Mapping Properties Of The General Integral Operator On The Classes $\mathcal{R}_k(\rho, b)$ And $\mathcal{V}_k(\rho, b)$

Trailokya Panigrahi, Gangadharan Murugusundaramoorthy

Received 23 May 2014

Abstract

In this paper, we investigate some mapping properties of two new subclasses of analytic function classes $\mathcal{R}_k(\rho, b)$ and $\mathcal{V}_k(\rho, b)$ under generalized integral operator. Several (known or new) consequences of the results are also pointed out.

1 Introduction

Let $A$ denote the family of all functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$. A function $f \in A$ is said to be starlike of complex order $b$ ($b \in \mathbb{C} \setminus \{0\}$) and type $\delta$ ($0 \leq \delta < 1$), denoted by $S^{\ast}_\delta(b)$ (see [6]) if and only if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > \delta \quad (z \in U).$$

A function $f \in A$ is said to be convex of complex order $b$ ($b \in \mathbb{C} \setminus \{0\}$) and type $\delta$ ($0 \leq \delta < 1$), denoted by $C_\delta(b)$ (see [6]) if and only if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > \delta \quad (z \in U).$$

For $b = 1$, $S^{\ast}_1(1) = S^{\ast}(\delta)$ and $C_1(1) = C(\delta)$, the classes of functions that are starlike of order $\delta$ and convex of order $\delta$ in $U$, respectively. Clearly,

$C(\delta) \subset S^{\ast}(\delta) \quad (0 \leq \delta < 1)$.

*Mathematics Subject Classification: 30C45.

†Department of Mathematics, School of applied sciences, KIIT university, Bhubaneswar-751024, Orissa, India

‡corresponding author, School of Advanced Science, VIT University, Vellore-632014, Tamil Nadu, India
Notice that $S^*_0(b) = S^*(b)$ and $C_0(b) = C(b)$, the classes considered earlier by Nasr and Aouf [8] and Wiatrowski [13].

Let $P_k(\rho)$ denote the class of functions $p : \mathcal{U} \rightarrow \mathbb{C}$, analytic in $\mathcal{U}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re(p(z)) - \rho}{1 - \rho} \right| \, d\theta \leq k\pi,$$  

(4)

where $z = re^{i\theta} \in \mathcal{U}$, $k \geq 2$ and $0 \leq \rho < 1$. For $\rho = 0$, we obtain the class $P_k$ defined and studied in [12]. For $\rho = 0$, $k = 2$, we obtain the well known class $P$ of functions with positive real part and the class $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than $\rho$. For $k > 2$, the functions in $P_k$ may not have positive real part. It is easy to see that $p \in P_k(\rho)$ if and only if there exists $h \in P_k$ such that

$$p(z) = (1 - \rho)h(z) + \rho.$$

Also, from (4), we note that $p \in P_k(\rho)$ if and only if there exists $p_1, p_2 \in P_k(\rho)$ such that

$$p(z) = \left( \frac{k}{4} + \frac{1}{2} \right)p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right)p_2(z).$$

It is well-known that the class $P_k(\rho)$ is a convex set (see [9]).

**DEFINITION. 1** A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_k(\rho, b)$ ($b \in \mathbb{C} \setminus \{0\}$) if and only if

$$1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \in P_k(\rho) \quad (k \geq 2, \ 0 \leq \rho < 1).$$  

(5)

Notice that $\mathcal{R}_k(0, 1) = \mathcal{R}_k$, which is the class of functions with bounded radius rotation.

**DEFINITION. 2** A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{V}_k(\rho, b)$ ($b \in \mathbb{C} \setminus \{0\}$) if and only if

$$1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \in P_k(\rho) \quad (k \geq 2, \ 0 \leq \rho < 1).$$  

(6)

We note that $\mathcal{V}_k(0, 1) \equiv \mathcal{V}_k$, the class of functions with bounded boundary rotation first discussed by Paatero [2]. It is clear that

$$f(z) \in \mathcal{V}_k(\rho, b) \iff zf'(z) \in \mathcal{R}_k(\rho, b).$$  

(7)

Recently, Frasin [7] introduced the following generalized integral operators:

Let $\alpha_i, \beta_i \in \mathbb{C}$ for all $i = 1, 2, 3, \ldots, n$, $n \in \mathbb{N}$, $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. Let $\mathcal{I}_\gamma^{\alpha_i, \beta_i} : \mathcal{A}^n \rightarrow \mathcal{A}$ be the integral operator defined by

$$\mathcal{I}_\gamma^{\alpha_i, \beta_i}(f_1, f_2, \ldots, f_n)(z) = \left\{ \int_0^z t^{\gamma - 1} \left( f_1'(t) \right)^{\alpha_1} \left( \frac{f_1(t)}{t} \right)^{\beta_1} \ldots \left( f_n'(t) \right)^{\alpha_n} \left( \frac{f_n(t)}{t} \right)^{\beta_n} \, dt \right\}^{\frac{1}{\alpha}}.$$  

(8)

where the power is taken as the principal one.

