

A Note On The Sparing Number Of The Sieve Graphs Of Certain Graphs*

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Abstract

Let \mathbb{N}_0 denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a given graph G is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. An IASI f of a graph G is said to be a weak IASI of G if $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A graph which admits a weak IASI may be called a weak IASI graph. The sparing number of a graph G is the minimum number of edges with singleton set-labels required for a graph G to admit a weak IASI. In this paper, we introduce the notion of k -sieve graphs of a given graph and study their sparing numbers.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [6, 13] and for different graph classes, we refer to [2]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The notion of a set-valued graph has been introduced in [1] as a graph, the labels of whose vertices and edges are the subsets of a given set. Since then, several studies have been done on set-valuations of graphs. The *sumset* of two non-empty sets A, B , denoted by $A + B$, is defined as $A + B = \{a + b : a \in A, b \in B\}$. Using the terminology of sumsets of sets, the notion of an integer additive set-indexer of a given graph is introduced in [4] as follows.

Let \mathbb{N}_0 denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An *integer additive set-indexer* (IASI, in short) of a graph G is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective.

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element.

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LEMMA 1.1 ([5]). Let A and B be two non-empty finite sets of non-negative integers. Then $\max(|A|, |B|) \leq |A + B| \leq |A| |B|$. Therefore, for any integer additive set-indexer f of a graph G , we have

$$\max(|f(u)|, |f(v)|) \leq |f^+(uv)| = |f(u) + f(v)| \leq |f(u)| |f(v)|,$$

where $uv \in E(G)$.

DEFINITION 1.2 ([5]). An IASI f of a graph G is said to be a *weak IASI* if

$$|f^+(uv)| = |f(u) + f(v)| = \max(|f(u)|, |f(v)|)$$

for all $u, v \in V(G)$. A graph which admits a weak IASI is called a *weak IASI graph*. A weak IASI f is said to be *weakly k -uniform IASI* if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer k .

If A and B are two non-empty sets of non-negative integers, then $|A + B| = |A|$ if and only if $|B| = 1$ and $|A + B| = |B|$ if and only if $|A| = 1$. Hence, we have the following Theorem 1.3.

THEOREM 1.3 ([5]). A graph G admits a weak IASI if and only if at least one end vertex of every edge of G has a singleton set-label.

DEFINITION 1.4 ([8]). A *mono-indexed element* (a vertex or an edge) of an IASI graph G is an element of G whose set-indexing number is 1. The *sparing number* of a graph G is defined to be the minimum number of mono-indexed edges required for G to admit a weak IASI and is denoted by $\varphi(G)$.

THEOREM 1.5 ([8]). An odd cycle C_n contains an odd number of mono-indexed edges and an even cycle contains an even number of mono-indexed edges.

THEOREM 1.6 ([8]). The sparing number of an odd cycle C_n is 1 and that of an even cycle is 0.

THEOREM 1.7 ([8]). The sparing number of a bipartite graph is 0.

THEOREM 1.8 ([8]). The sparing number of a complete graph K_n is $\frac{1}{2}(n-1)(n-2)$.

Now, recall the definition of graph powers.

DEFINITION 1.9 ([3]). The r -th power of a simple graph G is the graph G^r whose vertex set is V , two distinct vertices being adjacent in G^r if and only if their distance in G is at most r . The graph G^2 is referred to as the *square* of G , the graph G^3 as the *cube* of G .

The following is an important theorem on graph powers.

THEOREM 1.10 ([12]). If d is the diameter of a graph G , then G^d is a complete graph.

The admissibility of weak IASIs by certain graph classes and graph powers and the determination of their corresponding sparing numbers have been done in [10, 11, ?]. The admissibility of weak IASIs by the graph operations and certain graphs associated with the given IASI graphs have been discussed in [7] and [9]. As a continuation to these studies, in this paper, we discuss the sparing number of a particular type of graphs obtained by adding some edges to the given graphs according to certain rules.

2 Sparing Number of the k -Sieve of a Graph

Motivated by the terminology of graph powers, we introduce the notion of a k -sieve of a given graph as follows.

DEFINITION 2.1. A k -sieve graph or simply a k -sieve of a given graph G , denoted by $G^{(k)}$, is the graph obtained by joining the non-adjacent vertices of G which are at a distance k . A cycle obtained by joining two vertices of G , which are at a distance k in G , is called a k -ringlet of the graph G .

