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Mapping Properties Of The General Integral Operator On The Classes $\mathcal{R}_k(\rho, b)$ And $\mathcal{V}_k(\rho, b)$*

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Abstract

In this paper, we investigate some mapping properties of two new subclasses of analytic function classes $\mathcal{R}_k(\rho, b)$ and $\mathcal{V}_k(\rho, b)$ under generalized integral operator. Several (known or new) consequences of the results are also pointed out.

1 Introduction

Let $\mathcal{A}$ denote the family of all functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$. A function $f \in \mathcal{A}$ is said to be starlike of complex order b ($b \in \mathbb{C} \setminus \{0\}$) and type δ ($0 \le \delta < 1$), denoted by $\mathcal{S}^*_\delta(b)$ (see [6]) if and only if

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \delta \quad (z \in \mathcal{U}). \tag{2}$$

A function $f \in \mathcal{A}$ is said to be convex of complex order b ($b \in \mathbb{C} \setminus \{0\}$) and type δ ($0 \le \delta < 1$), denoted by $\mathcal{C}_\delta(b)$ (see [6]) if and only if

$$\Re\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \delta \quad (z \in \mathcal{U}). \tag{3}$$

For $b = 1$, $\mathcal{S}^*_\delta(1) = \mathcal{S}^*(\delta)$ and $\mathcal{C}_\delta(1) = \mathcal{C}(\delta)$, the classes of functions that are starlike of order δ and convex of order δ in $\mathcal{U}$, respectively. Clearly,

$$\mathcal{C}(\delta) \subset \mathcal{S}^*(\delta) \quad (0 \le \delta < 1).$$

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Notice that $\mathcal{S}_0^*(b) = \mathcal{S}^*(b)$ and $\mathcal{C}_0(b) = \mathcal{C}(b)$, the classes considered earlier by Nasr and Aouf [8] and Wiatrowski [13].

Let $P_k(\rho)$ denote the class of functions $p : \mathcal{U} \longrightarrow \mathbb{C}$, analytic in $\mathcal{U}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re(p(z)) - \rho}{1-\rho} \right| d\theta \leq k\pi, \tag{4}$$

where $z = re^{i\theta} \in \mathcal{U}$, $k \geq 2$ and $0 \leq \rho < 1$. For $\rho = 0$, we obtain the class P_k defined and studied in [12]. For $\rho = 0,\ k = 2$, we obtain the well known class P of functions with positive real part and the class $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ. For $k > 2$, the functions in P_k may not have positive real part. It is easy to see that $p \in P_k(\rho)$ if and only if there exists $h \in P_k$ such that

$$p(z) = (1-\rho)h(z) + \rho.$$

Also, from (4), we note that $p \in P_k(\rho)$ if and only if there exists $p_1, p_2 \in P_k(\rho)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

It is well-known that the class $P_k(\rho)$ is a convex set (see [9]).

DEFINITION. 1 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_k(\rho, b)$ $\ (b \in \mathbb{C} \setminus \{0\})$ if and only if

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \in P_k(\rho) \quad (k \geq 2,\ 0 \leq \rho < 1). \tag{5}$$

Notice that $\mathcal{R}_k(0,1) = \mathcal{R}_k$, which is the class of functions with bounded radius rotation.

DEFINITION. 2 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{V}_k(\rho, b)$ $\ (b \in \mathbb{C} \setminus \{0\})$ if and only if

$$1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \in P_k(\rho) \quad (k \geq 2,\ 0 \leq \rho < 1). \tag{6}$$

We note that $\mathcal{V}_k(0,1) \equiv \mathcal{V}_k$, the class of functions with bounded boundary rotation first discussed by Paatero [2]. It is clear that

$$f(z) \in \mathcal{V}_k(\rho, b) \Longleftrightarrow zf'(z) \in \mathcal{R}_k(\rho, b). \tag{7}$$

Recently, Frasin [7] introduced the following generalized integral operators:

Let $\alpha_i,\ \beta_i \in \mathbb{C}$ for all $i = 1, 2, 3, ..., n,\ n \in \mathbb{N},\ \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. Let $\mathcal{I}_\gamma^{\alpha_i, \beta_i} : \mathcal{A}^n \longrightarrow \mathcal{A}$ be the integral operator defined by

$$\begin{aligned} &\mathcal{I}_\gamma^{\alpha_i,\beta_i}(f_1, f_2, ..., f_n)(z) \\ &= \left\{ \int_0^z \gamma t^{\gamma-1} \left(f_1'(t)\right)^{\alpha_1} \left(\frac{f_1(t)}{t}\right)^{\beta_1} ......\left(f_n'(t)\right)^{\alpha_n} \left(\frac{f_n(t)}{t}\right)^{\beta_n} dt \right\}^{\frac{1}{\gamma}}, \end{aligned} \tag{8}$$

where the power is taken as the principal one.