Notice that, the integral operator $\mathcal{I}_\gamma^{\alpha_i, \beta_i}(f_1, f_2, \ldots, f_n)(z)$ generalizes several previously studied operators as follows:
• For $\alpha_i = 0$ for all $i = 1, 2, 3, ..., n$, the integral operator

$$\mathcal{I}^{0, \beta_i}(f_1, f_2, ..., f_n)(z) = \mathcal{I}_\gamma(f_1, f_2, ..., f_n)(z),$$

where

$$\mathcal{I}_\gamma(f_1, f_2, ..., f_n)(z) = \left\{ \int_0^z \gamma t^{\gamma-1} \left( \frac{f_1(t)}{t} \right)^{\beta_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\beta_n} \, dt \right\}^{\frac{1}{\gamma}}$$

is introduced and studied by Breaz and Breaz [3].

• For $\alpha_i = 0$ for all $i = 1, 2, 3, ..., n$ and $\gamma = 1$, the integral operator

$$\mathcal{I}^{0, \beta_i}(f_1, f_2, ..., f_n)(z) = \mathcal{F}_n(z),$$

where

$$\mathcal{F}_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\beta_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\beta_n} \, dt$$

is introduced and studied by Breaz and Breaz [3].

• For $\beta_i = 0$ for all $i = 1, 2, 3, ..., n$ and $\gamma = 1$, the integral operator

$$\mathcal{I}^{\alpha_i, 0}(f_1, f_2, ..., f_n)(z) = \mathcal{F}_{\alpha_1, \alpha_2, ..., \alpha_n}(z),$$

where

$$\mathcal{F}_{\alpha_1, \alpha_2, ..., \alpha_n}(z) = \int_0^z (f'_1(t))^{\alpha_1} \cdots (f'_n(t))^{\alpha_n} \, dt$$

is introduced and studied by Breaz et al. [5].

Recently, Breaz and Güney [4] considered the integral operators

$$\mathcal{F}_n(z) \text{ and } \mathcal{F}_{\alpha_1, \alpha_2, ..., \alpha_n}(z)$$

and obtained their properties on the classes $S^*_\gamma(b)$ and $C_\delta(b)$ of starlike and convex functions of complex order $b$ and type $\delta$. Later on Noor et al. [10] considered the same integral operators and investigated the mapping properties for the classes $V_k(\rho, b)$ and $R_k(\rho, b)$.

Motivated by the aforementioned work, in this paper, the author investigates some mapping properties of the classes $R_k(\rho, b)$ and $V_k(\rho, b)$ under generalized integral operator defined in (8) when $\gamma = 1$. The results obtain in this paper are generalized results of Breaz and Güney [4] and Noor et al. [10].

2 Main Results

We recall the following lemma which is useful for our investigation:
LEMMA 1 (see [11]). Let \( f(z) \in \mathcal{V}_k(\alpha), 0 \leq \alpha < 1 \). Then \( f(z) \in \mathcal{R}_k(\beta) \) where
\[
\beta = \frac{1}{4} \left[ (2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9} \right].
\] (12)

In this section we prove the following:

THEOREM 1. Let \( f_i, \phi_i \in \mathcal{R}_k(\rho, b) \) for all \( i = 1, 2, 3, \ldots, n \) with \( 0 \leq \rho < 1 \), \( b \in \mathbb{C} \setminus \{0\} \) and \( \alpha_i, \beta_i \in \mathbb{R}^+ \) for \( 1 \leq i \leq n \). If
\[
0 \leq (\rho - 1)n + (\rho - 1) \sum_{i=1}^{n} \beta_i + 1 < 1,
\] (13)
then the integral operator
\[
I_{\alpha_i, \beta_i}^0(f_1, f_2, \ldots, f_n)(z) = \int_0^z \Pi_{i=1}^{n} (f_i'(t))^{\alpha_i} \left( \frac{f_i(t)}{t} \right)^{\beta_i} dt
\] (14)
belong to the class \( \mathcal{V}_k(\chi, b) \) with
\[
\chi = 1 + (\rho - 1)n + (\rho - 1) \sum_{i=1}^{n} \beta_i.
\] (15)

