On The Growth Of An $E$-Valued Meromorphic Function And Its Derivative*

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Abstract

In this article, the relative growth of an $E$-valued meromorphic function and its derivative is studied and we obtain the bound for $T(r,f')/T(r,f)$ for an $E$-valued meromorphic function of finite order. We also extend the related results of S. K. Singh and H. S. Gopalakrishna [4] to $E$-valued meromorphic function. Our results are significant and much stronger than the result obtained by Z. Wu and Y. Chen [5].

1 Introduction


2 Basic Notions of Nevanlinna Theory in Infinite Dimensional Banach Space

Assume that $E$ is a infinite dimensional complex Banach space with a Schauder basis $\{e_j\}_{j=1}^\infty$ and $\mathbb{C}$ is a complex plane. Let $D = C_r = \{z : |z| < r\}$. An $E$-valued meromorphic function $f(z)$ in a domain $D \subset \mathbb{C}$ can be written as

$$f(z) = \sum_{j=1}^\infty f_j(z)e_j = (f_1(z), f_2(z), \ldots, f_j(z), \ldots),$$

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where each $f_j(z)$ is a complex-valued meromorphic functions in $D$. We now introduce the generalized quantities of the Nevanlinna theory (see [1]): For any $a \in E \cup \{\infty\}$, $n(r, a, f) = n(r, a)$ denotes the number of $a$-points of $f$ in $|z| \leq r$, counted with multiplicities and $n(r, \infty, f) = n(r, f)$ denote the number of poles of $f$ in $|z| \leq r$. Then we have the counting function of finite or infinite $a$-points as

$$N(r, a) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt,$$

$$N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt,$$

$$m(r, f) = m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|f(r e^{i\phi})\| d\phi,$$

$$m(r, a) = m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(r e^{i\phi})\| - a} d\phi, (a \neq \infty),$$

and

$$T(r, f) = m(r, f) + N(r, f),$$

where $\log^+ x = \max\{\log x, 0\}$. The volume function associated with $E$-valued meromorphic function $f$ is given by

$$V(r, a, f) = \frac{1}{2\pi} \int \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| d\sigma \wedge d\tau, \quad a \in E$$

and the curvature function is given by

$$V(r, 0, f') = G(r, f) = \int_0^r \frac{dt}{2\pi t} \int_{C_r} \Delta \log \|f'(\xi)\| d\sigma \wedge d\tau.$$

The order $\rho$ of an $E$-valued meromorphic function $f$ is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}$$

and the lower order $\lambda$ of $f$ is defined by

$$\lambda = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

We now define the following deficiencies as in [2]: For any $a \in E \cup \{\infty\}$, the number

$$\delta(a) = \delta(a, f) = \liminf_{r \to +\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{V(r, a) + N(r, a)}{T(r, f)}$$

is called the deficiency of the point $a$, a point $a$ with $\delta(a) > 0$ is called deficient.

The quantity

$$\theta(a) = \theta(a, f) = \liminf_{r \to +\infty} \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)}$$
is called the index of multiplicity of \( a \), and
\[
\Theta(a) = \Theta(a, f) = \liminf_{r \to +\infty} \frac{m(r, a) + N(r, a) - N(r, a)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{V(r, a) + N(r, a)}{T(r, f)}.
\]
In particular, we have
\[
\delta(\infty) = \liminf_{r \to +\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N(r, f)}{T(r, f)} \text{ since } V(r, \infty) = 0,
\]
\[
\theta(\infty) = \liminf_{r \to +\infty} \frac{N(r, f) - N(r, a)}{T(r, f)},
\]
\[
\Theta(\infty) = 1 - \limsup_{r \to +\infty} \frac{N(r, f)}{T(r, f)}.
\]
The quantity
\[
\delta_G = \delta_G(f) = \liminf_{r \to +\infty} \frac{G(r, f)}{T(r, f)}
\]
is called the Ricci Index of \( f(z) \).