Notice that, the integral operator $\mathcal{I}_\gamma^{\alpha_i,\beta_i}(f_1, f_2, ..., f_n)(z)$ generalizes several previously studied operators as follows:

- For $\alpha_i = 0$ for all $i = 1, 2, 3, ..., n$, the integral operator

$$\mathcal{I}_{\gamma}^{0,\beta_i}(f_1, f_2, ..., f_n)(z) = \mathcal{I}_{\gamma}(f_1, f_2, ..., f_n)(z),$$

where

$$\mathcal{I}_{\gamma}(f_1, f_2, ..., f_n)(z) = \left\{ \int_0^z \gamma t^{\gamma-1} \left(\frac{f_1(t)}{t} \right)^{\beta_1} .... \left(\frac{f_n(t)}{t} \right)^{\beta_n} dt \right\}^{\frac{1}{\gamma}} \quad (9)$$

is introduced and studied by Breaz and Breaz [3].

- For $\alpha_i = 0$ for all $i = 1, 2, 3, ..., n$ and $\gamma = 1$, the integral operator

$$\mathcal{I}_1^{0,\beta_i}(f_1, f_2, ..., f_n)(z) = \mathcal{F}_n(z),$$

where

$$\mathcal{F}_n(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\beta_1} .... \left(\frac{f_n(t)}{t} \right)^{\beta_n} dt \quad (10)$$

is introduced and studied by Breaz and Breaz [3].

- For $\beta_i = 0$ for all $i = 1, 2, 3, ..., n$ and $\gamma = 1$, the integral operator

$$\mathcal{I}_1^{\alpha_i,0}(f_1, f_2, ..., f_n)(z) = \mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n}(z),$$

where

$$\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} ..... (f_n'(t))^{\alpha_n} dt \quad (11)$$

is introduced and studied by Breaz et al. [5].

Recently, Breaz and Güney [4] considered the integral operators

$$\mathcal{F}_n(z) \text{ and } \mathcal{F}_{\alpha_1,\alpha_2,....,\alpha_n}(z)$$

and obtained their properties on the classes $\mathcal{S}_{\delta}^{*}(b)$ and $\mathcal{C}_{\delta}(b)$ of starlike and convex functions of complex order b and type δ. Later on Noor et al. [10] considered the same integral operators and investigated the mapping properties for the classes $\mathcal{V}_k(\rho, b)$ and $\mathcal{R}_k(\rho, b)$.

Motivated by the aforementioned work, in this paper, the author investigates some mapping properties of the classes $\mathcal{R}_k(\rho, b)$ and $\mathcal{V}_k(\rho, b)$ under generalized integral operator defined in (8) when $\gamma = 1$. The results obtain in this paper are generalized results of Breaz and Güney [4] and Noor et al. [10].

2 Main Results

We recall the following lemma which is useful for our investigation:

LEMMA 1 (see [11]). Let $f(z) \in \mathcal{V}_k(\alpha)$, $0 \leq \alpha < 1$. Then $f(z) \in \mathcal{R}_k(\beta)$ where

$$\beta = \frac{1}{4}\left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9}\right]. \tag{12}$$

In this section we prove the following:

THEOREM 1. Let $f_i,\ \phi_i \in \mathcal{R}_k(\rho, b)$ for all $i = 1, 2, 3, ..., n$ with $0 \leq \rho < 1$, $b \in \mathbb{C} \setminus \{0\}$ and $\alpha_i,\ \beta_i \in \mathbb{R}^+$ for $1 \leq i \leq n$. If

$$0 \leq (\rho - 1)n + (\rho - 1)\sum_{i=1}^{n} \beta_i + 1 < 1, \tag{13}$$

then the integral operator

$$\mathcal{I}_1^{\alpha_i,\ \beta_i}(f_1, f_2, ..., f_n)(z) = \int_0^z \Pi_{i=1}^n \left(f_i'(t)\right)^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\beta_i} dt \tag{14}$$

belong to the class $\mathcal{V}_k(\chi, b)$ with

$$\chi = 1 + (\rho - 1)n + (\rho - 1)\sum_{i=1}^{n} \beta_i. \tag{15}$$

PROOF. For the sake of simplicity, in the proof, we shall write $H(z)$ instead of $\mathcal{I}_1^{\alpha_i,\ \beta_i}(f_1, f_2, ..., f_n)(z)$. Differentiating (14) with respect to z, we obtain

$$\mathcal{H}'(z) = \Pi_{i=1}^n \left(f_i'(z)\right)^{\alpha_i} \left(\frac{f_i(z)}{z}\right)^{\beta_i}. \tag{16}$$