PROOF. For the sake of simplicity, in the proof, we shall write \( H(z) \) instead of \( I_{\alpha_i, \beta_i}^0(f_1, f_2, \ldots, f_n)(z) \). Differentiating (14) with respect to \( z \), we obtain
\[
H'(z) = \Pi_{i=1}^{n} (f_i'(z))^{\alpha_i} \left( \frac{f_i(z)}{z} \right)^{\beta_i}
\] (16)
Let us define
\[
\phi_i(z) = z(f_i'(z))^{\alpha_i}.
\] (17)
Clearly, \( \phi_i(z) \in \mathcal{A} \). Equation (16) becomes
\[
H'(z) = \Pi_{i=1}^{n} \frac{\phi_i(z)}{z^{n}} \left( \frac{f_i(z)}{z} \right)^{\beta_i}
\] (18)
Logarithmic differentiation of (18) yields
\[
\frac{\mathcal{H}''(z)}{\mathcal{H}'(z)} = \sum_{i=1}^{n} \left[ \beta_i \left( \frac{f_i'(z)}{f_i(z)} - \frac{1}{z} \right) + \left( \frac{\phi_i'(z)}{\phi_i(z)} - \frac{1}{z} \right) \right].
\] (19)
Multiplying (19) by \( z \) and simplifying we get
\[
1 + \frac{1}{b} \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} = 1 - n - \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \left\{ \frac{1}{b} \left( b \frac{z\phi_i'(z)}{\phi_i(z)} - 1 \right) \left( 1 + \frac{1}{b} \left( z\phi_i'(z) - 1 \right) \right) \right\}.
\] (20)
Adding and subtracting $\rho$ on the right hand side of (20) gives

$$
\left[ \left( 1 + \frac{1}{b} \mathcal{H}\,^n(z) \right) - \chi \right] = \sum_{i=1}^{n} \beta_i \left[ \left( 1 + \frac{1}{b} \left( \frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \rho \right]
+ \sum_{i=1}^{n} \left[ \left( 1 + \frac{1}{b} \left( \frac{\phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \rho \right],
$$

(21)

where $\chi$ is given by (15). Taking real part of (21) and after simplification, we get

$$\int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \left( \frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \rho \right] d\theta \leq \sum_{i=1}^{n} \beta_i \int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \left( \frac{\phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \rho \right] d\theta + \sum_{i=1}^{n} \int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \left( \frac{\phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \rho \right] d\theta.
$$

(22)

Since, by hypothesis, $f_i, \phi_i \in \mathcal{R}_k(\rho, b)$ for $1 \leq i \leq n$, we have

$$\int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \left( \frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \rho \right] d\theta \leq (1 - \rho)k\pi
$$

(23)

and

$$\int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \left( \frac{\phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \rho \right] d\theta \leq (1 - \rho)k\pi.
$$

(24)

Making use of (23) and (24) in (22), we have

$$\int_{0}^{2\pi} \mathcal{R} \left[ \left( 1 + \frac{1}{b} \mathcal{H}\,^n(z) \right) - \chi \right] d\theta \leq (1 - \chi)k\pi,
$$

(25)

where $\chi$ is given by (15). Hence $\mathcal{H}(z) \in \mathcal{V}_k(\chi, b)$. Thus, the proof of Theorem 1 is completed.

Put $\alpha_i = 0$ for all $i = 1, 2, 3, ..., n$ in Theorem 1. Notice that, in such case $\mathcal{H}(z) = \mathcal{F}_n(z)$ and $\phi_i(z) = z$ which shows

$$\frac{z \phi_i(z)}{\phi_i(z)} - 1 = 0.
$$

Therefore, from (21), it follows that

$$\left[ 1 + \frac{1}{b} \mathcal{F}_n^{\prime\prime}(z) \right] = \sum_{i=1}^{n} \beta_i \left[ \left( 1 + \frac{1}{b} \left( \frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \rho \right],
$$

where $\lambda = 1 + (\rho - 1) \sum_{i=1}^{n} \beta_i$. 

Hence we have the following Corollary 1.

**COROLLARY 1** (cf. [10, Theorem 2.1]). Let \( f_i(z) \in \mathcal{R}_k(\rho, b) \) for \( 1 \leq i < n \) with \( 0 \leq \rho < 1 \) and \( b \in \mathbb{C} \setminus \{0\} \). Also let \( \beta_i > 0 \), \( 1 \leq i \leq n \). If

\[
0 \leq (\rho - 1) \sum_{i=1}^{n} \beta_i + 1 < 1,
\]

then \( \mathcal{F}_n(z) \in \mathcal{V}_k(\lambda, b) \) with \( \lambda = (\rho - 1) \sum_{i=1}^{n} \beta_i + 1 \).

Next, we take \( \beta_i = 0 \), \( 1 \leq i \leq n \) in Theorem 1. In this case,

\[
\mathcal{H}(z) = \mathcal{F}_{\alpha_1, \alpha_2, \ldots, \alpha_n}(z) = \int_{0}^{z} \Pi_{i=1}^{n} (f_i'(z))^{\alpha_i},
\]

which implies \( \mathcal{H}'(z) = \Pi_{i=1}^{n} \frac{\phi_i(z)}{z^{\alpha_i}} \). Therefore,

\[
\frac{z^{H''(z)}}{H'(z)} = \sum_{i=1}^{n} \left( \frac{z^{\phi_i'(z)}}{\phi_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \frac{zf_i''(z)}{f_i(z)}.
\]

We have the following result due to Noor et al. [10].

