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Order Of Schlictness Of Certain Linear Sums*

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Abstract

In this paper, we find the largest real number ρ such that the real part of certain geometric quantity greater than ρ implies schlictness of certain linear sums.

1 Introduction

Let $\mathbb{H}$ denote the class of functions:

$$f(z) = z + a_2 z^2 + ... \tag{1}$$

which are holomorphic in the unit disk $E = \{z \in \mathbb{C}\colon |z| < 1\}$. Let $f, g \in \mathbb{H}$, we say that $f(z)$ is subordinate to $g(z)$ (written as $f \prec g$) if there exists a holomorphic function $w(z)$ (not necessarily schlict) in E, satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in E$. For example, it is well known that any Caratheodory function $p(z) = 1 + c_1 z + \cdots$ is subordinate to the Möbius function $L_0(z) = (1 + z)/(1 - z)$.

In [2], Babalola proved that holomorphic functions satisfying

$$Re\left\{(1-\lambda)\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} + \lambda\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \beta, \quad z \in E, \tag{2}$$

also satisfy

$$Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > 0, \quad z \in E,$$

for all λ such that $\lambda \leq \beta < 1$, and that if $\beta = \frac{1}{2}$, then for all $\lambda \geq 0$,

$$Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \frac{1}{2}, \quad 0 < \alpha \leq 1.$$

D^n is defined as $D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))'$ and for $n \geq 1$, schlictness is implied by both results.

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DEFINITION. We say $f \in T^{\alpha}_{n,\lambda}(\beta)$ if and only if $f(z)$ satisfies the inequality (2).

In this paper, we shall find the largest real number ρ such that

$$Re \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho, \quad z \in E$$

given that $f \in T^{\alpha}_{n,\lambda}(\beta)$ for $0 \leq \lambda \leq \beta < 1$. To do this we shall employ the technique of Briot-Bouquet differential subordination. A holomorphic function $p(z)$ is said to satisfy Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec h(z), \quad z \in E \tag{3}$$

for complex constants η and μ, and a complex function $h(z)$ with $h(0) = 1$ such that Re $[\eta h(z) + \mu] > 0$ in E. If the differential equation

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = h(z), \qquad q(0) = 1 \tag{4}$$

has the schlict solution $q(z)$ in E, then

$$p(z) \prec q(z) \prec h(z)$$

and $q(z)$ is the best dominant.

A schlict function $\tilde{q}(z)$ is said to be the best dominant of the differential subordination (3) if $q(z)$ and $\tilde{q}(z)$ are dominants of (3) and $\tilde{q}(z) \prec q(z)$ for all the dominants $q(z)$ of (3). For more on the technique of differential subordination, see [3,4,5,7].

We state and prove our main result in Section 3.

2 Preliminary Lemmas

In this section, we present Lemmas 1–4 as follows.

LEMMA 1 ([1]). Let $f \in \mathbb{H}$ and $\alpha > 0$ be real, if

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}$$

is independent of n for $z \in E$, then

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha \frac{D^{n+1} f(z)}{D^n f(z)}.$$

LEMMA 2 ([3]). Let η and μ be complex constants and $h(z)$ be a convex schlict function in E satisfying $h(0) = 1$ and Re $[\eta h(z) + \mu] > 0$. Suppose $p \in P$ satisfies the differential subordination (3). If the differential equation (4) has schlict solution $q(z)$

in E, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (3). Furthermore, the formal solution of (4) is given as

$$q(z) = \frac{zF'(z)}{F(z)} = \frac{\eta+\mu}{\eta}\left(\frac{H(z)}{F(z)}\right)^{\eta} - \frac{\mu}{\eta} \tag{5}$$

where

$$F(z)^{\eta} = \frac{\eta+\mu}{z^{\mu}} \int_0^z t^{\mu-1} H(t)^{\eta} dt$$

and

$$H(z) = z \exp\left(\int_0^z \frac{h(t)-1}{t} dt\right).$$

LEMMA 3 ([6]). Let v be a positive measure on $[0,1]$ and let h be a complex valued function defined on $E \times [0,1]$ such that $h(.,t)$ is holomorphic in E for each $t \in [0,1]$ for all $z \in E$. Suppose that $\mathrm{Re}[h(z,t)] > 0$, $h(-r,t)$ is real and

$$\mathrm{Re}\left[\frac{1}{h(z,t)}\right] \geq \frac{1}{h(-r,t)} \text{ for } |z| \leq r < 1 \text{ and } t \in [0,1].$$