Let us define

$$\phi_i(z) = z(f_i'(z))^{\alpha_i}. \tag{17}$$

Clearly, $\phi_i(z) \in \mathcal{A}$. Equation (16) becomes

$$\mathcal{H}'(z) = \Pi_{i=1}^n \frac{\phi_i(z)}{z^n} \left(\frac{f_i(z)}{z}\right)^{\beta_i}. \tag{18}$$

Logarithmic differentiation of (18) yields

$$\frac{\mathcal{H}''(z)}{\mathcal{H}'(z)} = \sum_{i=1}^{n} \left[\beta_i \left(\frac{f_i'(z)}{f_i(z)} - \frac{1}{z}\right) + \left(\frac{\phi_i'(z)}{\phi(z)} - \frac{1}{z}\right)\right]. \tag{19}$$

Multiplying (19) by z and simplifying we get

$$\begin{aligned} 1 + \frac{1}{b}\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} \quad &= \quad 1 - n - \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \left\{\beta_i \left[1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right]\right. \\ &\left. \quad +1 + \frac{1}{b}\left(\frac{z\phi_i'(z)}{\phi_i(z)} - 1\right)\right\}. \end{aligned} \tag{20}$$

Adding and subtracting ρ on the right hand side of (20) gives

$$\left[\left(1+\frac{1}{b}\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)}\right)-\chi\right]=\sum_{i=1}^{n}\beta_i\left[\left(1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right)-\rho\right]$$
$$+\sum_{i=1}^{n}\left[\left(1+\frac{1}{b}\left(\frac{z\phi_i'(z)}{\phi_i(z)}-1\right)\right)-\rho\right], \qquad (21)$$

where χ is given by (15). Taking real part of (21) and after simplification, we get

$$\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)}\right)-\chi\right]\right|d\theta$$
$$\leq\sum_{i=1}^{n}\beta_i\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right)-\rho\right]\right|d\theta$$
$$+\sum_{i=1}^{n}\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\left(\frac{z\phi_i'(z)}{\phi_i(z)}-1\right)\right)-\rho\right]\right|d\theta. \qquad (22)$$

Since, by hypothesis, $f_i,\ \phi_i\in\mathcal{R}_k(\rho,b)$ for $1\leq i\leq n$, we have

$$\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right)-\rho\right]\right|d\theta\leq(1-\rho)k\pi \qquad (23)$$

and

$$\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\left(\frac{z\phi_i'(z)}{\phi_i(z)}-1\right)\right)-\rho\right]\right|d\theta\leq(1-\rho)k\pi. \qquad (24)$$

Making use of (23) and (24) in (22), we have

$$\int_0^{2\pi}\left|\Re\left[\left(1+\frac{1}{b}\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)}\right)-\chi\right]\right|d\theta\leq(1-\chi)k\pi, \qquad (25)$$

where χ is given by (15). Hence $\mathcal{H}(z)\in\mathcal{V}_k(\chi,b)$. Thus, the proof of Theorem 1 is completed.

Put $\alpha_i=0$ for all $i=1,2,3,...,n$ in Theorem 1. Notice that, in such case $\mathcal{H}(z)=\mathcal{F}_n(z)$ and $\phi_i(z)=z$ which shows

$$\frac{z\phi_i'(z)}{\phi_i(z)}-1=0.$$

Therefore, from (21), it follows that

$$\left[1+\frac{1}{b}\frac{z\mathcal{F}_n''(z)}{\mathcal{F}_n'(z)}-\lambda\right]=\sum_{i=1}^{n}\beta_i\left[\left(1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right)-\rho\right],$$

where $\lambda=1+(\rho-1)\sum_{i=1}^{n}\beta_i$.

Hence we have the following Corollary 1.

COROLLARY 1 (cf. [10, Theorem 2.1]). Let $f_i(z) \in \mathcal{R}_k(\rho, b)$ for $1 \leq i < n$ with $0 \leq \rho < 1$ and $b \in \mathbb{C} \setminus \{0\}$. Also let $\beta_i > 0, \ \ 1 \leq i \leq n$. If

$$0 \leq (\rho - 1) \sum_{i=1}^{n} \beta_i + 1 < 1,$$

then $\mathcal{F}_n(z) \in \mathcal{V}_k(\lambda, b)$ with $\lambda = (\rho - 1) \sum_{i=1}^{n} \beta_i + 1$.