THEOREM 1 ([1]). (\( E \)-valued Nevanlinna’s first fundamental theorem) Let \( f(z) \) be an \( E \)-valued meromorphic mapping in \( C_R \). Then for \( 0 < r < R, a \in E, f(z) \neq a, \)
\[ T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log \| c_q(a) \| + \epsilon(r, a). \]
Here \( \epsilon(r, a) \) is a function such that \( |\epsilon(r, a)| \leq \log^+ \| a \| + \log 2, \epsilon(r, 0) \equiv 0, \) and \( c_q(a) \in E \) is the co-efficient of the first term in the Laurent series at the point \( a \).

THEOREM 2 ([1]). (\( E \)-valued Nevanlinna’s second fundamental theorem) Let \( f(z) \) be a non-constant \( E \)-valued meromorphic mapping of compact projection in \( C_R \) and \( a^{[k]} \in E \cup \{ \infty \} \ (k = 1, 2, \ldots, q) \) be \( q \geq 3 \) distinct finite or infinite points. Then
\[ \sum_{k=1}^{q} m(r, a^{[k]}) + G(r, f) \leq T(r, f) - N_1(r) + S(r), \]
where \( N_1(r) = N(r, 0, f') + 2N(r, f) - N(r, f') \) and
\[ G(r, f) = \int_0^r \frac{dt}{2\pi i} \int_{C_t} \Delta \log \| f'(\xi) \| \, d\sigma \wedge d\tau. \]
If \( R = +\infty \), then \( S(r) \) satisfies \( S(r) = O \left\{ \log T(r, f) \right\} + O(\log r) \) as \( r \to +\infty \) without exception if \( f(z) \) has finite order and otherwise as \( r \to +\infty \) outside a set \( J \) of exceptional intervals of finite measure \( \int_J dr < +\infty \). If \( 0 < R < +\infty \), then
\[ S(r) = O \left\{ \log^+ T(r, f) \right\} + O \left\{ \log \frac{1}{R - r} \right\}. \]
holds as \( r \to R \) without exception if \( f \) has finite order
\[
\rho = \limsup_{r \to R} \frac{\log T(r, f)}{\log(1/R - r)},
\]
and otherwise as \( r \to R \) outside of a set \( J \) exceptional intervals such that \( \int_J \frac{1}{r - \sigma} \, d\sigma < +\infty \). In all cases, the exceptional set \( J \) is independent of the choice of the finite points \( a^{[k]} \in E \) and of their number.

**THEOREM 3 ([2]).** (\( E \)-valued Nevanlinna deficiency relation) Let \( f(z) \) be an \( E \)-valued meromorphic function and admissible with the property of compact projection. Then the set \( \{ a \in E \cup \{ \infty \} : \Theta(a) > 0 \} \) is at most countable and summing over all such points
\[
\sum_a [\delta(a) + \theta(a)] + \delta_G \leq \sum_a \Theta(a) + \delta_G \leq 2.
\]

**THEOREM 4 (Lemma 3.1(A) of [1])** Let \( f(z) \) be an \( E \)-valued meromorphic function with the property of compact projection, and let
\[
S_1(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{\|f(re^{i\phi})\|}{\|f(re^{i\phi}')\|} \right) d\phi + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{|F(re^{i\phi})|}{|f(re^{i\phi}')|} \right) d\phi
\]
\[
+ p \log^+ \frac{2p}{\delta} - \log \|f'(z)\|.
\]
Then
\[
G(r) + \sum_{k=1}^{p+1} m(r, a^{[k]}) + N_1(r) \leq 2T(r, f) + S_1(r),
\]
where \( N_1(r) = N(r, 0, f') + 2N(r, f) - N(r, f') \) is the generalized counting function of multiple points, \( a^{[p]} = (a_1^{[p]}, \ldots, a_j^{[p]}, \ldots) (p \geq 2) \in E \) are distinct finite points, and
\[
F(z) = \sum_{\nu=1}^{p} \frac{1}{\|f(z) - a^{[\nu]}\|}.
\]

### 3 Main Results

S. K. Singh and H. S. Gopalkrishna [4] proved the following result:

**THEOREM 5.** If \( f \) is a non-constant meromorphic function of order \( \rho \) then
\[
\liminf_{r \to \infty} \frac{T(r, f')} {T(r, f)} \geq \sum_{a \in \mathcal{C}} \Theta(a, f)
\]
where \( r \to \infty \) without restriction if \( \rho \) is finite and \( r \to \infty \) outside an exceptional set of finite measure if \( \rho = +\infty \).
In [5], Z. Wu and Y. Chen proved the following result.