If

$$h(z) = \int_0^1 h(z,t) dv(t),$$

then

$$Re\left[\frac{1}{h(z)}\right] \geq \frac{1}{h(-r)}.$$

For real or complex numbers a, b, c with $c \neq 0, -1, -2, \dots$, the hypergeometric function is defined by

$${}_2F_1(a,b;c;z) = 1 + \frac{a.b}{c}.\frac{z}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \cdots. \tag{6}$$

We note that the series converges absolutely for $z \in E$ and hence represents an holomorphic function in E. The following identities associated with the hypergeometric series are well known.

LEMMA 4 ([6]). For real numbers a, b, c ($c \neq 0, -1, -2, \dots$), we have

$$\int_0^1 t^{b-1} \frac{(1-t)^{c-b-1}}{(1-tz)^a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \, {}_2F_1(a,b;c;z), \quad (c > b > 0),$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$$

3 Main Result

In this section, we present our main result.

THEOREM 1. Let $f \in T^{\alpha}_{n,\lambda}(\beta)$. If $\beta \geq \lambda \geq 0$, then

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$$

where

$$q(z) = \frac{\lambda(1-z)^{\frac{2\alpha(\beta-1)}{\lambda}}}{\alpha(1-\lambda)\int_0^1 s^{\frac{\alpha}{\lambda}-\alpha-1}(1-sz)^{\frac{2\alpha(\beta-1)}{\lambda}}ds} = 1 + q_1 z + \cdots$$

is the best dominant. Furthermore

$$\mathrm{Re}\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho$$

where

$$\rho = \left[{}_2F_1\left(1, \frac{2\alpha(1-\beta)}{\lambda}; \frac{\alpha}{\lambda} - \alpha + 1; \frac{1}{2}\right)\right]^{-1}.$$

PROOF. Since $f \in T^{\alpha}_{n,\lambda}(\beta)$, we see that

$$(1-\lambda)\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} + \lambda\frac{D^{n+1} f(z)}{D^n f(z)} = B_n(f;\lambda) \prec \frac{1+(1-2\beta)z}{1-z}. \tag{7}$$

Let

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = p(z). \tag{8}$$

Then $p(z)$ is holomorphic in E with $p(0) = 1$. So taking the logarithmic differentiations in both sides of (8), we have

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha + \frac{zp'(z)}{p(z)}$$

Using Lemma 1 and then simplify, we have

$$\frac{D^{n+1} f(z)}{D^n f(z)} = 1 + \frac{zp'(z)}{\alpha p(z)} \tag{9}$$

With (8) and (9) in (7), we have

$$\lambda + (1-\lambda)\left[p(z) + \frac{zp'(z)}{\alpha\left(\frac{1-\lambda}{\lambda}\right)p(z)}\right] = B_n(f;\lambda) \prec \frac{1+(1-2\beta)z}{1-z}.$$

If

$$\lambda + (1-\lambda)\left[q(z) + \frac{zq'(z)}{\alpha\left(\frac{1-\lambda}{\lambda}\right)q(z)}\right] = \frac{1+(1-2\beta)z}{1-z},$$

then

$$q(z)+\frac{zq'(z)}{\alpha(\frac{1-\lambda}{\lambda})q(z)}=\frac{1-\lambda+(1+\lambda-2\beta)z}{(1-\lambda)(1-z)}=h(z). \tag{10}$$

Then $h(0)=1$ and we take

$$\eta=\frac{\alpha(1-\lambda)}{\lambda}\quad\text{and}\quad\mu=0.$$

It is easily verified that $\eta h(z)+\mu$ has positive real part given that $0\le\lambda\le\beta$. Therefore, by Lemma 2, $p(z)$ satisfies the differential subordination (3) and hence

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\prec q(z)\prec h(z)$$

where $q(z)$ is the solution of differential equation (10) obtained as follows:

$$H(z)=z(1-z)^{\frac{2(\beta-1)}{(1-\lambda)}}.$$

Also, we have

$$F(z)=\left[\frac{\alpha(1-\lambda)}{\lambda}\int_0^z t^{\frac{\alpha}{\lambda}-\alpha-1}(1-t)^{\frac{2\alpha(\beta-1)}{\lambda}}dt\right]^{\frac{\lambda}{\alpha(1-\lambda)}}.$$