Next, we take $\beta_i = 0, \ 1 \leq i \leq n$ in Theorem 1. In this case,

$$\mathcal{H}(z) = \mathcal{F}_{\alpha_1, \alpha_2, \ldots, \alpha_n}(z) = \int_0^z \Pi_{i=1}^n \left(f_i'(z)\right)^{\alpha_i},$$

which implies $\mathcal{H}'(z) = \Pi_{i=1}^n \frac{\phi_i(z)}{z^n}$. Therefore,

$$\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} = \sum_{i=1}^{n} \left(\frac{z\phi_i'(z)}{\phi_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \frac{z f_i''(z)}{f_i'(z)}.$$

We have the following result due to Noor et al. [10].

COROLLARY 2 (cf. [10, Theorem 2.5]). Let $f_i(z) \in \mathcal{V}_k(\rho, b)$ for $1 \leq i \leq n$ with $0 \leq \rho < 1$ and $b \in \mathbb{C} \setminus \{0\}$. Also, let $\alpha_i > 0, \ \ 1 \leq i \leq n$. If

$$0 \leq (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $\mathcal{F}_{\alpha_1, \alpha_2, \ldots, \alpha_n}(z) \in \mathcal{V}_k(\lambda_1, b)$ with $\lambda_1 = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1$.

REMARK 1**.** Setting $k = 2$ in Corollary 1, we obtain the results of [4, Theorem 1] and [10, Corollary 2.2].

REMARK 2**.** Setting $k = 2$ in Corollary 2, we obtain another results of [4, Theorem 3] and [10, Corollary 2.6].

THEOREM 2. Let $f_i, \ \phi_i \in \mathcal{V}_k(\rho, 1)$ for $1 \leq i < n$ with $0 \leq \rho < 1$. Let $\alpha_i, \beta_i \in \mathcal{R}^+$, $1 \leq i \leq n$. If

$$0 \leq (\beta - 1) \sum_{i=1}^{n} (1 + \beta_i) + 1 < 1, \tag{26}$$

then $\mathcal{H}(z) \in \mathcal{V}_k(l, 1)$ with $l = 1 + (\beta - 1) \sum_{i=1}^{n} (1 + \beta_i)$ and β is given by (12).

PROOF: Proceeding as Theorem 1 with $b = 1$, we have

$$\begin{aligned} &\int_0^{2\pi} \left| \Re \left[1 + \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} - l \right] \right| d\theta \\ \leq \quad & \sum_{i=1}^{n} \beta_i \int_0^{2\pi} \left| \Re \ \left[\frac{z f_i'(z)}{f_i'(z)} - \beta \right] \right| d\theta + \sum_{i=1}^{n} \int_0^{2\pi} \left| \Re \ \left[\frac{z\phi_i'(z)}{\phi_i'(z)} - \beta \right] \right| d\theta. \end{aligned} \tag{27}$$

Since $f_i,\ \phi_i \in \mathcal{V}_k(\rho, l)$ for $1 \leq i \leq n$, and by using Lemma 1, we have

$$\sum_{i=1}^{n} \int_0^{2\pi} \left| \Re \left[\frac{z f_i'(z)}{f_i'(z)} - \beta \right] \right| d\theta \leq (1-\beta)k\pi \tag{28}$$

and

$$\sum_{i=1}^{n} \int_0^{2\pi} \left| \Re \left[\frac{z \phi_i'(z)}{\phi_i'(z)} - \beta \right] \right| d\theta \leq (1-\beta)k\pi, \tag{29}$$

where β is given by (12) with $\alpha = \rho$. Using (28) and (29) in (27), we obtain

$$\int_0^{2\pi} \left| \Re \left[1 + \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} - l \right] \right| d\theta \leq (1-l)k\pi. \tag{30}$$

Thus, $\mathcal{H}(z) \in \mathcal{V}_k(l, 1)$ with $l = 1 + (\beta - 1)\sum_{i=1}^{n}(1 + \beta_i)$. The proof of Theorem 2 is completed.

REMARK 3. For $\alpha_i = 0$ we obtain the result of ([10, Theorem 2.3]).

For $n = 1,\ \alpha_1 = 0,\ \beta_1 = 1, f_1 = f$ in Theorem 2, we get the following results due to [10].

COROLLARY 3.[10] Let $f(z) \in \mathcal{V}_k(\rho, 1)$. Then the Alexander operator $\mathcal{I}(z) = \int_0^z \frac{f(t)}{t} dt$ (see [1]) belongs to the class $\mathcal{V}_k(\beta)$, where β is given by (24).

REMARK 4. For $\rho = 0$ and $k = 2$ in the above Corollary 2, we have the well known result $f(z) \in \mathcal{C}(0) \Rightarrow \mathcal{I}(z) \in \mathcal{C}\left(\frac{1}{2}\right)$.

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