**THEOREM 6.** Let \( f(z) \) be an admissible \( E \)-valued meromorphic function of compact projection in \( \mathbb{C} \) of finite order and assume \( \sum a_\delta = 2 \). Then

\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty).
\]

Now in this article, we obtain a **THEOREM 5** for \( E \)-valued meromorphic function \( f(z) \) in modified form and also extend the related results of S. K. Singh and H. S. Gopalakrishna [4]. **THEOREM 6** is also proved as a consequence of our main result.

We prove the following main results.

**THEOREM 7.** Let \( f(z) \) be an admissible and non-constant \( E \)-valued meromorphic function of finite order \( \rho \) with compact projection then

\[
\sum_{a \in E} \Theta(a, f) + \delta_G \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)},
\]

where \( r \to +\infty \) without restriction if \( \rho \) is finite and \( r \to +\infty \) outside an exceptional set of finite measure if \( \rho = +\infty \).

To prove **THEOREM 7**, we first prove the following Lemma, which plays an prominent role in the proof of the **THEOREM 7**.

**LEMMA 1.** Let \( f(z) \) be a non-constant \( E \)-valued meromorphic function with the property of compact projection in \( C_r \) and

\[
a^{[\gamma]} = (a_1^{[\gamma]}, a_2^{[\gamma]}, \ldots, a_j^{[\gamma]}, \ldots) \quad (p \geq 2) \in E
\]

are finite or infinite distinct points then

\[
\sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) + N \left( r, \frac{1}{f'} \right) + G(r, f) \leq T(r, f') + S(r, f),
\]

where

\[
S(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ \{ F(re^{i\phi}) \| f''(\phi) \| \} \, d\phi - \log \| c' \| + p \log^+ \frac{2p}{\delta}
\]

and

\[
F(z) = \sum_{\nu=1}^{p} \frac{1}{\| f(z) - a^{[\nu]} \|}.
\]

**PROOF.** Following the proof of Lemma 3.1(A) in [1], we obtain the required result.
PROOF OF THEOREM 7. Let \( \{a^{[\mu]}\}, \mu = 1, 2, \ldots, \infty \) be an infinite sequence of distinct elements of \( E \), which includes every \( a \in E \) for which \( \Theta(a, f) > 0 \). Then

\[
\sum_{\mu=1}^{\infty} \Theta (a^{[\mu]}, f) = \sum_{a \in E} \Theta(a, f). \tag{1}
\]

We have

\[
\sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) + G(r, f) \leq T(r, f') - N \left( r, \frac{1}{f'} \right) + S(r, f).
\]

Adding \( \sum_{\mu=1}^{p} N \left( r, a^{[\mu]}, f \right) \) to both sides, we obtain

\[
\sum_{\mu=1}^{p} T(r, a^{[\mu]}, f) + G(r, f) \leq T(r, f') + \sum_{\mu=1}^{p} N \left( r, a^{[\mu]}, f \right) - N \left( r, \frac{1}{f'} \right) + S(r, f)
\]

where \( N \left( r, \frac{1}{f'} \right) \) is formed with the zeros of \( f' \) which are not zeros of any of \( f - a^{[\mu]} \) \( (\mu = 1, 2, \ldots, p) \). Since \( N \left( r, \frac{1}{f'} \right) \geq 0 \), we have

\[
\sum_{\mu=1}^{p} T(r, a^{[\mu]}, f) \leq T(r, f') + \sum_{\mu=1}^{p} N \left( r, a^{[\mu]}, f \right) - G(r, f) + S(r, f).
\]