Note that every k -ringlet of a graph is a cycle of length $k + 1$. The number of k -ringlets in a graph G is the number of distinct k -paths in G . The number of edges in $G^{(k)}$ that are not in G is the number of k -ringlets in G .

REMARK 2.2. Note that $G^{(2)} \cong G^2$, the square of the graph G . But, for $k > 2$, $G^{(k)}$ and G^k are non-isomorphic graphs. The sparing number of the square of certain graphs have already been studied and communicated. Hence, in this paper, we need to consider $k \geq 3$.

PROPOSITION 2.3. Let l be the length of a maximal path in G . If $k > l$, then the sparing number of the k -sieve of G is equal to the sparing number of G itself.

PROOF. Given that l is the length of a maximal path in G . Hence, for any pair of vertices x, y in G , $d(x, y) \leq l$. That is, there exists no vertex in G which is at a distance k from another vertex of G . Therefore, if $k > l$, then $G^{(k)} \cong G$. Hence, $\varphi(G^{(k)}) = \varphi(G)$.

Invoking the above result, we need to consider the integral values between 3 and l , including both, for k . If $k = l$, then the longest path of G becomes a cycle of length $l + 1$ in $G^{(k)}$.

We now proceed to determine the sparing number of the sieve graphs of certain other standard graphs. Let us begin with the path graphs.

THEOREM 2.4. Let P_n be a path of length n . Then, for odd integers k with $2 < k \leq n$, the sparing number of $P_n^{(k)}$ is 0 and for even integer k with $2 < k \leq n$, the

sparing number of $P_n^{(k)}$ is

$$\varphi(P_n^{(k)}) = \begin{cases} 2(lk+r) - 3 & \text{if } (lk+r)k + s = n \text{ where } s \leq r-2 \text{ and } r \geq 2, \\ 2(lk+r) - 2 & \text{if } (lk+r)k + s = n \text{ where } s = r-1 \text{ and } r \geq 1, \\ 2(lk+r) - 1 & \text{if } (lk+r)k + s = n \text{ where } s = r \text{ and } r \geq 0. \end{cases}$$

where l, k and s are non-negative integers.

PROOF. Let P_n be a path on $n+1$ vertices. Let $V = \{v_1, v_2, v_3, \dots, v_n, v_{n+1}\}$ be the vertex set of P_n . The proof is developed considering various possible cases as below.

Case 1 Let n be an odd integer. Label the vertices of P_n alternately by distinct singleton sets and distinct non-singleton sets. Then, each vertex v_i is adjacent to v_{i-1}, v_{i+1} and to the vertex v_{i+k} in $P_n^{(k)}$. Then, we have the following subcases.

Case 1-1 If $k = n$, then by Proposition 2.3, $P_n^{(k)} = C_{n+1}$. Since n is odd, $P_n^{(k)}$ is even cycle. Then, by Theorem 1.8, $P_n^{(k)}$ has no mono-indexed edges.

Case 1-2 Let $k < n$. Then, under the set-labeling we defined above, no two adjacent vertices simultaneously have singleton set-labels or non-singleton set-labels. Therefore, no edge in $P_n^{(k)}$ has no mono-indexed edges. Therefore, $P_n^{(k)}$ has no mono-indexed edges if $k \leq n$ is an odd integer.

Case 2 Let n be an even integer.

Case 2-1 If $k = n$, then by Proposition 2.3, $P_n^{(k)} = C_{n+1}$. Since n is even, $P_n^{(k)}$ is odd cycle. Then, by Theorem 1.8, $P_n^{(k)}$ has at least one mono-indexed edge. That is, $\varphi(P_n^{(k)}) = 1$.

Case 2-2 Let $k < n$. Then, there exists two integers r and s such that $rk + s \leq n$, where $0 \leq s < k$ and $r = 0, 1, 2, \dots$. We can label the vertices in such a way that no two adjacent vertices have non-singleton set-labels in the following way.

Label the vertices $v_1, v_3, v_5, \dots, v_{k-1}$ of $P_n^{(k)}$ by distinct non-singleton sets and label $v_2, v_4, v_6, \dots, v_k$ by distinct singleton sets. Since v_{k+1} is adjacent to v_1 , v_{k+1} can be labeled only by a singleton set which is not used for labeling any one of the preceding vertices. Then, the edge $v_k v_{k+1}$ is a mono-indexed edge. If $n \leq 2k$, then the only mono-indexed edge in $P_n^{(k)}$ is $v_k v_{k+1}$.

