed. Further, $F(x)$ is compact for all $x \in X$, from the fact that Y is compact. Hence by Fan-KMM theorem, we have

$$\bigcap_{x \in X} H(x) \neq \emptyset.$$

Thus there exists $y \in Y$ such that $G(x, y) \subset v + S$ for all $x \in X$, which implies that

$$\text{Min}_w G(x, y) \subset v + S.$$

Hence we have

$$v \in \text{Min}_w G(x, Y) - S \subset \bigcup_{x \in X} \text{Min}_w G(x, Y) - S \subset \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) - S$$

by Lemma 2.1.

EXAMPLE 3.1. Let $G : X \times Y \rightarrow 2^V$ be a set-valued mapping defined by

$$G(x, y) = [x + y, x - y] \times [x^2, x^2 + y^2]$$

for $(x, y) \in X \times Y$, where $X = [0, \frac{1}{2}] \cup [\frac{2}{3}, 1]$, $Y = [-1, 0]$, $V = \mathbb{R}^2$ and $S = \mathbb{R}_-^2 := \{(x, y) : x \geq 0, y \leq 0\}$. Then G is upper semicontinuous and $G(x, y)$ is compact for all $(x, y) \in X \times Y$. Moreover, G is S -concave in the second variable. In fact,

$$\begin{aligned} & G(x, ty_1 + (1-t)y_2) - (tG(x, y_1) + (1-t)G(x, y_2)) \\ &= [x^2, x^2 + (ty_1 + (1-t)y_2)^2] - (t[x^2, x^2 + y_1^2] + (1-t)[x^2, x^2 + y_2^2]) \\ &= \{0\} \times [0, t(t-1)(y_1 - y_2)^2] \\ &\in S \end{aligned}$$

for $x \in X$ and $y_1, y_2 \in Y$. On the other hand,

$$\text{Min} \bigcup_{x \in X, y \in Y} G(x, y) = \{(-1, 2)\}$$

and

$$\bigcup_{x \in X} \text{Min}_w G(x, Y) \subset \{(a, b) : -1 \leq a \leq 0 \text{ and } 1 \leq b \leq 2\}.$$

Hence

$$\text{Min} \bigcup_{x \in X, y \in Y} G(x, y) - \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) \in -S.$$

THEOREM 3.2. Let X be a nonempty compact subset of E and Y a nonempty compact convex subset of F . Let $G : X \times Y \rightarrow 2^V$ be an upper semicontinuous set-valued mapping with compact set-values and $g : X \rightarrow Y$ be a mapping. If the following conditions hold:

- (i) G is S -convex in the second variable, and
- (ii) for each $x \in X$,

$$\text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) \subset G(x, Y) + S.$$

Then

$$\text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y) \subset \text{Min} \bigcup_{x \in X, y \in Y} G(x, y) + S.$$

PROOF. Let $v \in \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, Y)$. Then

$$v \in G(x, Y) + S \text{ for each } x \in X. \quad (4)$$

Thus, for each $x \in X$, there exists $y \in Y$ such that $v \in G(x, y) + S$. Define a set-valued mapping

$$H : X \rightarrow 2^F$$

by $H(x) = \{y \in Y : G(x, y) \subset v - S\}$, then by the above argument, it is easily shown that $H(x) \neq \emptyset$ for all $x \in X$. Moreover, for all $x \in X$, $H(x)$ is closed. In fact, for any net $\{y_\alpha\}$ in $H(x)$ converging to y and any $z_\alpha \in G(x, y_\alpha)$ with $z_\alpha \in v - S$, there exist $z \in G(x, y)$ with $z \in v - S$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \rightarrow z$ by Proposition 2.1. Hence $y \in H(x)$, which says that $H(x)$ is closed and compact.

Now we show that H is a g -KKM mapping. If not, there exists a finite subset $\{x_i ; i = \overline{1, n}\}$ of X such that

$$\sum_{i=1}^n t_i g(x_i) \notin \bigcup_{i=1}^n H(x_i) \text{ for } t_i \in [0, 1] (i = \overline{1, n}) \text{ with } \sum_{i=1}^n t_i = 1.$$

By the definition of H ,

$$G \left(x_i, \sum_{j=1}^n t_j g(x_j) \right) \not\subset v - S \text{ for } i = \overline{1, n}.$$

Since G is S -convex in the second variable,

$$\sum_{j=1}^n t_j G(x_i, g(x_j)) \not\subset v - S,$$

which shows that

$$G(x_i, g(x_j)) \not\subset v - S (i, j = \overline{1, n}),$$

leading a contradiction to (4). Hence by Fan-KMM theorem, we have

$$\bigcap_{x \in X} H(x) \neq \emptyset.$$

Thus there exists $y \in Y$ such that $G(x, y) \subset v - S$ for all $x \in X$, which implies that

$$v \in G(x, y) + S \subset \bigcup_{x \in X, y \in Y} G(x, y) + S \subset \text{Min} \bigcup_{x \in X, y \in Y} G(x, y) + S$$

by Lemma 2.1.

REMARK 3.1. The same results can be obtained for a set-valued mapping $F : X \times Y \rightarrow 2^{\mathbb{R}}$ on a nonconvex domain X and a convex domain Y of \mathbb{R} , as corollaries.

REMARK 3.2. Putting $X = Y$ and $g = I$ in Theorem 3.1 and Theorem 3.2, we obtain the following results in [3] on convex domains $X \times X$;

$$\text{Max} \bigcup_{x \in X} \text{Min}_w G(x, X) \subset \text{Min} \bigcup_{x \in X} G(x, x) + S$$

and

$$\text{Min} \bigcup_{x \in X} G(x, x) \subset \text{Max} \bigcup_{x \in X} \text{Min}_w G(x, X) - S.$$

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