By an \( E \)-valued Nevanlinna’s first fundamental theorem, we have

\[
T(r, a, f) = T(r, f) - V(r, a, f) + O(1).
\]

Using this in the above equation, we obtain

\[
\sum_{\mu=1}^{p} \left[ T(r, f) - V(r, a^{[\mu]}, f) + O(1) \right] \leq T(r, f') + \sum_{\mu=1}^{p} N \left( r, a^{[\mu]}, f \right) - G(r, f) + S(r, f).
\]

We further obtain

\[
pT(r, f) \leq T(r, f') + \sum_{\mu=1}^{p} \left[ \sum_{\mu=1}^{p} \left( N \left( r, a^{[\mu]}, f \right) + V(r, a^{[\mu]}, f) \right) \right] - G(r, f) + S(r, f).
\]

Then

\[
p \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{\mu=1}^{p} \limsup_{r \to +\infty} \frac{N \left( r, a^{[\mu]}, f \right) + V(r, a^{[\mu]}, f)}{T(r, f)} - \liminf_{r \to +\infty} \frac{G(r, f)}{T(r, f)}
\]

\[
+ \limsup_{r \to +\infty} \frac{S(r, f)}{T(r, f)}.
\]
It follows that
\[ p \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{\mu=1}^{p} \left[ 1 - \Theta(a^{[\mu]}, f) \right] - \delta_G(f). \]

So
\[ \sum_{\mu=1}^{p} \Theta(a^{[\mu]}, f) + \delta_G(f) \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)}. \]

Letting \( p \to \infty \) and using (1), we get
\[ \sum_{a \in E} \Theta(a, f) + \delta_G(f) \leq \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \tag{2} \]

**COROLLARY 1.** Let \( f(z) \) be an admissible \( E \)-valued meromorphic function of finite order \( \rho \) with the property of compact projection such that
\[ \sum_{a \in \mathcal{E}} \Theta(a, f) + \delta_G = 2, \quad \mathcal{E} = E \cup \{\infty\}. \]

Then
\[ \lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f). \]

(ii) \[ 1 - \Theta(a, f) + \delta_G \leq \liminf_{r \to +\infty} \frac{V(r, a) + \overline{N}(r, a)}{T(r, f)} \leq \limsup_{r \to +\infty} \frac{V(r, a) + \overline{N}(r, a)}{T(r, f)} = 1 - \Theta(a, f). \]

**PROOF.** Given that
\[ \sum_{a \in \mathcal{E}} \Theta(a, f) + \delta_G = 2, \]
we have
\[ \sum_{a \in \mathcal{E}} \Theta(a, f) + \Theta(\infty, f) + \delta_G = 2. \]

It follows that
\[ \sum_{a \in \mathcal{E}} \Theta(a, f) + \delta_G = 2 - \Theta(\infty, f). \]

Using (2), we write
\[ \liminf_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \geq \sum_{a \in \mathcal{E}} \Theta(a, f) + \delta_G = 2 - \Theta(\infty, f). \]
On the other hand, we know that
\[
T(r, f') = m(r, f') + N(r, f') = m(r, \frac{f'}{f}) + m(r, f) + N(r, f')
\]
and
\[
\limsup_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \leq 1 + \limsup_{r \to +\infty} \frac{N(r, f)}{T(r, f)}.
\]
So
\[
\limsup_{r \to +\infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \Theta(\infty, f).
\]
Thus
\[
\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f).
\]