If $n > 2k$, label the vertices v_1, v_2, \dots, v_{k+1} as mentioned above and proceed labeling $v_{k+2}, v_{k+4}, \dots, v_{2k}$ by distinct non-singleton sets, that are not used for labeling before, and the vertices $v_{k+2}, v_{k+4}, \dots, v_{2k}$ by distinct singleton sets that are not used before for labeling. Then, $v_{k+1} v_{2k+1}$ is a mono-indexed edge. Since v_{k+2} is adjacent to v_{2k+2} , the vertex v_{2k+2} must be mono-indexed. Therefore, $v_{2k+1} v_{2k+2}$ is also a mono-indexed edge. Proceeding in this way, we arrive at the following cases.

Case 1 Let $s \leq r - 2$. Then, $r \geq 2$. Now, we can find a path $P' : v_k v_{k+1} v_{2k+1} v_{2k+2} v_{3k+2} v_{3k+3} \dots v_{(r-1)k+(r-2)} v_{(r-1)k+(r-1)}$, all of whose elements have singleton set-labels, containing all the mono-indexed edges of $P_n^{(k)}$. The length of the path P' is $2r - 3$.

Case 2 If $s = r - 1$, then $r \geq 1$. Now, there exists a path

$$P'' : v_k v_{k+1} v_{2k+1} v_{2k+2} v_{3k+2} v_{3k+3} \dots v_{(r-1)k+(r-1)} v_{rk+(r-1)},$$

all of whose edges are mono-indexed, containing all the mono-indexed edges of $P_n^{(k)}$. The length of the path P'' is $2r - 2$.

Case 3 If $s = r$, then the path

$$P''' : v_k v_{k+1} v_{2k+1} v_{2k+2} v_{3k+2} v_{3k+3} \dots v_{rk+(r-1)} v_{(r+1)k},$$

all of whose edges are mono-indexed, containing all the mono-indexed edges in $P_n^{(k)}$. Therefore, the length of P''' is $2r - 1$.

If $r = s = k$, then $rk + s = (k + 1)k$ and if $r = lk$ and $s = k$, then $rk + s = (lk + 1)k$ and hence we can proceed the labeling procedure in the same manner as explained above. Then, the result follows.

Figure 1 illustrates a weak IASI for the 4-sieve of the path of length 16. The mono-indexed edges are represented in dotted lines.

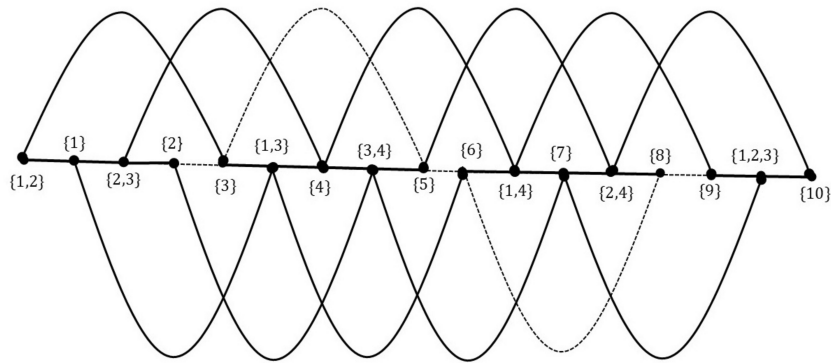


Figure 1: 4-sieve of a path with a weak IASI defined on it.

The above theorem arouses an interest in determining the sparing number of the k -sieve of a cycle. Here, note that a maximal path between any pair of vertices in a cycle C_n is $\lfloor \frac{n}{2} \rfloor$. Therefore, a k -sieve exists for C_n if and only if $n \geq 2k + 1$. Moreover, $C_{n(k)}$ is a 4-regular graph. Then, we have the following results.

THEOREM 2.5. Let C_n be a Cycle that admits a weak IASI. Then, for an odd integer k , $1 < k \leq l$,

$$\varphi(C_n^{(k)}) = \begin{cases} 0 & \text{if } C_n \text{ is an even cycle,} \\ k + 1 & \text{if } C_n \text{ is an odd cycle,} \end{cases}$$

where l is the length of a largest path in G .