Now from (5), we have

$$q(z)=\frac{H^\eta}{F^\eta}=\frac{\lambda}{\alpha(1-\lambda)Q(z)}$$

where

$$Q(z)=\int_0^1 s^{\frac{\alpha}{\lambda}-\alpha-1}\left(\frac{1-sz}{1-z}\right)^{\frac{2\alpha(\beta-1)}{\lambda}}ds$$

Next, we show that

$$\inf_{|z|<1}\{\mathrm{Re}(q(z))\}=q(-1),\ z\in E, \tag{11}$$

To prove (11), we show that

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\}\ge\frac{1}{Q(-1)}$$

and by Lemma 4 with some simplification, we get

$$Q(z)=\frac{\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(1,a;c;\frac{z}{z-1}\right)$$

where

$$a=\frac{2\alpha(1-\beta)}{\lambda},\quad b=\frac{\alpha}{\lambda}-\alpha\quad\text{and}\quad c=\frac{\alpha}{\lambda}-\alpha+1.$$

Hence, we see that

$$Q(z)=\int_0^1 h(z,s)dv(s)$$

where

$$h(z,s)=\frac{1-z}{1-(1-s)z},\ 0\le s\le 1$$

and

$$dv(s)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}s^{a-1}(1-s)^{c-a-1}ds$$

which is a positive measure on $[0,1]$.

It will be noted that $\operatorname{Re}\{h(z,s)\}>0$, $h(-r,s)$ is real for $0\le r<1$, and

$$\operatorname{Re}\left\{\frac{1}{h(z,s)}\right\}=\operatorname{Re}\left\{\frac{1-(1-s)z}{1-z}\right\}\ge\frac{1+(1-s)r}{1+r}=\frac{1}{h(-r,s)}$$

for $|z|\le r<1$ and $s\in[0,1]$. So using Lemma 3, we have

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\}\ge\frac{1}{Q(-r)},$$

and letting $r\to 1^-$, we obtain

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\}\ge\frac{1}{Q(-1)}.$$

Hence we have

$$\operatorname{Re}\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\}>q(-1),$$

that is,

$$\operatorname{Re}\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\}>\rho$$

where

$$\rho=\frac{\lambda}{\alpha(1-\lambda)Q(-1)}=\frac{\lambda}{\alpha(1-\lambda)}\left[2^a\int_0^1 s^{b-1}(1+s)^{-a}ds\right]^{-1}.$$

By Lemma 4, we see that

$$\rho=\frac{\lambda}{\alpha(1-\lambda)}\left[\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1\left(1,a;c;\frac{1}{2}\right)\right]^{-1}$$

can be simplify as

$$\rho=\left[{}_2F_1\left(1,\frac{2\alpha(1-\beta)}{\lambda};\frac{\alpha(1-\lambda)}{\lambda}+1;\frac{1}{2}\right)\right]^{-1}.$$

The bound ρ is the best possible.

4 Remark

We first remark that for $n \geq 1$, our result shows that holomorphic function $f(z)$, satisfying the geometric condition (2) defined by linear sum, is schlict of order ρ in the unit disk. Then by (6), we can write ρ in series form:

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \frac{2\alpha(1-\beta) + j\lambda}{\alpha(1-\lambda) + (j+1)\lambda},$$

from which it follows that $\lambda = 0$ yields $\rho = \beta$ for all α. The series can be rewritten as

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \left(\frac{\alpha(1-\beta) + (j+1)\lambda}{\alpha(1-\lambda) + (j+1)\lambda} + \frac{\alpha(1-\beta) - \lambda}{\alpha(1-\lambda) + (j+1)\lambda} \right).$$

Thus, if $\lambda \leq \alpha(1-\beta)$, we have

$$\frac{1}{\rho} > 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \frac{\alpha(1-\beta) + (j+1)\lambda}{\alpha(1-\lambda) + (j+1)\lambda} > 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} > 2.$$

It follows that $\rho > 1/2$ whenever $0 < \alpha \leq \frac{\beta}{1-\beta}$ which guarantees that $\lambda \leq \alpha(1-\beta)$ given that $\lambda \leq \beta$.

This deduction shows some improvement on the earlier result of Babalola [2] for λ in the real interval $[0, \beta]$.

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