(ii) Let \( a \in E \cup \{\infty\} \) and \( \{a[k]\}, k = 1, 2, \ldots, \infty \) be an infinite sequence of distinct elements of \( E \cup \{\infty\} \) which includes every \( b \in E \cup \{\infty\} \) such that \( b \neq a \) and \( \Theta(b, f) \neq 0 \). Then
\[
\sum_{k=1}^{\infty} \Theta(a[k], f) = \sum_{b \in E, b \neq a} \Theta(b, f) = 2 - \Theta(a, f). \tag{3}
\]
By \( E \)-valued Nevanlinna’s second fundamental theorem, we have
\[
(q-2)T(r, f) + G(r, f) \leq \sum_{k=1}^{q-1} \left[ V(r, a[k], f) + \frac{N(r, a[k], f)}{T(r, f)} \right]
\]
\[
+ \left[ V(r, a, f) + \frac{N(r, a, f)}{T(r, f)} \right] + S(r, f),
\]
\[
(q-2)T(r, f) \leq \sum_{k=1}^{q-1} \left[ V(r, a[k], f) + \frac{N(r, a[k], f)}{T(r, f)} \right] + \frac{G(r, f)}{T(r, f)} + \frac{S(r, f)}{T(r, f)},
\]
\[
(q-2) \leq \sum_{k=1}^{q-1} \limsup_{r \to +\infty} \frac{\left[ V(r, a[k], f) + \frac{N(r, a[k], f)}{T(r, f)} \right]}{T(r, f)}
\]
\[
+ \liminf_{r \to +\infty} \frac{\left[ V(r, a, f) + \frac{N(r, a, f)}{T(r, f)} \right]}{T(r, f)} - \liminf_{r \to +\infty} \frac{G(r, f)}{T(r, f)} + \limsup_{r \to +\infty} \frac{S(r, f)}{T(r, f)}.
\]
\[(q - 2) \leq \lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} + \sum_{k=1}^{q-1} [1 - \Theta \left( a^{[k]}, f \right)] - \delta_G, \]

\[(q - 2) + \delta_G \leq \lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} + (q - 1) - \sum_{k=1}^{q-1} \Theta \left( a^{[k]}, f \right), \]

\[\delta_G - 1 \leq \lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} - \sum_{k=1}^{q-1} \Theta \left( a^{[k]}, f \right). \]

So

\[\lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} \geq \sum_{k=1}^{q-1} \Theta \left( a^{[k]}, f \right) + \delta_G - 1. \]

Let \( q \to \infty \) and using (3), we get

\[\lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} \geq \sum_{k=1}^{\infty} \Theta \left( a^{[k]}, f \right) + \delta_G - 1 \]

\[= 2 - \Theta(a, f) + \delta_G - 1 = 1 - \Theta(a, f) + \delta_G. \]

On the other hand, by the definition of \( \Theta(a, f) \), we have

\[\lim_{r \to +\infty} \sup \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} = 1 - \Theta(a, f). \]

Thus

\[1 - \Theta(a, f) + \delta_G \leq \lim_{r \to +\infty} \inf \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} \]

\[\leq \lim_{r \to +\infty} \sup \frac{V(r, a, f) + \overline{N}(r, a, f)}{T(r, f)} = 1 - \Theta(a, f). \]

**COROLLARY 2** Let \( f(z) \) be a admissible \( E \)-valued meromorphic function of finite order \( \rho \) with the property of compact projection such that

\[\sum_{a \in E} \delta(a, f) + \delta_G = 2. \]

Then

\[\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty, f). \]

**PROOF.** We know that \( \delta(a, f) \leq \Theta(a, f), \forall a \in E \cup \{\infty\} = E \) and

\[\sum_{a \in E} \delta(a, f) + \delta_G \leq \sum \Theta(a, f) + \delta_G \leq 2. \]
Given $\sum \delta(a, f) + \delta_G = 2$. Then $\sum \Theta(a, f) + \delta_G = 2$. We observe that

$$\sum \delta(a, f) + \delta_G = \sum \Theta(a, f) + \delta_G = 2.$$ 

Then

$$\sum_{a \in \mathbb{E}} \delta(a, f) = \sum_{a \in \mathbb{E}} \Theta(a, f).$$

So

$$\delta(a, f) = \Theta(a, f) \quad \forall a \in \mathbb{E}.$$ 

By using Corollary 1(i), we have

$$\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f) = 2 - \delta(\infty, f).$$ 

So

$$\lim_{r \to +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty, f)$$

References