PROOF. Let k be an odd integer. Then, every k -subcycle of $C_n^{(k)}$, obtained by joining the vertices of C_n which are at a distance k in C_n , is an even cycle of length $k + 1$. Let C' be such an even cycle of length $k + 1$ in $C_n^{(k)}$, which has exactly one edge, say e' , which is not in C_n . If C_n is an even cycle, then by Theorem 1.6, it need not contain mono-indexed edges. Therefore, as a result of Theorem 1.5, e' can not be mono-indexed. Therefore, C' does not contain any mono-indexed edges. If C_n is an odd cycle, then by Theorem 1.6, it must have one mono-indexed edge. If C' contains this mono-indexed edge of C_n , then as a result of Theorem 1.5, e' must be mono-indexed. There exist such k cycles containing this mono-indexed edge of C_n . Therefore, the sparing number of $C_n^{(k)}$ is $k + 1$. The proof is complete.

Figure 2 illustrates Theorem 2. The first subfigure is the 3-sieve of an even cycle with a weak IASI defined on it and the second subfigure is 3-sieve of an odd cycle with a weak IASI on it. Mono-indexed edges in the second graph are represented by dotted lines.

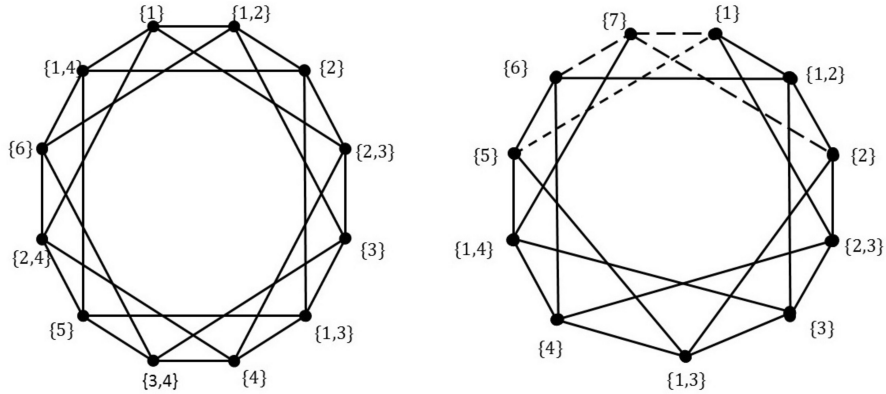


Figure 2: 3-sieve of C_{12} and a 3-sieve of C_{12} with weak IASIs defined on them.

Next let us consider the case when k is an even integer.

THEOREM 2.6 Let C_n be a cycle of length n . For an even integer k ; $2 < k \leq n$, the sparing number of $C_n^{(k)}$ is

$$\varphi(P_n^{(k)}) = \begin{cases} 3 & \text{if } n = 2k, \\ 2 \left[(lk + r) - 2 \left\lfloor \frac{(l-1)k + (r-1)}{2} \right\rfloor \right] & \text{if } n = lk + r, \\ 2l & \text{if } n = l(k + 1), \end{cases}$$

where l, k and r are non-negative integers such that $l \geq 2$ and $r < l$.

PROOF. First let $n = 2k$. Then $C_n^{(k)}$ is a cubic graph. Let us begin the labeling process by labeling the first vertex v_1 by a non-singleton set and then label the following vertices alternatively by distinct singleton sets and distinct non-singleton sets. Then, the vertex v_k is a mono-indexed vertex. Being adjacent to the vertex v_1 , v_{k+1} must also be mono-indexed. That is, the edge $v_k v_{k+1}$ is mono-indexed. Now, label the vertex v_{k+2} by a non-singleton set and then label the following vertices alternatively by distinct singleton sets and distinct non-singleton sets. Here, the vertex v_{2k-1} is mono-indexed. Since, v_{2k} is adjacent to the vertex v_1 , v_{2k} must be mono-indexed. Therefore, the edge $v_{2k-1} v_{2k}$ is mono-indexed. Also, the edge $v_k v_{2k}$ is also mono-indexed. Therefore, the number of mono-indexed edges in this case is 3.

Note that if $n > 2k$ the k -sieve of every cycle is a 4-regular graph, for any integer k . Then we have the following cases.

Case 1 Assume that $n = lk + r$; $r < l$, l and r being positive integers and $l \geq 2$. Then, the total number of edges in $C_n^{(k)}$ is

$$|E(C_n^{(k)})| = \frac{1}{2} \sum d(v) = 2(lk + r).$$

Now, label the vertex v_1 by a non-singleton set and then label the remaining vertices by distinct singleton sets and distinct non-singleton sets such that no two adjacent vertices having non-singleton set-labels. Then, the last $k + 1$ vertices must be 1-uniform, as each of them are adjacent to one vertex having a non-singleton set-label. Out of the remaining $(l - 1)k + (r - 1)$ vertices, $\lfloor \frac{(l-1)k+(r-1)}{2} \rfloor$ vertices have non-singleton set-label. The number of edges that are not mono-indexed is $4 \lfloor \frac{(l-1)k+(r-1)}{2} \rfloor$. The total number of mono-indexed edges is $2[(lk + r) - 2 \lfloor \frac{(l-1)k+(r-1)}{2} \rfloor]$.