**COROLLARY 2** (cf. [10, Theorem 2.5]). Let \( f_i(z) \in \mathcal{V}_k(\rho, b) \) for \( 1 \leq i \leq n \) with \( 0 \leq \rho < 1 \) and \( b \in \mathbb{C} \setminus \{0\} \). Also, let \( \alpha_i > 0 \), \( 1 \leq i \leq n \). If

\[
0 \leq (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,
\]

then \( \mathcal{F}_{\alpha_1, \alpha_2, \ldots, \alpha_n}(z) \in \mathcal{V}_k(\lambda_1, b) \) with \( \lambda_1 = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 \).

**REMARK 1.** Setting \( k = 2 \) in Corollary 1, we obtain the results of [4, Theorem 1] and [10, Corollary 2.2].

**REMARK 2.** Setting \( k = 2 \) in Corollary 2, we obtain another results of [4, Theorem 3] and [10, Corollary 2.6].

**THEOREM 2.** Let \( f_i, \phi_i \in \mathcal{V}_k(\rho, 1) \) for \( 1 \leq i < n \) with \( 0 \leq \rho < 1 \). Let \( \alpha_i, \beta_i \in \mathcal{R}^+ \), \( 1 \leq i \leq n \). If

\[
0 \leq (\beta - 1) \sum_{i=1}^{n} (1 + \beta_i) + 1 < 1,
\]

then \( \mathcal{H}(z) \in \mathcal{V}_k(l, 1) \) with \( l = 1 + (\beta - 1) \sum_{i=1}^{n} (1 + \beta_i) \) and \( \beta \) is given by (12).

**PROOF:** Proceeding as Theorem 1 with \( b = 1 \), we have

\[
\int_{0}^{2\pi} \left| \mathbb{R} \left[ 1 + \frac{z^{H''}(z)}{H'(z)} - l \right] \right| d\theta \leq \sum_{i=1}^{n} \beta_i \int_{0}^{2\pi} \left| \mathbb{R} \left[ \frac{zf_i''(z)}{f_i(z)} - \beta \right] \right| d\theta + \sum_{i=1}^{n} \int_{0}^{2\pi} \left| \mathbb{R} \left[ \frac{z^{\phi_i'(z)}}{\phi_i(z)} - \beta \right] \right| d\theta.
\]
Since \( f_i, \phi_i \in V_k(\rho, l) \) for \( 1 \leq i \leq n \), and by using Lemma 1, we have
\[
\sum_{i=1}^{n} \int_{0}^{2\pi} \left| \mathbb{R} \left[ \frac{zf_i'(z)}{f_i'(z)} - \beta \right] \right| \, d\theta \leq (1 - \beta)k\pi \tag{28}
\]
and
\[
\sum_{i=1}^{n} \int_{0}^{2\pi} \left| \mathbb{R} \left[ \frac{z\phi_i'(z)}{\phi_i'(z)} - \beta \right] \right| \, d\theta \leq (1 - \beta)k\pi, \tag{29}
\]
where \( \beta \) is given by (12) with \( \alpha = \rho \). Using (28) and (29) in (27), we obtain
\[
\int_{0}^{2\pi} \left| \mathbb{R} \left[ 1 + \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} - l \right] \right| \, d\theta \leq (1 - l)k\pi. \tag{30}
\]
Thus, \( \mathcal{H}(z) \in V_k(l, 1) \) with \( l = 1 + (\beta - 1)\sum_{i=1}^{n}(1 + \beta_i) \). The proof of Theorem 2 is completed.

**REMARK 3.** For \( \alpha_i = 0 \) we obtain the result of ([10, Theorem 2.3]).

For \( n = 1, \alpha_1 = 0, \beta_1 = 1, f_1 = f \) in Theorem 2, we get the following results due to [10].

**COROLLARY 3.** [10] Let \( f(z) \in V_k(\rho, 1) \). Then the Alexander operator \( \mathcal{I}(z) = \int_{0}^{z} \frac{f(t)}{t} \, dt \) (see [1]) belongs to the class \( V_k(\beta) \), where \( \beta \) is given by (24).

**REMARK 4.** For \( \rho = 0 \) and \( k = 2 \) in the above Corollary 2, we have the well known result \( f(z) \in C(0) \Rightarrow \mathcal{I}(z) \in C \left( \frac{1}{2} \right) \).

**Acknowledgements.** We record our sincere thanks to the referees for the valuable suggestions to improve our results.

**References**