Case 2 Assume that $n = l(k + 1)$, l being a positive integer. Let \mathfrak{C} be a partition of $V(G)$, where each set in \mathfrak{C} contains exactly $k + 1$ vertices. Therefore, each set in \mathfrak{C} consists of exactly $\frac{k}{2}$ vertices have non-singleton set-labels and $1 + \frac{k}{2}$ mono-indexed vertices in $C_n^{(k)}$. Therefore, the number of vertices having non-singleton set-labels is $l \frac{k}{2}$. Therefore, the number of edges that are not mono-indexed is $2lk$. The number edges in $C_n^{(k)}$ is

$$|E(C_n^{(k)})| = \frac{1}{2} \sum d(v) = 2l(k + 1).$$

Therefore, the number of mono-indexed edges in $C_n^{(k)}$ is $2l(k + 1) - 2lk = 2l$.

Figure 3 illustrates a weak IASI for the 4-sieve of a cycle on 20 vertices.

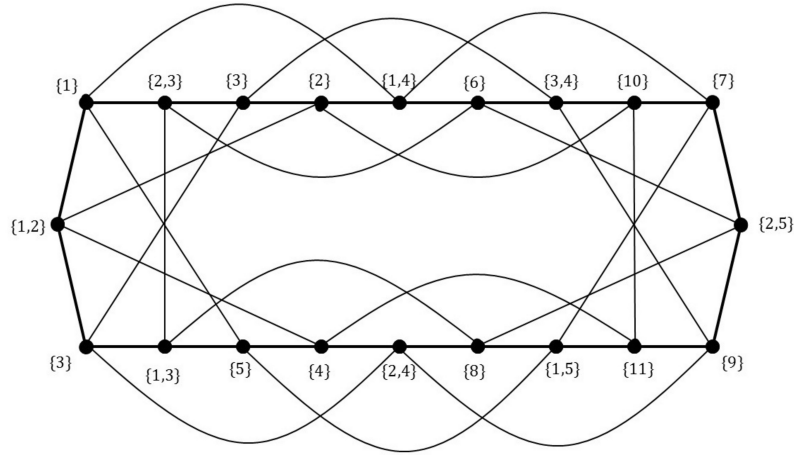


Figure 3: 4-sieve of C_{20} which is a weak IASI graph.

3 Conclusion

It can be observed that a complete graph K_n can have a k -sieve graph as every vertex of K_n is at a distance 1 from all other vertices of G . Similarly, a complete bipartite graph $K_{m,n}$ (or a complete r -partite graph, K_{n_1,n_2,\dots,n_r} , for $r > 2$) also does have a k -sieve graph, for $k \geq 3$, as any two vertices in $K_{m,n}$ (or in K_{n_1,n_2,\dots,n_r}) are at a distance at most 2.

Let k be an odd integer. If G be a tree, then $G^{(k)}$ is a graph all of whose cycles are of length $k + 1$, an even integer. Then, $G^{(k)}$ is a bipartite graph. Therefore, by Theorem 1.7, the number of mono-indexed edges in $G^{(k)}$ is also 0.

If G be a bipartite graph (containing cycles), then G has no odd cycles. Then, since every k -ringlet of G is an even cycle, every cycle in $G^{(k)}$ is of even length. Therefore, $G^{(k)}$ is also a bipartite graph, for odd k . Therefore, by Theorem 1.7, the number of mono-indexed edges in $G^{(k)}$ is also 0.

But, for even integers k , to determine the sparing number of $G^{(k)}$, for a graph G which is bipartite (cyclic or acyclic), we need to use Theorem 2.4, Theorem 2.5 and Theorem 2.6, for distinct paths and cycles in G .

Evaluating the sparing number of the k -sieves of bipartite graphs, Eulerian graphs, armed crown graphs etc. are some of the open problems in this area. Determining the sparing number of the k -sieves of graph operations and graph products are also worth for further exploration.

More properties and characteristics of weak IASIs, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain to be settled. All these facts highlight a great scope for further studies in this area.

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